Short(er) SLPs in group recognition

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(joint work with Ákos Seress)

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Constructive recognition

Problem

Let $\mathcal{G}$ be S_n or $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{PGL}_n(\mathbb{F}_q)$ and

$$M_1, \ldots, M_k \in \mathcal{G}.$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- a procedure that, given $M \in \mathcal{G}$,
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If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

Recursion

We build a so-called composition tree:

$$\begin{tikzcd}
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Up arrows: inclusions
Down arrows: homomorphisms

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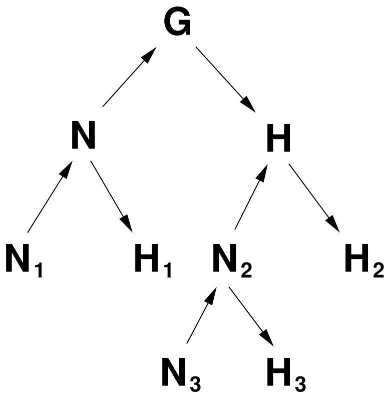

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Down arrows: homomorphisms

Recognising image and kernel suffices

Let $\varphi : G \to H$ be an epimorphism and assume that both H and the kernel $N = \langle N_1, \ldots, N_m \rangle$ of φ are already recognised constructively (assume the N_i are expressed in terms of the M_j).

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7. If $M \notin G$, then at least one step does not work.

Long SLPs

Typical examples:

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Learning from base and strong generators

The same groups with stabiliser chains:

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Typical elements in W give SLPs of length $\approx$ 500.

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We compare lengths of SLPs:

	Stabiliser chain		Composition tree	
	in strong			in gens
G	15			900
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$\Longrightarrow$ "nice generators"

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Problems with recursion

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Recall: Generators of H were images of those of G.

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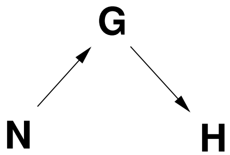

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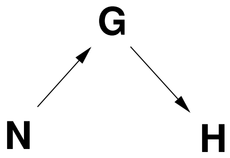

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Note: The first allows to compute N once H is recognised!

Constructive recognition revisited

Problem — new formulation

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Having recognised H in this sense, we can:

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- put together the nice generators for G.

If we remember how we created the generators for N, then we have recognised G constructively:

- Using H and N we can test membership in G,
- express elements as SLPs in the nice generators,
- and, given preimages of the original generators of G under some homomorphism, we can find preimages of the nice generators.

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Note that a stabiliser chain for G has

- length 132 and 2512 strong generators,
- typical SLP in the strong generators: $\approx$ 2700 lines,
- typical SLP in the original generators: $\approx$ 12000 lines.