Some Calculations regarding Foulkes' Conjecture

Max Neunhöffer

max.neunhoeffer@math.rwth-aachen.de

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Notation

(the following was presented on the blackboard)

joint work with J. Müller

$M_n := \{1, 2, \ldots, n\}$, $S_n := \{\pi : M_n \to M_n$ bijective$\}$

$\pi \cdot \varphi$ means first π, then φ for mappings throughout

$$S_m \,\text{wr}\, S_n := (\underbrace{S_m \times \cdots \times S_m}_{n \text{ factors}}) \rtimes S_n \text{ (wreath product)}$$

$\Longrightarrow |S_m \,\text{wr}\, S_n| = (m!)^n \cdot n!$, $S_m \,\text{wr}\, S_n \leq S_{m \cdot n}$

$\Omega_{m,n} := S_m \,\text{wr}\, S_n \backslash S_{m \cdot n} = \{(S_m \,\text{wr}\, S_n) \cdot \pi \mid \pi \in S_{m \cdot n}\}$

Foulkes' conjecture

Lemma:

If $m > n$, then $|S_n \,\mathrm{wr}\, S_m| < |S_m \,\mathrm{wr}\, S_n|$.

Conjecture:

Let $m > n$. Then the permutation module $\mathbb{Q}\Omega_{m,n}$ is a submodule of the permutation module $\mathbb{Q}\Omega_{n,m}$.
(as $\mathbb{Q}S_{m \cdot n}$-modules)

Implementation of Action of $S_{m \cdot n}$ on $\Omega_{m,n}$

$G := S_{m \cdot n}$, $H := S_m \operatorname{wr} S_n$, $S := S_n$, $U := S_m \times \cdots \times S_m \lhd H$

$V := V^{(m,n)} :=$

$\{v : M_{m \cdot n} \to M_n \mid v$ takes every value exactly m times$\}$

$\to$ store as vectors of length $m \cdot n$ with entries in M_n.

We get transitive actions

of S on V by $V \times S \to V$, $(v, \pi) \mapsto v \cdot \pi$ and

of G on V by $V \times G \to V$, $(v, \psi) \mapsto \psi^{-1} \cdot v =: v\psi$.

These actions commute:

$((v \cdot \pi)\psi) = \psi^{-1} \cdot v \cdot \pi = (v\psi) \cdot \pi$ for all v, π, ψ

$\Longrightarrow G$ acts on the set of S-orbits, let $\Omega := V/S$.

$\Longrightarrow \operatorname{Stab}_G(v_1) = U$ with $v_1 := [1, \ldots, 1, 2, \ldots, 2, \ldots, n, \ldots, n]$

and $\operatorname{Stab}_G(v_1 S) = H \Longrightarrow$ this is the action on $\Omega_{m,n}$.

Implementation of Action of $S_{m \cdot n}$ on $\Omega_{m,n}$

Def.: We call the lexicographically smallest vector in each S-orbit S-minimal.

This means: In an S-minimal vector the first entry is a 1, the first entry bigger than 1 is 2. The first entry bigger than 2 is a 3 etc.

Algorithm: Identify $\Omega_{m,n}$ with $\{v \in V^{(m,n)} \mid v$ is $S -$ minimal$\}$

Act with an element $\psi \in G$ by:

1. $v' := v\psi = \psi^{-1} \cdot v$
2. S-minimalize the result v' ($\longrightarrow$ doable in $O(m \cdot n)$)

This can all be implemented efficiently on a computer.

We need typically 1 byte per entry or $m \cdot n$ bytes per vector.

The Idea of Black and List

Construct a $\mathbb{Q}G$-module homomorphism

$$\varphi^{(m,n)} : \mathbb{Q}\Omega_{m,n} \to \mathbb{Q}\Omega_{n,m}$$

and show injectivity. For $v \in \Omega_{m,n}$ set

$$\begin{aligned} I(v) := \{w \in \Omega_{n,m} \quad | \quad & \forall (i,j) \in M_n \times M_m \ \exists! k \in M_{m\cdot n} \text{ s.t.} \\ & v(k) = i \text{ and } w(k) = j\} \end{aligned}$$

and $\varphi^{(m,n)}(v) := \sum_{w \in I(v)} w \in \mathbb{Q}\Omega_{n,m}$.

Example: $m = 3$, $n = 2$, $v = v_1 = [1,1,1,2,2,2]$

$$\overset{\varphi^{(3,2)}}{\longmapsto} \quad [\overbrace{1,2,3}^{\text{S-minimal}},\underbrace{1,2,3}_{\text{all permutations}}] + [1,2,3,1,3,2] + \cdots$$

The Idea of Black and List

This is a $\mathbb{Q}G$-module homomorphism.

Black & List (1989):

$\varphi^{(m,n)}$ injective $\Longrightarrow$ Foulkes' conjecture is true for m, n.

Lemma: $m > n$ and $\varphi^{(m,n+1)}$ injective $\Longrightarrow$ $\varphi^{(m,n)}$ injective.

Theorem: $\varphi^{(m,n)}$ injective for certain $m \geq n > 1$,
$\Longrightarrow$ Foulkes' conjecture true for all (m, r) with $1 \leq r \leq n$.

Observation: $\varphi^{(3,3)}$ and $\varphi^{(2,2)}$ injective.

Juby Jacob, Jürgen Müller (2003): $\varphi^{(4,4)}$ injective

What about $\varphi^{(5,5)}$???

End.-Ring of a Permutation Module

Let G act on Ω, $v_1 \in \Omega$, $H := \mathrm{Stab}_G(v_1)$.

$$\Omega = v_1 H \cup v_2 H \cup \cdots \cup v_l H \quad \text{(disjoint)}$$

for some $v_2, \ldots, v_l \in \Omega$. Then

$$\Omega \times \Omega = (v_1, v_1)G \cup (v_1, v_2)G \cup \cdots \cup (v_1, v_l)G \quad \text{(disjoint)}$$

are the G-orbits in $\Omega \times \Omega$ (diagonal action). The Schur basis of $\mathrm{End}(\mathbb{Q}\Omega)$ consists of matrices $A^{(1)}$, $A^{(2)}$, . . . , $A^{(l)}$ with

$$A^{(i)}_{\omega,\omega'} := \begin{cases} 1 & \text{if } (\omega, \omega') \in (v_1, v_i)G \\ 0 & \text{otherwise} \end{cases}$$

(with respect to the natural basis, column convention).

$\varphi^{(m,m)}$ expressed in the Schur basis

$\varphi^{(m,m)}$ is an endomorphism of $\mathbb{Q}\Omega_{m,m}$.
To express it in the Schur basis, look at first row of matrix:
Which vectors $v \in \Omega_{m,m}$ have the property, that

$$v_1 = [1, 1, \ldots, 1, 2, \ldots, 2, \ \ldots \ , m, \ldots, m]$$

occurs in $\varphi^{(m,m)}(v)$?
Exactly those of the form

$$v_2 := [1, 2, 3, \ldots, m, \underbrace{1, 2, 3, \ldots, m}_{\text{permuted}}, \ \ldots \ , \underbrace{1, 2, 3, \ldots, m}_{\text{permuted}}]$$

This is exactly the H-orbit $v_2 H \Longrightarrow$ matrix of $\varphi^{(m,m)}$ is $A^{(2)}$.

Use regular representation of End$(\mathbb{Q}\Omega)$

We compute using the left regular representation:

$$A^{(2)} \cdot A^{(j)} =: \sum_{k=1}^{l} p_{2,j,k} \cdot A^{(k)}$$

Use structure constants of End$(\mathbb{Q}\Omega_{m,m})$.

Structure constants with respect to the Schur basis are intersection numbers:

$$p_{2,j,k} = |v_2 H g_k^{-1} \cap v_{j^*} H|$$

where $g_1, \ldots, g_l \in G$ with $v_i = v_1 g_i$ ($*$ some involution).

So: Run through $v_2 H$, apply g_k^{-1}, recognize H-orbit, count.

Size of the Problem

$|G| = 25! \approx 1.5 \cdot 10^{25}$ $\qquad |H| = |S_5 \,\mathrm{wr}\, S_5| \approx 3 \cdot 10^{12}$

$|\Omega_{5,5}| \approx 5 \cdot 10^{12}$, one vector uses 25 bytes

$\Longrightarrow 1.3 \cdot 10^{14}$ bytes ≈ 130 Terabyte!

$|v_2 H| \approx 2 \cdot 10^8$, $l = 1856$ (character theory).

So: $1856 \cdot 2 \cdot 10^8 \approx 3.8 \cdot 10^{11}$ operations,
for each: recognize H-orbit.

$\Longrightarrow$ DOABLE by parallelization!

But how can we recognize the H-orbit a vector v lies in?

(without storing the full orbit!)

A Trick

Let $U := S_5 \times \cdots \times S_5 \lhd H$.
Consider first all vectors in $V := V^{(5,5)}$.
Def.: We again call the lexicographically smallest vector in each U-orbit U-minimal.
Idea: Only store U-minimal vectors.
Problem: Action is perm. of entries $+$ S-minimalization.

Lemma:
v S-minimal $\Longrightarrow$ the U-minimalization of v is S-minimal.

So:
(1) permute, (2) S-minimalize, (3) U-minimalize, (4) lookup.
$\rightarrow$ get S-minimal and U-minimal vector
(there are only 2298891 of those).

Those we can classify beforehand into H-orbits.

The Computation

Precomputation:

- Enumerate all U- and S-minimal vectors in $\Omega_{5,5}$.
- Determine their distribution into the 1856 H-orbits.
- Compute at the same time permutations $g_1, \ldots, g_{1856}$.

Main Computation (parallel, distribute data):

Run (parallelized) through $v_2 H$, apply all g_i^{-1}, and do:

- S-minimalize
- U-minimalize
- lookup H-orbit
- count

The Result

≈ 20 modern PCs ≈ 12 hours CPU time (special C-program).

$\rightarrow$ get 1856×1856 matrix for $A^{(2)}$ in left-regular representation with respect to Schur basis of End$(\mathbb{Q}\Omega_{5,5})$.

RESULT:

$\varphi^{(5,5)}$ is NOT INJECTIVE!

Bibliography

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