Generalisations of Small Cancellation Theory

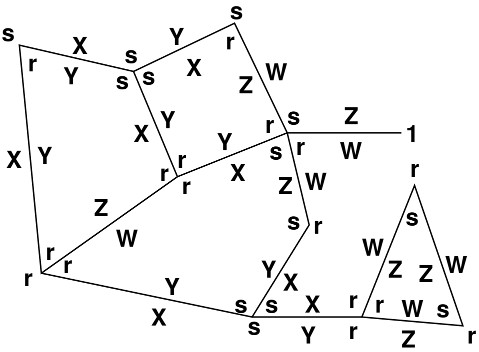
Max Neunhöffer

joint work with Jeffrey Burdges, Stephen Linton,

Richard Parker and Colva Roney-Dougal

NBSAN meeting St Andrews, 9 April 2013

We draw connected finite graphs in the plane and label them:



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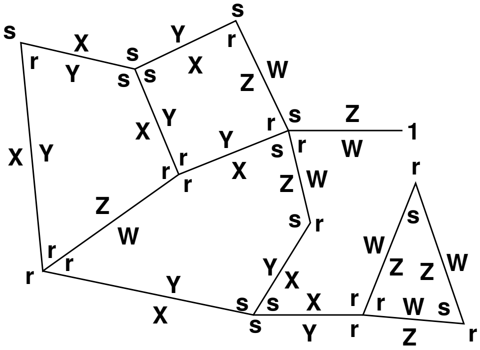

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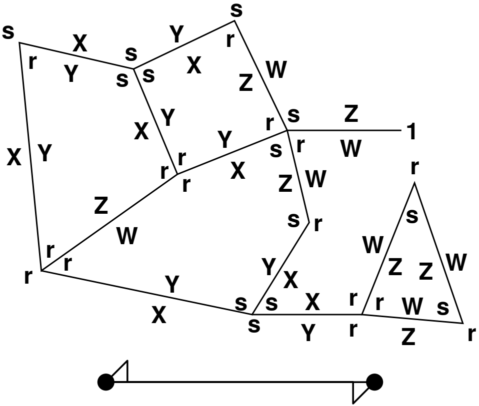

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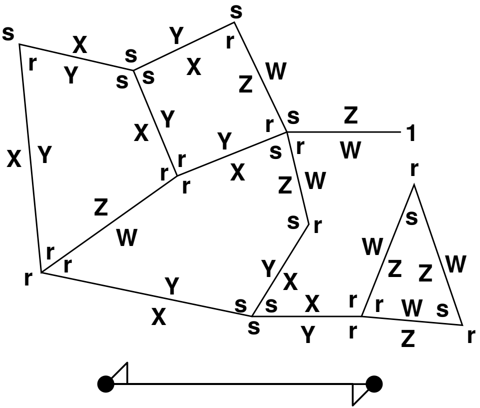

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We view each edge as a pair of opposite directed edges: half-edges.

Each half-edge is labelled at the start vertex and along the half-edge.

The diagram boundary problem

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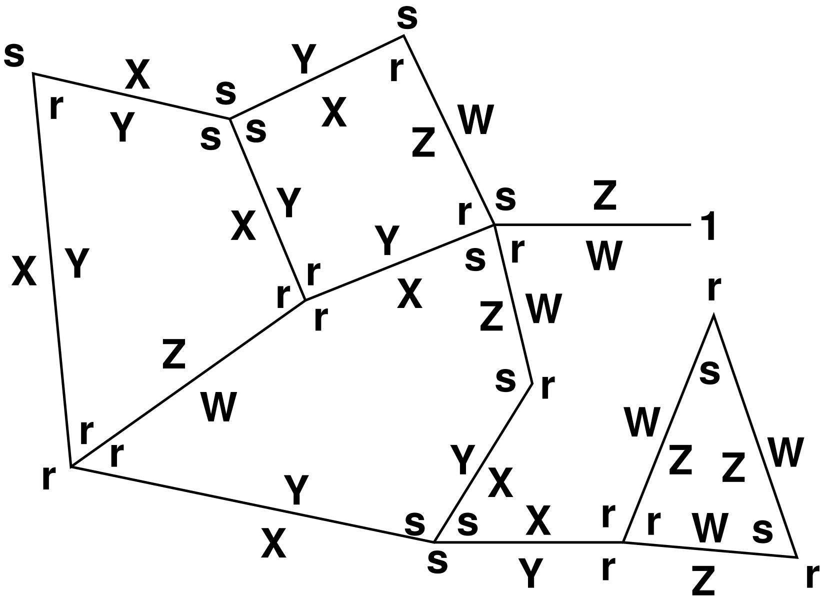
Algorithmically devise a procedure that decides for any cyclic word w, whether or not there is a diagram such that

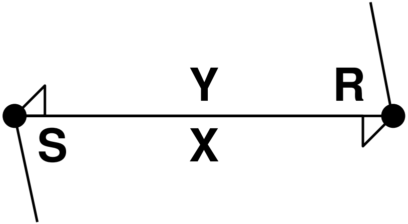
- *every internal region is labelled by a relator, and*
- *the external boundary is labelled by w.*


s
s Y X r
s X s r Y
Y s X Z W
s r Z
X Y Y r 1
X Y s r W
r r X
r r Z W
Z r
W s r
W
r W Z Z
r r Y X
Y
s s X r r W s
X s Y r Z r

Rules for the labels

We label every half-edge with two symbols,

- one for the corner to the right of where it starts, and
- one for the right hand side of it:



We now need rules for the corner labels and the edge labels.

Definition (Corner structures)

A corner structure is a set S with a subset $S_+ \subset S$, such that $S_0 := S \,\dot\cup\, \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in S_+ \text{ for } x, y \in S, \text{ then } yx \in S_+.$$

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Lemma (Cyclicity)

Let S be a corner structure, if $s_1 s_2 \cdots s_k \in S_+$, then all rotations $s_i s_{i+1} \cdots s_k s_1 s_2 \cdots s_{i-1} \in S_+$.

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The corner labels are from a corner structure S, a vertex is valid, if the clockwise product of its corner labels is an acceptor.

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l	·	·	·	l	s	·

Note: $rl = e$ and $lr = s$, cyclicity, “inverses”, two idempotents.

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(For the experts:
This is a generalisation of the rules of van Kampen diagrams.)

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- $(\ell_S(e_1), \ell_X(e_1), \ldots, \ell_S(e_k), \ell_X(e_k))^{\circlearrowleft} \in R$ for every clockwise cycle $(e_1, e_2, \ldots, e_k)^{\circlearrowleft}$ of half-edges around an internal face.

Let $\langle S; X \mid R \rangle$ be a presentation, that is:

- S is a corner structure,
- X is an edge alphabet and
- R is a set of relators in S and X.

Problem (Diagram boundary problem)

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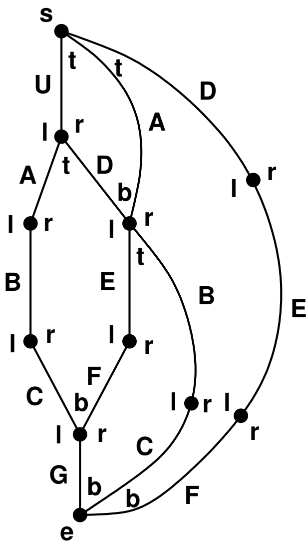
If there is a linear $\mathcal{D}$, we call $\langle S; X \mid R \rangle$ hyperbolic.

With K_6 we can do rewrite systems, if no rewrite has an empty side:



$$S = \begin{array}{c||c|c|c|c|c|c} & s & t & e & b & r & l \\ \hline s & \cdot & s & \cdot & \cdot & \cdot & \cdot \\ t & s & t & \cdot & \cdot & \cdot & l \\ e & \cdot & \cdot & \cdot & e & \cdot & \cdot \\ b & \cdot & \cdot & e & b & r & \cdot \\ r & \cdot & r & \cdot & \cdot & \cdot & e \\ l & \cdot & \cdot & \cdot & l & s & \cdot \end{array}, \quad S_+ = \{s, e\}$$

$X = \{A, B, C, D, E, F, G, U\}$ ($\bar{\ }$ is id_X)

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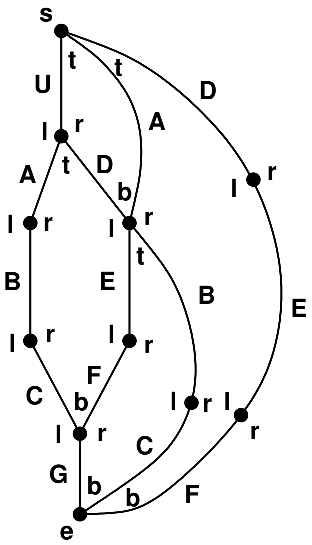

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	s	t	e	b	r	l
s	·	s	·	·	·	·
t	s	t	·	·	·	l
e	·	·	·	e	·	·
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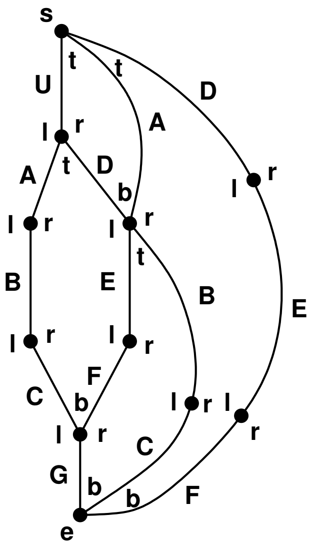

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b	$\cdot$	$\cdot$	e	b	r	$\cdot$
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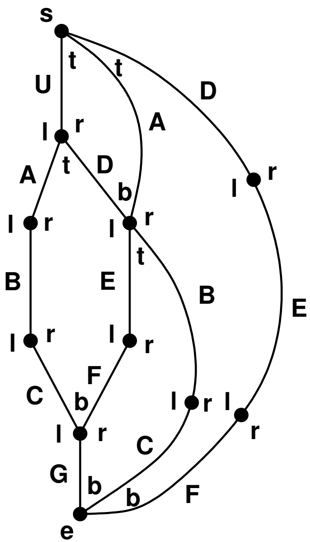

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S accepts $st^* + eb^* + rt^*lb^*$ and all rotations.

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Euler’s formula

The total sum of our combinatorial curvature is always $+1$.

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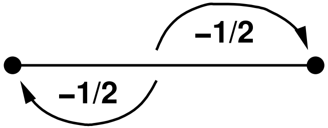
In Phase 1 Tom moves the negative curvature to the vertices:



A vertex with valency $v \geq 3$ will now have $+1 - \frac{v}{2} < 0$.

Idea (Curvature redistribution — Officers)

We redistribute the curvature locally in a conservative way.
We call a curvature redistribution scheme an "officer".

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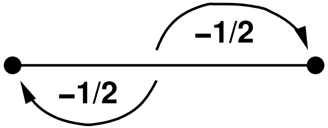
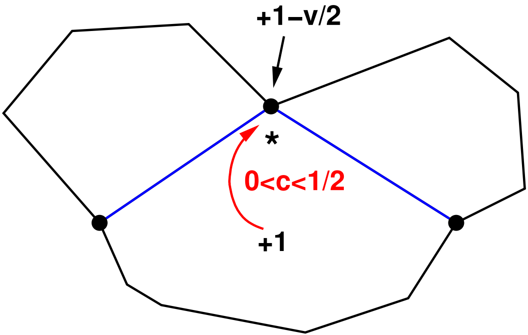


A vertex with valency $v \geq 3$ will now have $+1 - \frac{v}{2} < 0$.
Faces still have $+1$, edges now have 0.

In Phase 2 Tom moves the negative curvature to the vertices:



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A corner value c of Tom depends on two edges that are adjacent on a face.

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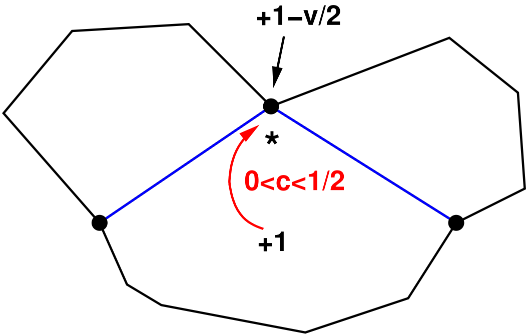

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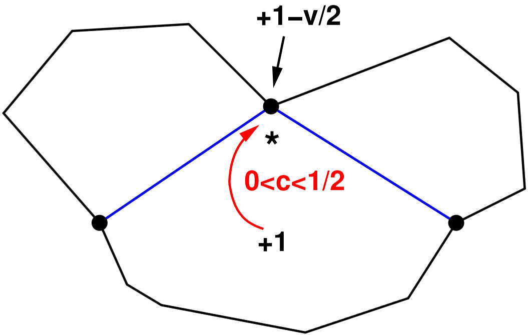

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- every internal face has $< -\varepsilon$ curvature (for some explicit $\varepsilon > 0$),
- every vertex has ≤ 0 curvature.
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If $S \geq 0$ then there is a $j \in L$ such that for all $i \in \mathbb{N}$ the partial sum

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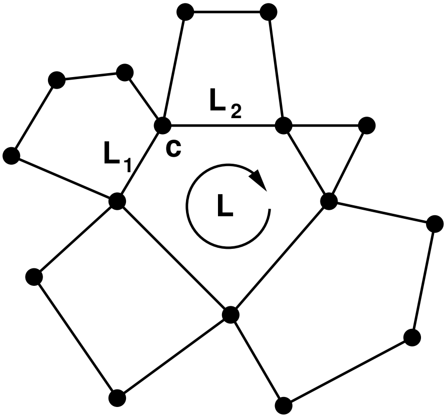
Corollary

Assume that there are $k \in \mathbb{N}$ and $\varepsilon \geq 0$ such that for all $j \in L$ there is an $i \leq k$ with $s_{j,i} < -\varepsilon$, then $S < -\varepsilon \cdot \ell / k$.

Sunflower

To show that every internal face has curvature $< -\varepsilon$:



Use Goes Up and Stays Up on $\frac{L_1+L_2}{2L} - c$.

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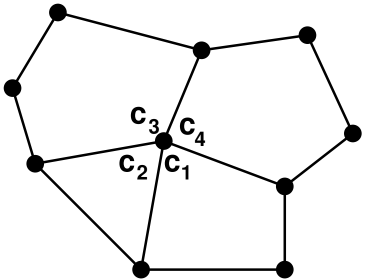

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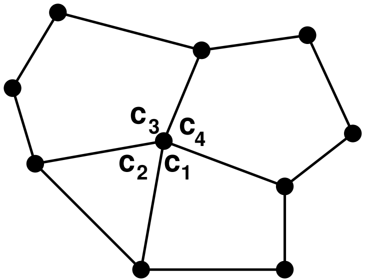

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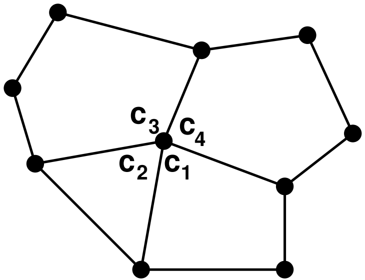

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This terminates: higher valencies tend to be negatively curved anyway.

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and try again. If $\langle S, X; R \rangle$ is not hyperbolic, this will not work.