

COMPLETING THE BRAUER TREES FOR THE SPORADIC SIMPLE LYONS GROUP

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ABSTRACT. In this paper we complete the Brauer trees for the sporadic simple Lyons group Ly in characteristics 37 and 67. The results are obtained using tools from computational representation theory, in particular a new condensation technique, and with the assistance of the computer algebra systems MeatAxe and GAP.

1. INTRODUCTION

1.1. In this paper we complete the Brauer trees for the sporadic simple Lyons group Ly in characteristics 37 and 67. The results are stated in Section 2, and will also be made accessible in the character table library of the computer algebra system GAP and electronically in [1]. The shape of the Brauer trees as well as the labelling of nodes up to algebraic conjugacy of irreducible ordinary characters had already been found in [8, Section 6.19.], while here we complete the trees by determining the labelling of the nodes on their real stems and their planar embedding; proofs are given in Section 4. Together with the results in [8, Section 6.19.] for the other primes dividing the group order, this completes all the Brauer trees for Ly .

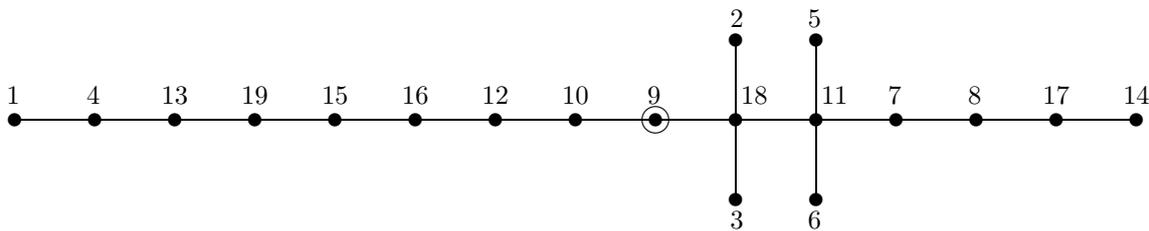
Our main computational workhorse is fixed point condensation, which originally was invented for permutation modules in [20], but has been applied to different types of modules as well. To our knowledge, the permutation module we have condensed is the largest one for which this has been accomplished so far. The theoretical background of the idea of condensation is described in Section 3. Details of the particular condensation technique used, and of some other computational tricks, are given in Section 5. For the heavy part of our computations we have built upon and modified the implementation described in [11]. In all parts of our work, we have made substantial use of the computer algebra systems MeatAxe [18] and GAP [6]. Due to our standard setting, see Section 1.2, to write down the set of Brauer characters in the case $p = 67$ we have to know the 67-modular reduction of $\sqrt{37} \in \mathbb{R}^+$. The latter depends on the Conway polynomial $C_{67,18}$, which at our request was computed in [12]. As again some tricks and heavy computation are involved in finding $C_{67,18}$, we indicate in Section 2.2 what would be known without having $C_{67,18}$ at hand.

We remark that a method similar to the one described here has also been used to solve similar, admittedly much smaller sized, problems for the sporadic simple

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TABLE 1. The Brauer tree for $p = 37$.

Thompson group Th [4], for the sporadic simple Rudvalis group Ru and its double cover $2.Ru$ [19], as well as for the sporadic simple O’Nan group ON and its triple cover $3.ON$ [15]. The latter results completed all the Brauer character tables for $3.ON$ and $2.Ru$ in all characteristics dividing the respective group order.

1.2. The standard setting. We assume the reader is familiar with block theory and decomposition theory of finite groups as well as with the Brauer-Dade theory of blocks of cyclic defect, see e.g. [5, Sections IV. and VII.].

Throughout we use the standard choice of p -modular systems (L, R, F) as described in [9, Introduction]. Here, L is an algebraic number field, R is a discrete valuation ring in L , with maximal ideal $\mathfrak{m} \triangleleft R$ and residue class field $F := R/\mathfrak{m}$ of characteristic p . Let $\overline{} : R \rightarrow F$ denote the natural epimorphism. Assume $\zeta_{p,n} := \exp(2\pi i/(p^n - 1)) \in R \subset L \subset \mathbb{C}$ for some $n \in \mathbb{N}$. Then the minimum polynomial of $\overline{\zeta_{p,n}} \in F$ over \mathbb{F}_p is the n -th Conway polynomial $C_{p,n} \in \mathbb{F}_p[X]$, where again we assume the reader is familiar with the compatibility properties of the $C_{p,n}$, when n varies.

2. RESULTS

In this section, we state the Brauer trees for Ly for the cases $p = 37$ and $p = 67$. In both cases, the principal p -block has defect 1 and is the only block of positive defect. The nodes of the Brauer tree are labelled by $1, 2, \dots$, where for each case we indicate the corresponding irreducible ordinary characters of Ly according to the numbering given in [3, p.174]. The labelling of the nodes of the Brauer trees and their planar embedding depend on the definition of the conjugacy classes of Ly in terms of the chosen pair of standard generators for Ly , see Section 4.3. For the planar embedding we use the convention given in [8, Section 1.2.].

2.1. The case $p = 37$.

Table 1	1	2	3	4	5	6	7	8	9	10	
[3]	1	2	3	4	7	8	11	12	24	25	33

Table 1	11	12	13	14	15	16	17	18	19
[3]	39	40	41	42	43	47	48	49	52

The exceptional node 9 has multiplicity 2. The sets of algebraically conjugate ordinary characters are: $\{2, 3\}$ and $\{5, 6\}$ consisting of pairs of complex conjugate characters; and $\{11, \dots, 15\}$ and $\{16, 17\}$ consisting of real characters. The Brauer tree is given in Table 1.

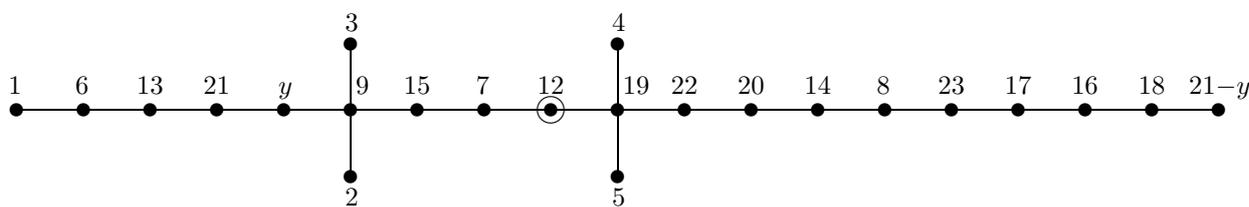
TABLE 2. The Brauer tree for $p = 67$; for y see Section 2.2.2.2. The case $p = 67$.

Table 2	1	2	3	4	5	6	7	8	9	10	11	12		
[3]	1	2	3	5	6	11	20	21	22	24	25	26	27	28

Table 2	13	14	15	16	17	18	19	20	21	22	23
[3]	29	30	31	32	39	40	41	42	43	44	50

The exceptional node 12 has multiplicity 3. The sets of algebraically conjugate ordinary characters are: $\{2, 3\}$ and $\{4, 5\}$ consisting of pairs of complex conjugate characters; and $\{8, 9\}$, $\{10, 11\}$, $\{13, 14\}$, $\{15, 16\}$, and $\{17, \dots, 21\}$ consisting of real characters. We remark that there is a mistake in [8, p.271] concerning the relative position of the characters $\{8, 9\}$ and $\{13, 14\}$ on the Brauer tree. The Brauer tree is given in Table 2, where only the value of $y \in \{10, 11\}$ depends on the Conway polynomial $C_{67,18}$, see Section 1. As $X^2 - 37 = (X - 38)(X - 29) \in \mathbb{F}_{67}[X]$, we have $\sqrt{37} = 38 \in \mathbb{F}_{67}$ or $\sqrt{37} = 29 \in \mathbb{F}_{67}$ anyway. The former case leads to $y = 10$, while the latter leads to $y = 11$. Using the Conway polynomial $C_{67,18} = 2 + 13X + 59X^2 + 6X^3 + 51X^4 + 29X^5 + 28X^6 + 55X^7 + 33X^8 + 18X^9 + 52X^{10} + 63X^{11} + X^{12} + X^{18}$ computed in [12], we find $\sqrt{37} = 38 \in \mathbb{F}_{67}$, hence $y = 10$, for our standard choice of 67-modular system.

3. CONDENSATION

3.1. Let θ be a principal ideal domain or a field, and A be a θ -algebra, which is a finitely generated θ -free θ -module. Let $\mathbf{mod}_{\theta}\text{-}A$ be the category of finitely generated and θ -free right A -modules, where for $V, W \in \mathbf{mod}_{\theta}\text{-}A$ the homomorphism set is the set of all $\alpha \in \text{Hom}_A(V, W)$ such that $\text{im}(\alpha) \leq W$ is a θ -pure submodule.

Let $e \in A$ be an idempotent, i.e. $0 \neq e = e^2$. Then the additive exact functor $? \otimes_A Ae \cong \text{Hom}_A(eA, ?)$ from $\mathbf{mod}_{\theta}\text{-}A$ to $\mathbf{mod}_{\theta}\text{-}eAe$ is called the *condensation functor with respect to e* , see [7, Section 6.2]. The image $Ve \in \mathbf{mod}_{\theta}\text{-}eAe$ of some $V \in \mathbf{mod}_{\theta}\text{-}A$ under this functor is called the *condensed module* of V . Note that under this functor an A -homomorphism defined on V is simply mapped to its restriction to Ve .

3.2. Let θ be a field and $S \in \mathbf{mod}_{\theta}\text{-}A$ simple, then either $Se = \{0\}$ or $Se \in \mathbf{mod}_{\theta}\text{-}eAe$ is also simple, since for $0 \neq v, v' \in Se$ we have $v \cdot eae = v'$ for some $a \in A$. The following Lemma shows that we can distinguish non-isomorphic simple modules in $\mathbf{mod}_{\theta}\text{-}A$ by their condensed modules, if the latter are different from $\{0\}$. Note that we do not assume that $Se \neq \{0\}$ holds for all $S \in \mathbf{mod}_{\theta}\text{-}A$.

Lemma. Let θ be a field and let $S, S' \in \mathbf{mod}_{\theta}\text{-}A$ be simple, such that $Se \neq \{0\} \neq S'e$. Then $S \cong S'$ in $\mathbf{mod}_{\theta}\text{-}A$ if and only if $Se \cong S'e$ in $\mathbf{mod}_{\theta}\text{-}eAe$.

Proof. We only need to show the ‘if’ part. Assume that $Se \cong S'e$ in $\mathbf{mod}_\theta\text{-}eAe$ and choose a decomposition of e as a sum of pairwise orthogonal primitive idempotents in A . As $\text{Hom}_A(eA, S) \cong Se \neq \{0\}$, there is a summand, e_S say, such that e_SA is a projective indecomposable module with $e_SA/\text{rad}(e_SA) \cong S$. Applying the condensation functor with respect to the idempotent $e_S \in eAe$, we obtain $Se_S \cong S'e_S$ in $\mathbf{mod}_{\theta-e_SAe_S}$. Hence we have $\{0\} \neq S'e_S \cong \text{Hom}_A(e_SA, S')$, thus $S' \cong S$ in $\mathbf{mod}_{\theta-A}$. $\#$

3.3. Let $\theta := R$, where R is as in Section 1.2. Let $A_L := A \otimes_R L$ and $A_F := A \otimes_R F$. If $\hat{e} \in A \subset A_L$ is an idempotent, then $e := \bar{\hat{e}} \in A_F$ also is. The exact additive functors $\text{Hom}_A(\hat{e}A, ?) \otimes_R L$ and $\text{Hom}_{A_L}(\hat{e}A_L, ? \otimes_R L)$ from \mathbf{mod}_{R-A} to $\mathbf{mod}_{L-\hat{e}A_L\hat{e}}$ are equivalent, as well as the exact additive functors $\text{Hom}_A(\hat{e}A, ?) \otimes_R F$ and $\text{Hom}_{A_F}(eA_F, ? \otimes_R F)$ from \mathbf{mod}_{R-A} to \mathbf{mod}_{F-eA_Fe} .

This means that, given $\hat{V} \in \mathbf{mod}_{L-A_L}$ with R -form $V \in \mathbf{mod}_{R-A}$, the p -modular reduction $\bar{V}\hat{e}$ of the condensed module of V can be identified with the condensed module $\bar{V}e$ of the p -modular reduction of V . In this sense we speak of the condensed module in \mathbf{mod}_{F-eA_Fe} of a representation of A_L .

3.4. **Fixed point condensation.** We are going to apply a particular condensation functor, so-called *fixed point condensation*. Keeping the notation from Section 3.3, let $A = R[G]$ be the group algebra of a finite group G and $e = e_K := \bar{\hat{e}} \in F[G]$, where $\hat{e} = \hat{e}_K := |K|^{-1} \sum_{k \in K} k \in R[K] \subseteq R[G]$, where in turn K is a subgroup of G whose order $|K|$ is not divisible by the characteristic p of F .

Let Ω be a finite set acted on by G , and $F[\Omega]$ be the corresponding $F[G]$ -permutation module. Then the condensed module $F[\Omega]e$ can be described as follows. Let $\{\Omega_i; 1 \leq i \leq r\}$ be the set of K -orbits on Ω , and $\Omega_i^+ := \sum_{\omega \in \Omega_i} \omega \in F[\Omega]$ be the orbit sums. Then $\{\Omega_i^+; 1 \leq i \leq r\}$ is an F -basis of $F[\Omega]e$, and for $g \in G$ the action of $ege \in eF[G]e$ on $F[\Omega]e$ is given as

$$\Omega_i^+ \cdot ege = \sum_{1 \leq j \leq r} a_{ij}(g) \cdot |\Omega_j|^{-1} \cdot \Omega_j^+, \text{ where } a_{ij}(g) = |\{\omega \in \Omega_i; \omega g \in \Omega_j\}|,$$

and where we consider $a_{ij}(g)$ and $|\Omega_j|$ as elements of F . Hence to find the action of ege , we have to find the K -orbits $\{\Omega_i\}$, their lengths $|\Omega_i|$, and the $a_{ij}(g) \in \mathbb{Z}$. Note that this does not depend on the particular choice of F , and that an analogous description holds for F replaced by R or L and e replaced by \hat{e} .

We are going to apply a new variant of fixed point condensation of permutation modules, the so-called *direct condensation* technique, whose basic idea has been invented in [17]. It has subsequently been modified, extending its range of applicability, in [11]. Here, we have built upon and modified the latter implementation, see Sections 4.4 and 5.2.

3.5. Let $V \in \mathbf{mod}_{F-F[G]}$ and φ_V be its Brauer character, viewed as an L -valued class function on the p' -classes of G . Then φ_V can be extended to a class function $\tilde{\varphi}_V$ on the whole of G as follows. For $g \in G$ let $g_p, g_{p'} \in G$ denote its p -part and p' -part, respectively, and let $\tilde{\varphi}_V(g) = \varphi_V(g_{p'})$. As $F[K]$ is a semisimple algebra, we have $\dim_F(Ve) = \langle (\varphi_V)_K, 1_K \rangle_K = \langle \tilde{\varphi}_V, 1_K^G \rangle_G$, where $\langle \cdot, \cdot \rangle_G$ denotes the usual scalar product for class functions on G .

Let $W \in \mathbf{mod}_{L-L[G]}$ and χ_W be its ordinary character. In view of the remarks in Section 3.3, we have $\dim_F(\bar{W}e) = \dim_L(W\hat{e}) = \langle (\chi_W)_K, 1_K \rangle_K = \langle \chi_W, 1_K^G \rangle_G$.

Writing φ_V is a \mathbb{Z} -linear combination of ordinary characters restricted to the p' -classes of G , the dimension $\dim_F(Ve)$ can be computed in terms of ordinary characters. As the blocks under consideration here are described by Brauer trees, these linear combinations can directly be read off from the trees.

3.6. The trace formula. To solve the algebraic conjugacy problems, we apply the following idea. Let $V \in \mathbf{mod}_{F-F}[G]$ and φ_V be its Brauer character. If t_V and t_{Ve} are the usual F -valued trace functions, we have

$$t_{Ve}(ege) = |K|^{-1} \cdot \sum_{k \in K} t_V(gk) = |K|^{-1} \cdot \sum_{k \in K} \overline{\tilde{\varphi}_V(gk)}.$$

If Ve is explicitly given, the trace $t_{Ve}(ege)$ can be evaluated explicitly as well. On the other hand, $t_{Ve}(ege)$ can be computed, if φ_V is known and if it is known to which conjugacy classes of G the elements of the coset gK belong.

4. PROOFS

4.1. A permutation representation. Let $G = Ly$ for short. We are looking for a permutation representation of G , such that its permutation character has sufficiently many constituents belonging to the principal p -blocks, for both cases $p = 37$ and $p = 67$. Let $5^3:L_3(5) \cong H < G$ be a maximal subgroup of index 1 113 229 656, see [3, p.174], let Ω be the set of right cosets of H in G , and χ_Ω the character of $\mathbb{C}[\Omega]$. Note that both $p = 37$ and $p = 67$ do not divide the group order $|H|$, hence both $\mathbb{F}_{37}[\Omega]$ and $\mathbb{F}_{67}[\Omega]$ are projective $\mathbb{F}_{37}[G]$ - and $\mathbb{F}_{67}[G]$ -modules, respectively.

Let V be the absolutely irreducible 111-dimensional representation of G over \mathbb{F}_5 . It was constructed in [14] and is accessible electronically in [22] as representing matrices for a pair $\mathcal{G} = \{g_1, g_2\}$ of standard generators, in the sense of [21]. In [22] we also find words in \mathcal{G} yielding a subgroup conjugate to H . These were obtained by a random search among subgroups generated by an involution and an element of class 3A, see [3, p.174]. We find the submodule structure of the restriction V_H of V to H , using the algorithms in [13] implemented in the `MeatAxe`. In particular, it turns out that V_H has a simple socle S of dimension 10. As $H < G$ is a maximal subgroup and G acts irreducibly on V , we conclude that H is the stabilizer in G of S . Thus the action of G on the set S^G of subspaces of V is equivalent to its action on Ω , and we may identify Ω with S^G .

4.2. A condensation subgroup. As we are going to condense the permutation module $F[\Omega]$ of dimension $\sim 10^9$, we need a condensation subgroup of order at least $\sim 10^5$ to obtain a condensed module of a dimension small enough, such that its structure can be analysed using the `MeatAxe`. Here a subgroup $2.A_9 \cong K < G$, having order 362 880, springs to mind. In particular, K is contained in a maximal subgroup $2.A_{11} \cong L < G$, for which we also find a generating set as words in \mathcal{G} in [22]. Such words are found e.g. using the method described in [2] for finding involution centralizers. We then find standard generators for L , which are preimages of standard generators of A_{11} . The latter are A_{11} -conjugate to the pair $\{(1, 2, 3), (3, 4, 5, 6, 7, 8, 9, 10, 11)\}$ in the natural permutation representation of A_{11} . From that generators $\{k_1, k_2\}$ for K are found as words in \mathcal{G} .

Using the ordinary character tables of H , K and G , accessible in `GAP`, and its library functions dealing with conjugacy class fusions and scalar products between characters, the fusions of the conjugacy classes of H and K into those of G are

TABLE 3. Characters, multiplicities and condensed dimensions.

χ	m	d									
1	1	1	15	.	24	28	1	56	41	1	120
2	.	.	16	.	16	29	2	88	42	1	120
3	.	.	17	.	25	30	2	88	43	1	120
4	1	4	18	1	32	31	1	82	44	.	116
5	.	1	19	.	48	32	1	82	45	.	110
6	.	1	20	1	47	33	2	81	46	1	126
7	.	.	21	1	57	34	1	87	47	1	133
8	.	.	22	1	57	35	.	78	48	1	133
9	1	1	23	1	52	36	2	109	49	.	140
10	.	5	24	1	56	37	.	102	50	.	148
11	2	14	25	1	56	38	1	98	51	1	151
12	1	20	26	1	56	39	1	120	52	1	172
13	.	9	27	1	56	40	1	120	53	1	196
14	2	18									

determined, as well as the multiplicities m of the irreducible ordinary characters χ of G in χ_Ω . Taking the remarks in Section 3.5 into account, the dimensions d of the condensed modules of the irreducible ordinary representations of G , with respect to the condensation subgroup K , can also be computed as scalar products. The results are given in Table 3. In particular, the condensed module $F[\Omega]e$, where $e = e_K$, has dimension 3207, independent of the particular choice of F .

4.3. A class distribution. We are going to apply the formula given in Section 3.6 to the element $g_3 := (g_1g_2)^3g_2 \in G$, which has order 67. We have to find the distribution of the elements in the coset g_3K into the conjugacy classes of G .

The conjugacy classes of cyclic subgroups of G are defined by group theoretic data, i.e. by element orders, centralizer orders, and normalizer orders, as can be checked in [3, p.174]. Words in \mathcal{G} giving representatives for the conjugacy classes of cyclic subgroups are accessible in [22]. For our purposes, we have to distinguish algebraically conjugate classes. To do this we use traces and ranks of certain elements of $\mathbb{Z}[G]$ on V as well as on the absolutely irreducible 651-dimensional representation W over \mathbb{F}_3 , which was constructed in [10] and is accessible in [22], and on one of the algebraically conjugate 2480-dimensional absolutely irreducible representations X , \tilde{X} over \mathbb{F}_5 , which are 5-modular reductions of ordinary representations. The latter have been constructed on occasion of this paper, using the `MeatAxe`, as constituents of the exterior square of V ; they are also accessible in [22]. To distinguish X and \tilde{X} , we define the conjugacy class 22A of G to be the class containing the particular element of order 22 given in [22] as a word in \mathcal{G} . Using this, X is the 5-modular reduction of the representation affording the ordinary character χ_2 .

Details are given in Table 4, where t and r denote trace and rank, respectively, and where we also give the values of the corresponding extended Brauer characters, see Section 3.5, using the notation of [3]. We find the numbers of elements belonging to the different conjugacy classes as shown in Table 5; for details of the computations see Section 5.1.

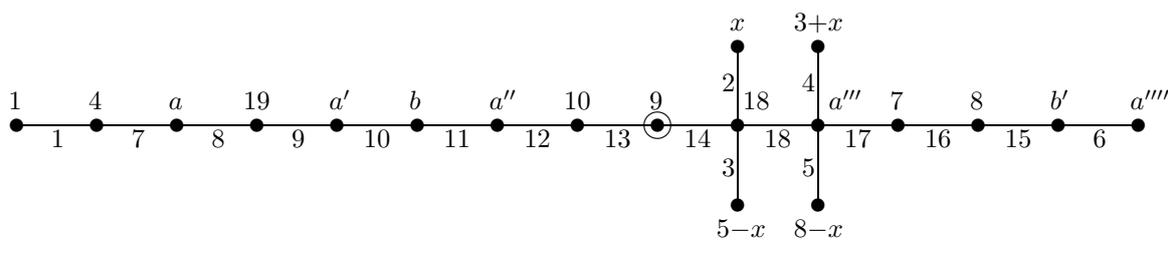
4.4. Applying condensation. We condense the elements g_1 , g_2 and g_3 , where $\mathcal{G} = \{g_1, g_2\}$ is as in Section 4.1, and $g_3 = (g_1g_2)^3g_2 \in G$ is as in Section 4.3;

TABLE 4. Definition of the conjugacy classes of Ly .

	$\tilde{\varphi}_V$	$\tilde{\varphi}_W$	$\tilde{\varphi}_X$	V , over \mathbb{F}_5	W , over \mathbb{F}_3	X , over \mathbb{F}_5
1A	111	651	2480			
2A	-1	11	-16			
3A	-24	651	104	$t(g) = 1$		
3B	3	651	-4	$t(g) = 3$		
4A	3	-5	0			
5A	111	26	-20	$t(g) = 1$	$t(g) = 2$	
5B	111	1	5	$t(g) = 1$	$t(g) = 1$	
6A	8	11	8	$t(g) = 3$	$r((g-1)^2) = 426$	
6B	-1	11	-4	$t(g) = 4$	$r((g-1)^2) = 430$	
6C	-1	11	2	$t(g) = 4$	$r((g-1)^2) = 429$	
7A	-1	0	2			
8A	-3	-1	0	$t(g) = 2$		
8B	1	-1	0	$t(g) = 1$		
9A	0	651	-1			
10A	-1	6	4	$t(g) = 4$	$t(g) = 0$	
10B	-1	1	-1	$t(g) = 4$	$t(g) = 1$	
11A	1	2	b_{11}			$t(g) = 1$
11B	1	2	**			$t(g) = 3$
12A	0	-5	0	$t(g) = 0$		
12B	-3	-5	0	$t(g) = 2$		
14A	-1	4	-2			
15A	-24	26	4	$t(g) = 1$	$t(g) = 2$	
15B	3	26	1	$t(g) = 3$	$t(g) = 2$	
15C	-24	1	-1	$t(g) = 1$	$t(g) = 1$	
18A	2	11	-1			
20A	3	0	0			
21A	*	0	-1	$t(g) = 1$		
21B	$-b_{21}$	0	-1	$t(g) = 0$		
22A	-1	0	$-b_{11}$			$t(g) = 4$
22B	-1	0	**			$t(g) = 2$
24A	0	-1	0	$t(g) = 0, t(g^2) = 0$		
24B	$1 + r_6$	-1	0	$t(g) = 0, t(g^2) = 2$		
24C	*	-1	0	$t(g) = 2, t(g^2) = 2$		
25A	111	1	0			
28A	3	2	0			
30A	8	6	-2	$t(g) = 3$		
30B	-1	6	1	$t(g) = 4$		
31A	$e_{31} * 4 \& 8 \& 16$	0	0	$t(g) = 1, t(g^2) = 4$		
31B	*2	0	0	$t(g) = 4, t(g^2) = 3$		
31C	*4	0	0	$t(g) = 3, t(g^2) = 2$		
31D	*8	0	0	$t(g) = 2, t(g^2) = 2$		
31E	*16	0	0	$t(g) = 2, t(g^2) = 1$		
33A	-2	2	b_{11}			$t(g) = 1$
33B	-2	2	**			$t(g) = 3$
37A	0	$4 + b_{37}$	1	$t(g) = 0$	$t(g) = 1$	
37B	0	*	1	$t(g) = 0$	$t(g) = 0$	
40A	-3	$4 + r_{10}$	0	$t(g) = 2$	$t(g) = 2$	
40B	-3	*	0	$t(g) = 2$	$t(g) = 0$	
42A	$-3 - b_{21}$	4	1	$t(g) = 2$		
42B	*	4	1	$t(g) = 3$		
67A	$-1 - c_{67}$	$3 - c_{67}$	1	$t(g) = 1$		
67B	*2	*2	1	$t(g) = 3$		
67C	*4	*4	1	$t(g) = 4$		

TABLE 5. The class distribution of g_3K

class	#	class	#	class	#	class	#
1A	0	10A	95	21B	8663	31D	11721
2A	0	10B	7092	22A	16387	31E	11659
3A	0	11A	5406	22B	16657	33A	10935
3B	0	11B	5496	24A	15173	33B	11115
4A	16	12A	1205	24B	15054	37A	9927
5A	0	12B	5067	24C	15194	37B	9775
5B	89	14A	2106	25A	14522	40A	9183
6A	4	15A	172	28A	13023	40B	9008
6B	172	15B	3976	30A	3975	42A	8707
6C	1105	15C	4891	30B	4088	42B	8589
7A	2172	18A	20394	31A	11561	67A	5413
8A	765	20A	9035	31B	11660	67B	5269
8B	3776	21A	8767	31C	11731	67C	5429
9A	6661						

TABLE 6. Partial information for $p = 37$, see Section 4.5.

for details of the computations see Section 5.2. Note that by Section 3.4 this essentially amounts to finding the $a_{ij}(g) \in \mathbb{Z}$, hence we do not have to specify the field F beforehand.

4.5. The case $p = 37$. The partial information on the Brauer tree known from [8, p.268] is shown in Table 6, where $\{x, x'\} = \{2, 3\}$, and $\{a, a', a'', a''', a''''\} = \{11, \dots, 15\}$, and $\{b, b'\} = \{16, 17\}$. Hence there are 480 possible cases left. We also give labels to the edges of the Brauer tree, for future reference. The dimensions d of the condensed modules of the irreducible modular representations of G can be computed from this information; for those in the principal block they are given in Table 7, where the numbering is as given in the Brauer tree in Table 6. We also give the multiplicities m of the corresponding Brauer characters in the principal block component of the permutation character χ_Ω . The dimensions of the condensed modules of the representations not in the principal block and the multiplicities of the corresponding ordinary irreducible characters are already given in Table 3.

We now specify $F := \mathbb{F}_{37}$. Using the `MeatAxe`, the condensed module $F[\Omega]e$, acted on by the F -algebra $F[\mathcal{E}]$ generated by $\mathcal{E} := \{eg_1e, eg_2e, eg_3e\}$, turns out to have the following constituents, where we denote the constituents by their dimension and a trailing letter, and their multiplicities by exponents:

$$1a^2, 1b^1, 3a^2, 7a^3, 7b^3, 13a^2, 18a^2, 27a^2, 29a^4, 32a^1, 47a^1, \\ 52a^3, 52b^1, 55a^2, 57a^1, 57b^1, 65a^2, 68a^2, 87a^1, 88a^2, 88b^2, 98a^1,$$

TABLE 7. Brauer characters and condensed dimensions for $p = 37$.

φ	m	d	φ	m	d	φ	m	d
1	2	1	7	2	3	13	4	29
2	.	.	8	2	117	14	2	27
3	.	.	9	2	55	15	2	13
4	1	.	10	2	65	16	3	7
5	1	.	11	2	68	17	3	7
6	2	120	12	3	52	18	1	113

$$109a^2, 113a^1, 117a^2, 120a^2, 126a^1, 151a^1, 164a^1, 168a^1, 196a^1.$$

Here the constituents $164a$ and $168a$ split over \mathbb{F}_{37^2} and \mathbb{F}_{37^3} , respectively, while all the others are absolutely irreducible.

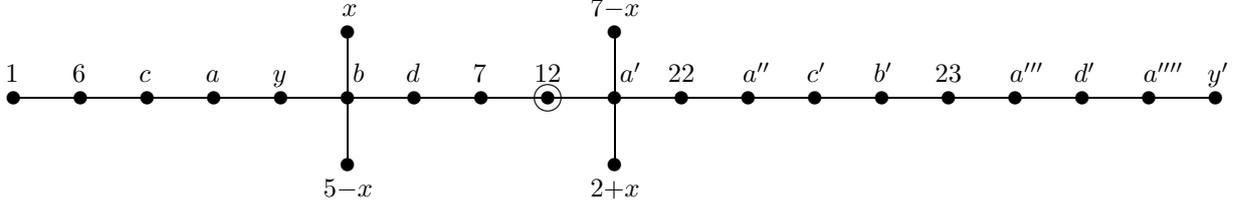
If φ is an irreducible Brauer character occurring in χ_Ω , which is afforded by the simple $F[G]$ -module S_φ , then $S_\varphi e$ either is equal to $\{0\}$ or is a simple $eF[G]e$ -module, see Section 3.1. But as we have $F[\Omega]e$ given as an $F[\mathcal{E}]$ -module only, where $F[\mathcal{E}]$ might be strictly smaller than $eF[G]e$, we can only try to determine the constituents of $(S_\varphi e)|_{F[\mathcal{E}]}$ and their multiplicities. This is done by comparing the dimensions and multiplicities of the constituents of $F[\Omega]e$ found by the **MeatAxe** with the data given in Tables 7 and 3. These considerations show that for all $S_\varphi e \neq \{0\}$ the restriction $(S_\varphi e)|_{F[\mathcal{E}]}$ is simple as well, and that this indeed gives a bijection from the set of irreducible Brauer characters φ occurring in χ_Ω , such that $S_\varphi e \neq \{0\}$, and the constituents found by the **MeatAxe**. Furthermore this bijection is uniquely determined, up to the fact that it maps $\{\varphi_{16}, \varphi_{17}\}$ to $\{7a, 7b\}$.

Recall that $F[\Omega]$ is a projective $F[G]$ -module. By decomposing χ_Ω into projective indecomposable characters and using the multiplicities given in Table 7, we conclude that in a given decomposition of $F[\Omega]$ into projective indecomposable summands both the projective covers P_{16} of φ_{16} and P_{17} of φ_{17} occur with multiplicity 1. Furthermore, all the other projective indecomposable summands do not have φ_{16} and φ_{17} as constituents. The **MeatAxe**, together with the peakword technique described in [13], shows that there is an $F[\mathcal{E}]$ -submodule $U_1 \leq (P_{16} \oplus P_{17})e$ of dimension 34 having a simple head and a simple socle isomorphic to $7a$ and containing the constituent $7b$ with multiplicity 1, and an $F[\mathcal{E}]$ -submodule $U_2 \leq (P_{16} \oplus P_{17})e$ of dimension 134 having a simple head and a simple socle isomorphic to $7b$ and containing the constituent $7a$ with multiplicity 1. As $P_{16}e$ and $P_{17}e$ have dimension 34 and 134, respectively, we have $U_1 \oplus U_2 = (P_{16} \oplus P_{17})e$. By the Krull–Schmidt Theorem we conclude $U_1 \cong (P_{16}e)|_{F[\mathcal{E}]}$ and $U_2 \cong (P_{17}e)|_{F[\mathcal{E}]}$. Thus the above bijection maps φ_{16} to $7a$ and φ_{17} to $7b$.

We find the following traces $t_{S_\varphi e}(eg_3e) \in F$ of the action of eg_3e :

$$\begin{array}{c|cccccccccccccccccccc} \varphi & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\ \hline t_{S_\varphi e} & 1 & . & . & . & . & 10 & 1 & 36 & 5 & 9 & 14 & 19 & 13 & 20 & 36 & 25 & 29 & 6 \end{array}$$

This gives the left hand side of the trace formula in Section 3.6 for the condensed modules corresponding to the φ_i . Using the class distribution found in Section 4.3, we compute the right hand side of the formula for the 480 cases left. It turns out that there are precisely two cases consistent with the actual traces found by the **MeatAxe**; these are the Brauer tree printed in Section 2.1, for which we have $x = 2$, and the tree obtained by reflecting the latter one at its real stem, for which we have $x = 3$. Hence it remains to determine the planar embedding of the Brauer tree.

TABLE 8. Partial information for $p = 67$, see Section 4.7.

4.6. The planar embedding. We still let $F := \mathbb{F}_{37}$. As both pairs $\varphi_{1,2}$ and $\varphi_{3,4}$ condense to the zero module, the condensation subgroup K is not suitable to determine the planar embedding of the Brauer tree. Instead, we use another condensation subgroup $\tilde{K} \cong 2.(A_6 \times A_5).2$ of order 86 400, which is also contained in the maximal subgroup $2.A_{11} \cong L < G$, and repeat the steps described in Sections 4.2, 4.3 and 4.4. The condensed module $F[\Omega]\tilde{e}$, where $\tilde{e} = e_{\tilde{K}}$, has dimension 13 257. Note that due to this large dimension it would not have been feasible to analyse $F[\Omega]\tilde{e}$, instead of $F[\Omega]e$, as completely as it was necessary to find the labelling of the nodes of the Brauer tree in Section 4.5.

In particular, we find that $S_{\varphi_{16}}\tilde{e}$, $S_{\varphi_{17}}\tilde{e}$ and $S_{\varphi_{18}}\tilde{e}$ have dimension 16, 20 and 477, respectively, and that $S_{\overline{\chi}_7}\tilde{e}$ and $S_{\overline{\chi}_8}\tilde{e}$ both have dimension 1, where $\overline{\chi}_7$ and $\overline{\chi}_8$ denote the 37-modular reductions of the ordinary characters χ_7 and χ_8 , see Section 3.3. Furthermore, $\tilde{e}g_3\tilde{e}$ acts by multiplication with $21 \in F$ and $26 \in F$ on $S_{\overline{\chi}_7}\tilde{e}$ and $S_{\overline{\chi}_8}\tilde{e}$, respectively. In a given decomposition of $F[\Omega]$ into projective indecomposable summands P_{17} occurs with multiplicity 1, and all the other projective indecomposable summands do not have $\overline{\chi}_7$ and $\overline{\chi}_8$ as constituents. Hence the constituents $S_{\overline{\chi}_7}\tilde{e}$ and $S_{\overline{\chi}_8}\tilde{e}$ of $F[\Omega]\tilde{e}$ occur with multiplicity 1 and even are constituents of the $\tilde{e}F[G]\tilde{e}$ -submodule $P_{17}\tilde{e} \leq F[\Omega]\tilde{e}$. By the submodule structure of projective indecomposable modules in blocks of cyclic defect, there is a unique minimal submodule $U \leq P_{17}\tilde{e}$ having both $S_{\overline{\chi}_7}\tilde{e}$ and $S_{\overline{\chi}_8}\tilde{e}$ as constituents. Furthermore U has dimension 499, and is uniserial with ascending composition series $S_{\varphi_{17}}\tilde{e}, S_{\varphi_5}\tilde{e}, S_{\varphi_{18}}\tilde{e}, S_{\varphi_4}\tilde{e}$.

We consider $F[\Omega]\tilde{e}$ as a module for the F -algebra $F[\tilde{\mathcal{E}}]$ being generated by $\tilde{\mathcal{E}} := \{\tilde{e}g_1\tilde{e}, \tilde{e}g_3\tilde{e}\}$, and let $N := \ker_{F[\Omega]\tilde{e}}(\tilde{e}g_3\tilde{e} - 21\tilde{e})$. The MeatAxe finds that N has dimension 1, that the $F[\tilde{\mathcal{E}}]$ -module $\tilde{N} := N \cdot F[\tilde{\mathcal{E}}] \leq F[\Omega]\tilde{e}$ has dimension 499 and that it is uniserial with ascending composition series $20a, 1a, 477a, 1b$, where $\tilde{e}g_3\tilde{e}$ acts by multiplication with $26 \in F$ and $21 \in F$ on $1a$ and $1b$, respectively. Furthermore, the MeatAxe finds $\ker_{F[\Omega]\tilde{e}/\tilde{N}}(\tilde{e}g_3\tilde{e} - 21\tilde{e}) = \{0\}$, hence $\tilde{e}g_3\tilde{e} - 21\tilde{e} \in F[\tilde{\mathcal{E}}]$ is a peakword on $F[\Omega]\tilde{e}$ for the constituent $1b$, see [13]. From that we conclude that \tilde{N} is contained in U , and hence $\tilde{N} = U$ is an $\tilde{e}F[G]\tilde{e}$ -submodule of $P_{17}\tilde{e}$. Thus we have $(S_{\overline{\chi}_7}\tilde{e})|_{F[\tilde{\mathcal{E}}]} \cong 1b \cong (S_{\varphi_4}\tilde{e})|_{F[\tilde{\mathcal{E}}]}$, hence $x = 2$.

4.7. The case $p = 67$. The partial information on the Brauer tree known from [8, p.271] is shown in Table 8, where $\{x, x'\} = \{2, 3\}$, and $\{b, b'\} = \{8, 9\}$, and $\{y, y'\} = \{10, 11\}$, and $\{c, c'\} = \{13, 14\}$, and $\{d, d'\} = \{15, 16\}$, and $\{a, a', a'', a''', a''''\} = \{17, \dots, 21\}$. Hence there are 3840 possible cases left. We find the Brauer tree by analysing $F[\Omega]e$, where we now specify $F = \mathbb{F}_{67}$, using similar techniques as described in Sections 4.5 and 4.6, and again spare the details here.

5. COMMENTS ON THE COMPUTATIONS

In this section we give some details of how the computations were done. We concentrate on the calculations for the condensation subgroup K , see Sections 4.3 and 4.4. The condensation subgroup \tilde{K} , see Section 4.6, has been treated similarly.

5.1. Finding the class distribution. As described in Section 4.3, we have to determine how the elements of the coset g_3K distribute into the conjugacy classes of G . To do this we have to compute representing matrices for the $|K| = 362880$ elements in the coset g_3K on the modules V , W and X , and then to compute some traces or ranks. But to keep $|K|$ many (2480×2480) -matrices over \mathbb{F}_5 in memory, $\sim 7.5 \times 10^{11}$ Bytes would be necessary. Hence we want to obtain a reasonable number of matrices which have to be stored simultaneously.

Let $\mathcal{K} = \{k_1, \dots, k_t\}$ be a fixed generating set for K . In a precomputation we use an orbit algorithm to enumerate the elements of K , starting with $1 \in K$. This yields a Schreier tree for K with respect to \mathcal{K} , whose shape depends on the particular strategy employed in the orbit algorithm. Given a Schreier tree, we define a valuation v on its vertices as follows. If the vertex g is a leaf, we let $v(g) = 1$. Otherwise, if g_1, \dots, g_s are the immediate successors of g in the Schreier tree, we let $v(g) = \max\{v(g_i); 1 \leq i \leq s\}$ if this maximum is assumed exactly once, and $v(g) = 1 + \max\{v(g_i); 1 \leq i \leq s\}$ if it is assumed more than once. In the recursive run through the Schreier tree, which is used to find the class distribution, at each vertex we are now able to work through the most expensive subtree last. Hence for each matrix representation considered, besides representing matrices for \mathcal{K} we have to store no more than $v(1)$ matrices at the same time. Hence our aim is to find a Schreier tree such that $v(1)$ is reasonably small.

To find a suitable Schreier tree, we enumerate a regular K -orbit in V , as its elements are in bijection with the elements of K . The algorithm we use is a modification of the PubCrawl algorithm presented in [16, Section 5.], see Table 9. Note that T is a list which collects the orbit during the algorithm. New vectors are appended to T within the loop that runs through T . Of course, the loop must also run through these new vectors. The idea is to use a standard breadth-first orbit algorithm as an outer loop to run through the whole orbit, but each time we find a new element of the orbit, i.e. a new vertex of the Schreier tree, before going on with the standard algorithm, we try to attach a ‘thread’, i.e. a lengthy path without branching points, to the corresponding vertex in the Schreier tree.

We choose the generating set \mathcal{K} as follows. We fix a 2-Sylow subgroup \tilde{U} of K as a helper subgroup, hence $|\tilde{U}| = 2^7$, and let \mathcal{K} be the union of the generating set $\{k_1, k_2\}$ of K , see Section 4.2, and a set of generators of \tilde{U} . Thus we end up with a Schreier tree such that $v(1) = 3$.

As $|\tilde{U}|$ is coprime to the characteristics of V , W and X , we choose bases exhibiting the semisimplicity of $V|_{\tilde{U}}$, $W|_{\tilde{U}}$ and $X|_{\tilde{U}}$. As the constituents of these modules have dimension at most 8, this considerably reduces the amount of time needed for a matrix multiplication with one of the generators of \tilde{U} . Furthermore, using the MeatAxe and the algorithms in [13], $X|_K$ turns out to be a direct sum of 14 indecomposable modules, the largest of which has dimension 560. Hence we adapt the basis of X , such that it also exhibits a direct sum decomposition of $X|_K$.

Building the Schreier tree, running through the elements of g_3K , and computing representing matrices, traces and ranks is easily done using the new fast finite field

TABLE 9. Modified orbit algorithm.

```

 $T := [1];$ 
for  $g$  in  $T$  do
  for  $k$  in  $\mathcal{K}$  do
     $h := g \cdot k;$ 
    if  $h$  not in  $T$  then
      Append  $h$  to  $T;$ 
       $i := 1;$ 
      while  $i \leq t$  do
         $h' := h \cdot k_i;$ 
        if  $h'$  not in  $T$  then
          Append  $h'$  to  $T;$ 
           $h := h';$ 
           $i := 1;$ 
        else
           $i := i + 1;$ 

```

arithmetic of GAP, which is based on the ideas of the arithmetic of the MeatAxe. Using a Pentium III 800 processor, this needed ~ 60 hours of CPU time.

5.2. Applying condensation. As described in Section 4.4, we want to compute the action of the elements $\{eg_1e, eg_2e, eg_3e\}$ on the condensed module $F[\Omega]e$, where the permutation module $F[\Omega]$ is given by the action of $\{g_1, g_2\}$ on the set $\Omega = S^G$ of subspaces of V of dimension 10, see Section 4.1. As Ω is not yet known, it has to be enumerated first, and subsequently we have to compute the integers $a_{ij}(g)$, see 3.4, for all $\omega \in \Omega$ and $g \in \{g_1, g_2, g_3\}$. But to store a subspace of V of dimension 10 we need 370 Bytes, thus to store the whole orbit Ω of length $[G : H]$ we would need $\sim 4 \times 10^{11}$ Bytes. Hence we can only afford to store $\sim 1/400$, say, of the elements of Ω , which fit into $\sim 10^9$ Bytes. We use a modification of the ideas invented in [11].

We choose a helper subgroup $U \leq K \leq G$, such that its elements can be enumerated, and objects representing their action on Ω can be kept in memory. The basic idea now is to modify the standard breadth-first orbit algorithm for G , such that Ω is enumerated piecewise, namely K -orbit by K -orbit, where these in turn are enumerated U -orbit by U -orbit, keeping track of how the U -orbits fall into K -orbits. Hence for $\omega \in \Omega$ we are reduced to finding out whether we have already encountered its orbit ω^U , and if this is the case, which of the orbits already encountered it is.

To do this, we choose a helper U -set Θ , such that there is a homomorphism $q : \Omega|_U \rightarrow \Theta$ of U -sets. Furthermore we assume that the elements of Θ can be enumerated completely. Let $\{\Theta_i; 1 \leq i \leq s\}$ be the U -orbits on Θ . For each $1 \leq i \leq s$ we choose a *minimal* element $\vartheta_i \in \Theta_i$, e.g. by using an injective function on Θ into a totally ordered set, which is quickly evaluated on the elements of Θ . An element $\omega \in \Omega$ is called *q-minimal*, if $q(\omega)$ is minimal. It is exactly the q -minimal elements which are stored in a table during the enumeration of the whole of Ω ; to recover elements quickly, we of course use a hashing technique. If we are able to find a homomorphism q such that most of the $\{\Theta_i\}$ as above are regular U -orbits, then $\sim 1/|U|$ of the elements of Ω will be q -minimal. Hence in this case we will be content with a subgroup U of order $|U| \sim 400$.

Furthermore, let $\kappa : \Theta \rightarrow \mathcal{P}(U)$, where $\mathcal{P}(U)$ denotes the power set of U , be defined as follows. For $\vartheta \in \Theta_i$, we let $\kappa(\vartheta) = \{u \in U; \vartheta \cdot u = \vartheta_i\}$, which therefore is a left

coset of $\text{Stab}_U(\vartheta_i)$ in U . Since we assume that the elements of U and those of Θ can be enumerated, we are able to compute all the sets $\kappa(\vartheta)$ explicitly. If we want to check whether for some $\omega \in \Omega$ the orbit ω^U has been encountered before, we try to look up $\omega \cdot u$, for an arbitrary $u \in \kappa(q(\omega))$, in the table. If it is there we have encountered ω^U before, if it is not we store all of the elements $\{\omega \cdot u; u \in \kappa(q(\omega))\}$ in the table.

Having fixed the subgroup U , we try to find a suitable U -set Θ . We look for U -modules X_1 and X_2 , each of dimension 11, such that there are epimorphisms of U -modules $q_i : V_U \rightarrow X_i$ and $\ker(q_1) \cap \ker(q_2) < V$ is of codimension 22. Let Θ_i be the set all subspaces of X_i of codimension 1, and Θ'_2 be the set of all proper subspaces of X_2 of codimension at least 2. As Ω consists of subspaces of V of dimension 10, there is a homomorphism of U -sets $q : \Omega \rightarrow \Theta := \Theta_1 \cup \Theta_2 \cup \Theta'_2$, defined by $q(\omega) = q_1(\omega)$, if $q_1(\omega) \in \Theta_1$, and $q(\omega) = q_2(\omega)$ otherwise.

A simple calculation shows the reason for this choice. As X_i has dimension 11 over \mathbb{F}_5 , there are $(5^{11} - 1)/(5 - 1) \sim 1.2 \times 10^7$ and $(5^{11} - 1)(5^{11} - 5)/((5^2 - 1)(5^2 - 5)) \sim 5 \times 10^{12}$ subspaces of X_i of codimension 1 and 2, respectively, and even more of higher codimension. Thus it is only possible to enumerate the subspaces of X_i of codimension 1, and hence the above mentioned preprocessing is only done for $\Theta_1 \cup \Theta_2$. By going over to the dual space $X_i^* = \text{Hom}_{\mathbb{F}_5}(X_i, \mathbb{F}_5)$, we need 4 Bytes to store one of these subspaces of X_i , which still means $\sim 9.8 \times 10^7$ Bytes for the whole of $\Theta_1 \cup \Theta_2$. This means that all subspaces of V mapped by q into $\Theta_1 \cup \Theta_2$ are dealt with as explained above, but those mapped into Θ'_2 simply have to be stored, hence for these we do not save memory at all. The memory requirements are estimated as follows.

There are $\prod_{i=0}^9 (5^{111} - 5^i)/(5^{10} - 5^i)$ subspaces of V of dimension 10, but amongst them only $\prod_{i=0}^9 (5^{111} - 5^{100+i})/(5^{10} - 5^i)$ intersect trivially with $\ker(q_1)$. Hence $\sim 1/20$ of these subspaces are not mapped into Θ_1 by q_1 . If we assume Ω to consist of a uniformly distributed random sample of subspaces of V of dimension 10, this amounts to $\sim 1/20 \cdot [G : H] \sim 5.5 \times 10^7$ subspaces. To store this many subspaces, at the cost of 370 Bytes each, we would need $\sim 2 \times 10^{10}$ Bytes. This shows the need for a second map q_2 . Then only $\sim 1/400$ of the elements of Ω are expected to be mapped by q into Θ'_2 , which means $\sim 2.7 \times 10^6$ elements or $\sim 10^9$ Bytes, which still means that we need as much memory for these elements of Ω as for its q -minimal elements. We remark that indeed the memory requirements in the actual computations fitted well into this picture.

Despite these serious constraints, we were lucky to find a suitable maximal subgroup $3^2:2A_4 \cong \overline{U} < A_9 \cong K/Z(K)$ of order 216. It is uniquely defined up to conjugacy in A_9 , see [3, p.37]. We let $Z(K) \times \overline{U} \cong U < K$ be the preimage of \overline{U} with respect to the natural epimorphism $K \rightarrow A_9$, hence $|U| = 432$. Again we choose a basis for V exhibiting the semisimplicity of $V|_U$. As the constituents of $V|_U$ have dimension at most 16, this considerably reduces the amount of time needed for a matrix multiplication with one of the elements of U . Using the `MeatAxe`, together with the algorithms in [13], we find epimorphic images X_1 and X_2 of V_U , where $X_1 \cong 1^- \oplus 2 \oplus 8$ and $X_2 \cong 3 \oplus 8^-$. Here $Z(K)$ acts non-trivially on the constituents 1^- and 8^- , and trivially on the others. Hence U acts faithfully on both X_1 and X_2 , and $Z(K)$ acts non-trivially on subspaces of X_1 and X_2 of codimension 1. The average orbit length of U on subspaces of codimension 1 turned out to be ~ 405 for X_1 , and ~ 415 for X_2 .

With these preparations, we adjusted the implementation described in [11], which allows for massive parallelization, accordingly. We are grateful to the University of St. Andrews for allowing us to use their PC cluster to run these computations. Using 50 Pentium II 450 processors, the computations needed ~ 13 hours of elapsed time, hence ~ 650 hours of CPU time.

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