Finding normal subgroups

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The problem

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

- Assume no more knowledge about G or N.
- I shall tell you soon why we want to do this.
- We are looking for a randomised algorithm.
- Assume we can generate uniformly distributed random elements in G.
- "High probability" means for the moment "higher than $1/[G : N]$".

Matrix groups ...

Let $\mathbb{F}_q$ be the field with q elements and

$$\mathrm{GL}_n(\mathbb{F}_q) := \{M \in \mathbb{F}_q^{n \times n} \mid M \text{ invertible}\}$$

Given: $M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q)$

Then the M_i generate a group $G \leq \mathrm{GL}_n(\mathbb{F}_q)$.

It is finite, we have $|\mathrm{GL}_n(\mathbb{F}_q)| = q^{n(n-1)/2} \prod_{i=1}^{n} (q^i - 1)$

What do we want to determine about G?

- The group order $|G|$
- Membership test: Is $M \in \mathrm{GL}_n(\mathbb{F}_q)$ in G?
- Homomorphisms $\varphi : G \to H$?
- Kernels of homomorphisms? Is G simple?
- Comparison with known groups
- (Maximal) subgroups?
- ...

Constructive recognition

Problem

Let $\mathbb{F}_q$ be the field with q elements and

$$M_1, \ldots, M_k \in \mathrm{GL}_n(\mathbb{F}_q).$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathrm{GL}_n(\mathbb{F}_q)$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as word in the M_i.
- The runtime should be bounded from above by a polynomial in n, k and $\log q$.
- A Monte Carlo Algorithm is enough. (Verification!)

If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

What is a reduction?

Let $G := \langle M_1, \ldots, M_k \rangle \le \mathrm{GL}_n(\mathbb{F}_q)$.

A reduction is a group homomorphism

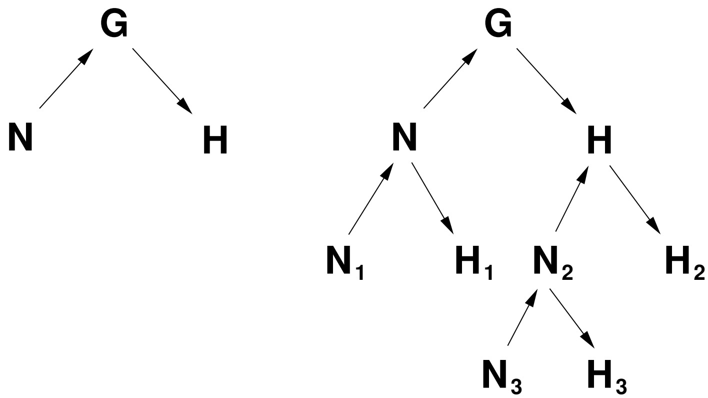
$$\begin{array}{rcccl} \varphi : & G & \to & H & \\ & M_i & \mapsto & P_i & \quad \text{for all } i \end{array}$$

with the following properties:

- $\varphi(M)$ is explicitly computable for all $M \in G$
- φ is surjective: $H = \langle P_1, \ldots, P_k \rangle$
- H is in some sense "smaller"
- or at least "easier to recognise constructively"
- e.g. $H \le S_m$ or $H \le \mathrm{GL}_{n'}(\mathbb{F}_{q'})$ with $n' \log q' < n \log q$

Recursion: composition trees

We get a tree:



Up arrows: inclusions

Down arrows: homomorphisms

Old idea, substantial improvements are still being made

Reduction in the imprimitive case

One case, in which we want to find a reduction, is:

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We can compute the reduction once N is found.

Since we can compute normal closures, our initial problem is exactly, what we need to do.

Things we can do in matrix groups

We can efficiently:

- store and compare elements
- form products and inverses,
- act on vectors, subspaces and matrices,
- compute element orders
- produce uniformly distributed random elements
- use previously assembled data about groups and representations
- compute normal closures (at least Monte Carlo).

The latter means that for $H < G$, we can compute some elements that generate with high probability the smallest normal subgroup of G containing H.

Blind descent (Babai, Beals)

Let $1 \neq x, y \in G$ and G non-abelian.

Assume at least one of x, y is contained in a non-trivial proper normal subgroup.

We do not know which!

Aim: Produce $1 \neq z \in G$ that is contained in a non-trivial proper normal subgroup.

1. Consider $c := [x, y] := x^{-1}y^{-1}xy$,
   if $c \neq 1$, we take $z := c$.
2. If $c = 1$, the elements x and y commute.
   If $x \in Z(G)$, take $z := x$.
3. Compute generators $\{y_i\}$ for $Y := \langle y^G \rangle$.
   - If some $c_i := [x, y_i] \neq 1$, then take $z := c_i$ as in 1.
   - Otherwise $g \in C_G(Y)$ but $g \notin Z(G)$, thus $Y \neq G$, we take $z := y$.

A first try

Algorithm 1 (Babai, Beals)

Initialize $1 \neq x := \textsc{RandomElement}(G)$

Repeat K times:

1. $y := \textsc{RandomElement}(G)$
2. $o := \textsc{Order}(y)$
3. $p :=$ some prime divisor of o
4. $y' := y^{o/p}$ has order p
5. $x := \textsc{BlindDescent}(x, y')$

Return x

✓

What is the Involution Jumper?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.
repeat
$y := \text{RandomElement}(G)$
$c := x^{-1}y^{-1}xy$ **and** $o := \text{Order}(c)$
if o is even **then**
return $c^{o/2}$
else
$z := y \cdot c^{(o-1)/2}$ **and** $o' := \text{Order}(z)$
if o' is even **then**
return $z^{o'/2}$
until patience lost
return FAIL

Note: If $xy = yx$ then $c = 1_G$ and $o = 1$ and $z = y$.
But this happens rarely.

What does the Involution Jumper do?

Input: $G = \langle g_1, \ldots, g_k \rangle$ and an involution $x \in G$.

- If it does not fail, it returns an involution $\tilde{x} \in G$.
- $x\tilde{x} = \tilde{x}x$
- Every involution of $C_G(x)$ occurs with probability > 0.
- Using product replacement to produce random elements, this is a practical method for
  - permutation groups,
  - matrix groups and
  - projective groups,

  if nothing goes wrong.
- It needs an involution to start with.

Jumping between classes

Notation: Let x^G denote the conjugacy class of x in G.

Lemma

Let $x, a \in G$ be involutions and $g \in G$. Then

$$\mathit{Prob}(\mathit{IJ}(x) \in a^G) = \mathit{Prob}(\mathit{IJ}(x^g) \in a^G).$$

or equivalently

Lemma

Let $x \in G$ be an involution. Then the distribution of $\mathit{IJ}(x)^G$ only depends on x^G and not on the choice of x in x^G.

Proof: $f(x, y) :=$

$$\begin{cases} [x,y]^k & \text{if } \mathrm{ORDER}([x,y]) = 2k \\ (y[x,y]^k)^l & \text{if } \mathrm{ORDER}([x,y]) = 2k+1 > 1 \text{ and} \\ & \quad \mathrm{ORDER}([y[x,y]^k]) = 2l \\ y^k & \text{if } xy = yx \text{ and } \mathrm{ORDER}(y) = 2k \end{cases}$$

and we have $f(x^g, y^g) = f(x, y)^g$ whenever f is defined. ✓

A Markov chain $\mathcal{M}$

The states are the conjugacy classes of involutions in G.

The transition is done as follows: At a class a^G:

- Pick an arbitrary involution $x \in a^G$.
- Compute $\tilde{x} := IJ(x)$ until $\tilde{x} \neq \textsc{Fail}$.
- Next state is $\tilde{x}^G$.

By the lemma, the distribution of the class $\tilde{x}^G$ does not depend on the choice of x.

Theorem

The above Markov chain $\mathcal{M}$ is irreducible and aperiodic and thus has a stationary distribution in which every state has non-zero probability.

Back to the original question

Problem

Let $1 < N \lhd G = \langle g_1, \ldots, g_k \rangle$ be a finite group and N be a normal subgroup.
Produce a non-trivial element of N as a word in the g_i with "high probability".

- If we find an involution in G to start with
- and N contains at least one involution class,

the IJ will eventually jump onto an involution class in N.

✓

A better try

Algorithm 2

Initialize $1 \neq x := \textsc{RandomElement}(G)$ and
$z := \textsc{RandomInvolution}(G)$

Repeat K times:

1. $y := \textsc{RandomElement}(G)$
2. $o := \textsc{Order}(y)$
3. For a few prime divisors p of o do:
   - $y' := y^{o/p}$ has order p
   - $x := \textsc{BlindDescent}(x, y')$
4. $z := \textsc{InvolutionJumper}(G, z)$
5. $x := \textsc{BlindDescent}(x, z)$

Return x

Examples

In practice, the IJ works extremely well in many cases:

G	N	# hops*
$S_5 \wr S_{10}$	$S_5^{\times 10}$	1.91
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	1.17
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$	$\mathrm{Sp}(6,3) \otimes 1$	1.83

* average number of IJ hops needed to reach N.

Running Algorithm 2 (with $K = 5$) also works nicely:

G	N	succ.
$S_5 \wr S_{10}$	$S_5^{\times 10}$	100%
$\mathrm{GL}(3,3) \wr S_6 < \mathrm{GL}(18,3)$	$\mathrm{GL}(3,3)^{\times 6}$	100%
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$	$\mathrm{Sp}(6,3) \otimes 1$	100%

(here we have done 100 runs)

Reductions for imprimitive matrix groups

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

$$V|_N = W_1 \oplus W_2 \oplus \cdots \oplus W_k,$$

all W_i are invariant under N, and G permutes the W_i transitively. Then there is a reduction $\varphi : G \to S_k$.

We use Algorithm 2, for the result x, do:

- compute the normal closure $M := \langle x^G \rangle$,
- use the MeatAxe to check whether $V|_M$ is reducible,
- if $x \in N$, we find a reduction.

Possible problems

The InvolutionJumper is in trouble, if at least one of the following happens:

- we do not easily find an involution in G (like for example in $\mathrm{SL}(2, 2^n)$ for big n),
- the involution classes of N have a small probability in the limit distribution (when does this happen?),
- the Markov chain does not converge quick enough to its limiting distribution (how quick does it converge?),
- the Involution Jumper returns FAIL too often (when does this happen?),
- N has odd order.

Fortunately: Centralisers of involutions seem to contain enough involutions.