# Short(er) SLPs in group recognition

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# Constructive recognition

## Problem

Let $\mathcal{G}$ be $S_n$ or $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{PGL}_n(\mathbb{F}_q)$ and

$$M_1, \ldots, M_k \in \mathcal{G}.$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

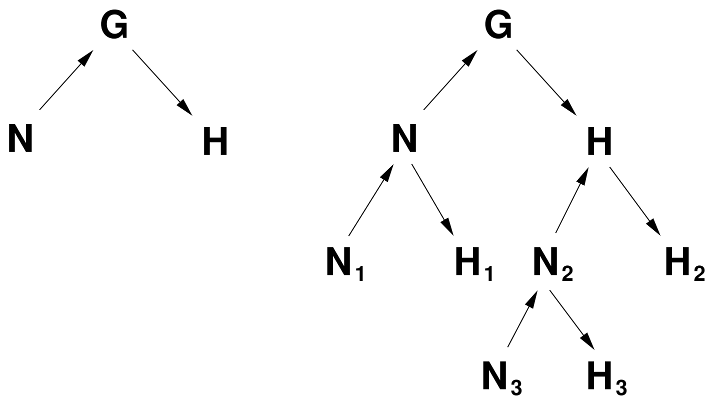
- The group order $|G|$ and
- a procedure that, given $M \in \mathcal{G}$,
  - decides, whether or not $M \in G$ and
  - if so, expresses $M$ as an SLP in the $M_i$.
- The runtime should be bounded from above by a polynomial in $n$, $k$ and $\log q$.
- A Monte Carlo Algorithmus is enough. (Verification!)

If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

# Recursion

We build a so-called composition tree:




Up arrows: inclusions
Down arrows: homomorphisms

# Recognising image and kernel suffices

Let $\varphi : G \to H$ be an epimorphism and assume that both $H$ and the kernel $N = \langle N_1, \ldots, N_m \rangle$ of $\varphi$ are already recognised constructively (assume the $N_i$ are expressed in terms of the $M_j$).

Then we have recognised $G$ constructively:

$|G| = |H| \cdot |N|$. And for $M \in \mathcal{G}$:

1. map $M$ with $\varphi$ onto $\varphi(M) \in H = \langle P_1, \ldots, P_k \rangle$,
2. express $\varphi(M) = \mathrm{SLP}_1(P_1, \ldots, P_k)$,
3. evaluate the same SLP: $M' := \mathrm{SLP}_1(M_1, \ldots, M_k)$,
4. get element $M' \in G$ such that $M \cdot M'^{-1} \in N$,
5. express $M \cdot M'^{-1} = \mathrm{SLP}_2(N_1, \ldots, N_m)$,
6. get $M$ as SLP in the $M_i$ and $N_j$:

   $$\Rightarrow M = \mathrm{SLP}_2(N_1, \ldots, N_m) \cdot \mathrm{SLP}_1(M_1, \ldots, M_k).$$

7. If $M \notin G$, then at least one step does not work.

# Long SLPs

Typical examples:

(1) $G := (2 \times 2^{1+8}) : U_4(2) : 2 < \mathrm{GL}_{78}(2)$

(7th maximal subgroup of the sporadic simple group $\mathrm{Fi}_{22}$)

$G$ has 53 084 160 elements, generated by 2 elements.

Composition tree of depth 8 with 3 non-trivial leaves.

Typical elements in $G$ give SLPs of length $\approx$ 900.

(2) $W := S_{12} \wr S_5 < S_{60}$

$W$ has 3 025 980 091 991 082 565 958 286 705 898 291 200 000 000 000 elements and is generated by 12 elements.

Composition tree of depth 4 with 6 non-trivial leaves.

Typical elements in $W$ give SLPs of length $\approx$ 10000.

# Learning from base and strong generators

The same groups with stabiliser chains:

$$G := (2 \times 2^{1+8}) : U_4(2) : 2 < S_{3510}$$

(7th maximal subgroup of the sporadic simple group $\mathrm{Fi}_{22}$)

Stabiliser chain of length 4 with 14 strong generators.

Typical elements in $G$ give SLPs of length $\approx$ 15.

$$W := S_{12} \wr S_5 < S_{60}$$

Stabiliser chain of length 55 with 434 strong generators.

Typical elements in $W$ give SLPs of length $\approx$ 500.

# Comparison

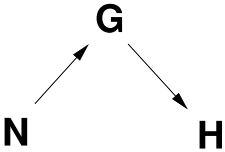
We compare lengths of SLPs:

| | Stabiliser chain | | Composition tree | |
|---|---|---|---|---|
| | in strong | in gens | in nice | in gens |
| $G$ | 15 | 290 | 15 | 900 |
| $S_{12} \wr S_5$ | 500 | 4300 | 300 | 10000 |

We want to change the generating system!

$\Longrightarrow$ "nice generators"

# Problems with recursion




Recall: Generators of $H$ were images of those of $G$.

Having changed the generators in $H$,

we can no longer find preimages!

Solution: Nice generators of $G$ are

- preimages of the nice generators of $H$ together with
- nice generators of $N$.

Note: The first allows to compute $N$ once $H$ is recognised!

# Constructive recognition revisited

## Problem — new formulation

Let $\mathcal{G}$ be $S_n$ or $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{PGL}_n(\mathbb{F}_q)$ and

$$M_1, \ldots, M_k \in \mathcal{G}.$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$,
- new nice generators $G = \langle N_1, \ldots, N_m \rangle$ and
- a procedure that, given $M \in \mathcal{G}$,
  - decides, whether or not $M \in G$ and
  - if so, expresses $M$ as an SLP in the $N_j$ and
- another procedure that, given preimages $\hat{M}_1, \ldots, \hat{M}_k$ of the $M_i$ under some homomorphism onto $G$, produces preimages of the nice generators.

If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

# Recursion works again

Having recognised $H$ in this sense, we can:

- ask $H$ to generate preimages of its nice generators,
- compute generators for $N$,
- recursively recognise $N$ and
- put together the nice generators for $G$.

If we remember how we created the generators for $N$, then we have recognised $G$ constructively:

- Using $H$ and $N$ we can test membership in $G$,
- express elements as SLPs in the nice generators,
- and, given preimages of the original generators of $G$ under some homomorphism, we can find preimages of the nice generators.

# Example

Let

$$G := S_{12} \wr M_{12} < S_{144}$$

$G$ has about $10^{109}$ elements and is generated by 27 elements.

Composition tree of depth 5 with 13 non-trivial leaves.

Typical elements in $G$ give

SLPs of length $\approx$ 800 in 33 nice generators.

SLPs of length $\approx$ 40000 in 27 original generators!

Note that a stabiliser chain for $G$ has

- length 132 and 2512 strong generators,
- typical SLP in the strong generators: $\approx$ 2700 lines,
- typical SLP in the original generators: $\approx$ 12000 lines.