Generalisations of Small Cancellation Theory

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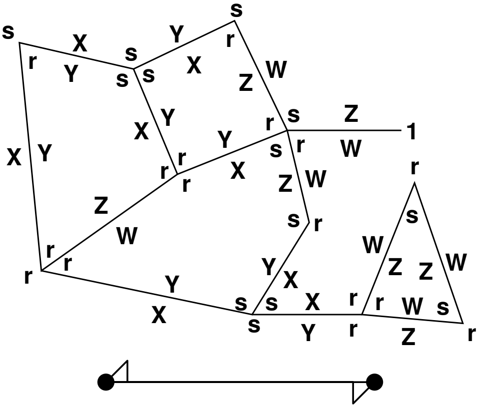
We draw connected finite graphs in the plane and label them:



Faces are oriented clockwise.

We view each edge as a pair of opposite directed edges: half-edges.

Each half-edge is labelled at the start vertex and along the half-edge.

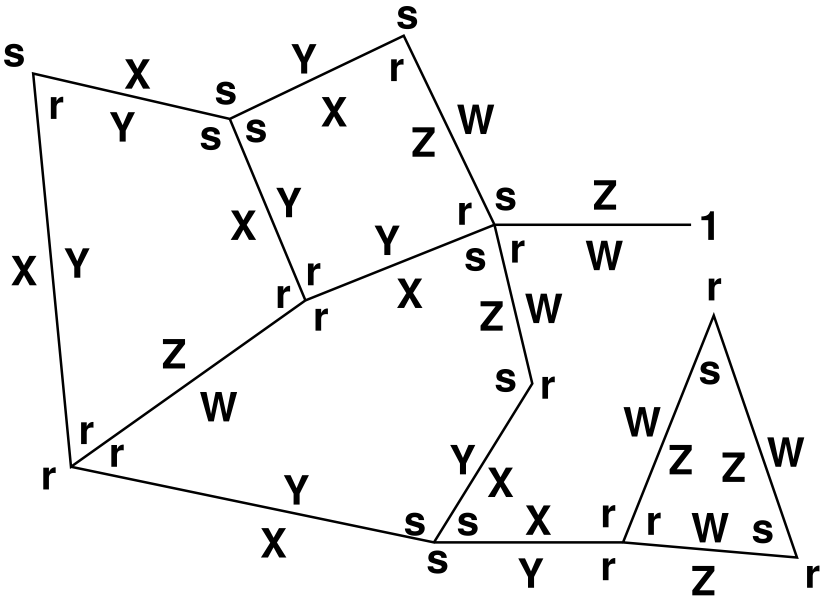
The diagram boundary problem

Let R be a finite set of cyclic words, called relators.

Problem (Diagram boundary problem)

Algorithmically devise a procedure that decides for any cyclic word w, whether or not there is a diagram such that

- *every internal region is labelled by a relator, and*
- *the external boundary is labelled by w.*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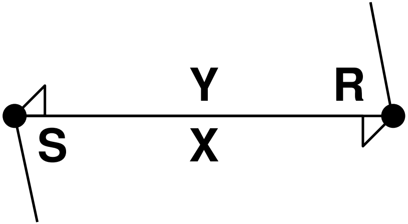
Rules for the labels

We label every half-edge with two symbols,

- one for the corner to the right of where it starts, and
- one for the right hand side of it:



We now need rules for the corner labels and the edge labels.

Definition (Corner structures)

A corner structure is a set S with a subset $S_+ \subset S$, such that $S_0 := S \dot{\cup} \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in S_+ \text{ for } x, y \in S, \text{ then } yx \in S_+.$$

The elements in S_+ are called acceptors.
Usually we will have: for all $x \in S$ there is a $y \in S$ with $xy \in S_+$.

Lemma (Cyclicity)

Let S be a corner structure, if $s_1 s_2 \cdots s_k \in S_+$, then all rotations $s_i s_{i+1} \cdots s_k s_1 s_2 \cdots s_{i-1} \in S_+$.

Vertex rules

The corner labels are from a corner structure S, a vertex is valid, if the clockwise product of its corner labels is an acceptor.

Examples of corner structures

- Let G be a group. Let $P := G$ and $P_+ := \{1\}$.
- Let $G_1, \ldots, G_k$ be groups. Let $Q := \dot{\bigcup} G_i$ and $Q_+ := \{1_{G_i} \mid 1 \leq i \leq k\}$. Elements of a single G_i multiply as before. Products across factors are all 0.
- Take any groupoid, undefined products are 0, identities accept.
- $K_6 := \{s, t, e, b, r, l\}$, $K_{6+} := \{s, e\}$,

	s	t	e	b	r	l
s	·	s	·	·	·	·
t	s	t	·	·	·	l
e	·	·	·	e	·	·
b	·	·	e	b	r	·
r	·	r	·	·	·	e
l	·	·	·	l	s	·

Note: $rl = e$ and $lr = s$, cyclicity, “inverses”, two idempotents.

Definition (Edge alphabet)

An edge alphabet is a set X with an involution $\overline{} : X \to X$.

(This is actually a special case of a corner structure.)

Edge rules

The edge labels are from an edge alphabet, a pair of half-edges forming an edge with labels X and Y is valid, if $Y = \overline{X}$.

(For the experts:
This is a generalisation of the rules of van Kampen diagrams.)

Let S be a corner structure and X be an edge alphabet.

Definition (Set of relators)

A set of relators R is a finite set of cyclic alternating words in S and X.

Definition (Valid diagram)

Let R be a set of relators in S and X. A valid diagram is:
a finite plane graph with half-edge set $\hat{E}$ and a labelling function
$\ell : \hat{E} \to S \times X, e \mapsto (\ell_S(e), \ell_X(e))$, such that

- $\ell_S(e_1) \cdot \ell_S(e_2) \cdot \ell_S(e_3) \cdot \cdots \cdot \ell_S(e_k) \in S_+$ for every clockwise cyclic sequence $e_1, e_2, \ldots, e_k$ of half-edges leaving the same vertex,
- $\ell_X(e) = \overline{\ell_X(e')}$ for all edges $\{e, e'\}$ consisting of half-edges e, e',
- $(\ell_S(e_1), \ell_X(e_1), \ldots, \ell_S(e_k), \ell_X(e_k))^{\circlearrowleft} \in R$ for every clockwise cycle $(e_1, e_2, \ldots, e_k)^{\circlearrowleft}$ of half-edges around an internal face.

Let $\langle S; X \mid R\rangle$ be a presentation, that is:

- S is a corner structure,
- X is an edge alphabet and
- R is a set of relators in S and X.

Problem (Diagram boundary problem)

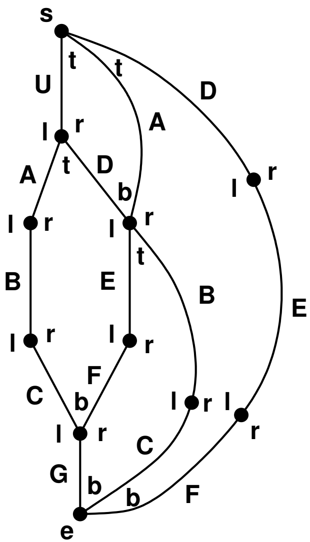
Algorithmically devise a procedure that decides for any cyclic alternating word w in S and X whether or not there is a valid diagram such that the external face is labelled by w.

Problem (Isoperimetric inequality)

Algorithmically find and prove a function $\mathcal{D} : \mathbb{N} \to \mathbb{N}$, such that for every cyclic alternating word w in S and X of length $2k$ that is the boundary label of a valid diagram, there is one with at most $\mathcal{D}(k)$ internal faces.

If there is a linear $\mathcal{D}$, we call $\langle S; X \mid R\rangle$ hyperbolic.

With K_6 we can do rewrite systems, if no rewrite has an empty side:



$$S = \begin{array}{c||c|c|c|c|c|c} & s & t & e & b & r & l \\ \hline s & \cdot & s & \cdot & \cdot & \cdot & \cdot \\ t & s & t & \cdot & \cdot & \cdot & l \\ e & \cdot & \cdot & \cdot & e & \cdot & \cdot \\ b & \cdot & \cdot & e & b & r & \cdot \\ r & \cdot & r & \cdot & \cdot & \cdot & e \\ l & \cdot & \cdot & \cdot & l & s & \cdot \end{array}, \quad S_+ = \{s, e\}$$

$X = \{A, B, C, D, E, F, G, U\}$ ($\bar{\ }$ is id_X)

This encodes $UABCG \to DEF$ using:

$$\{ABC \to DE, UD \to A, EFG \to BC\}$$

$ABC \to DE$ is encoded as $(bCrBrAtDlE)^{\circlearrowleft}$,
we "prove" $(sUlAlBlClGeFrErD)^{\circlearrowleft}$.

S accepts $st^* + eb^* + rt^*lb^*$ and all rotations.

Other Applications

These diagrams and their two fundamental problems encode

- the word problem in quotients of the free group,
- the word problem in quotients of the modular group,
- the word problem for relative presentations (relative to one subgroup gives a Howie diagram)
- the rewrite decision problem for rewrite systems, in which no side of a rewrite is empty,
- the word problem in finite semigroup presentations,
- jigsaw-puzzles in which you can use arbitrarily many copies of each piece,
- the word problem in monoids?
- computations of non-deterministic Turing machines?
- etc. ???

You just have to chose the right corner structure and edge alphabet!

Combinatorial Curvature

Find “pieces”, and remove vertices of valency 1 and 2:

- compute the finite list of all possible edges,
- this produces a new edge alphabet, edges now have different lengths, refer to original edges as mini-edges,
- denote the new set of half-edges in a diagram by $\hat{E}$.

Combinatorical curvature: We endow

- each vertex with $+1$ unit of combinatorial curvature,
- each edge with -1 unit of combinatorial curvature and
- each internal face with $+1$ unit of combinatorial curvature.

Euler's formula

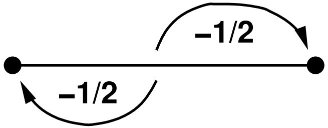
The total sum of our combinatorial curvature is always $+1$.

Idea (Curvature redistribution — Officers)

We redistribute the curvature locally in a conservative way.
We call a curvature redistribution scheme an "officer".

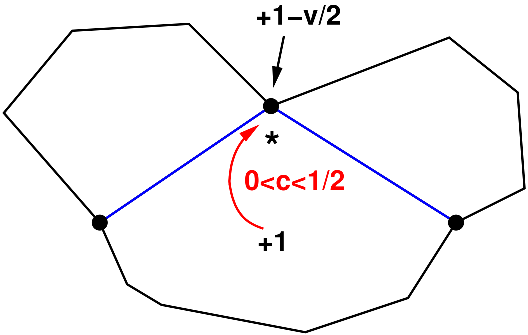
Here, I want to describe our "Officer Tom":

In Phase 1 Tom moves the negative curvature to the vertices:



A vertex with valency $v \geq 3$ will now have $+1 - \frac{v}{2} < 0$.
Faces still have $+1$, edges now have 0.

In Phase 2 Tom moves the negative curvature to the vertices:



Corner values for Tom

A corner value c of Tom depends on two edges that are adjacent on a face. Tom moves c units of curvature from the face to the vertex.
The default value for c is $1/6$ if the vertex can have valency 3 and $1/4$ otherwise.

Tom — and officers in general — want to redistribute the curvature, such that for all permitted diagrams after redistribution

- every internal face has $< -\varepsilon$ curvature (for some explicit $\varepsilon > 0$),
- every vertex has ≤ 0 curvature.
- every edge has 0 curvature,
- every face with more than one external edge has ≤ 0 curvature.

Consequence

$\Longrightarrow$ All the positive curvature is on faces touching the boundary once.

Facts:

- All boundaries of diagrams have a permitted diagram as proof.
- The total positive curvature $\leq n$ (boundary length).
- Let $F := \#$internal faces, then

$$1 < n - F \cdot \varepsilon \quad \Longrightarrow \quad F < \varepsilon^{-1} \cdot n \quad \Longrightarrow \quad \text{hyperbolic}$$

Let $L := \{1, 2, \ldots, \ell\}$ and $a_1, a_2, \ldots, a_\ell \in \mathbb{R}$ and $S := \sum_{m \in L} a_m$.
Define $\pi_L : \mathbb{Z} \to L$ such that $z \equiv \pi_L(z) \pmod{\ell}$.

Lemma (Goes up and stays up)

If $S \geq 0$ then there is a $j \in L$ such that for all $i \in \mathbb{N}$ the partial sum

$$s_{j,i} := \sum_{m=0}^{i-1} a_{\pi_L(j+m)} \geq 0.$$

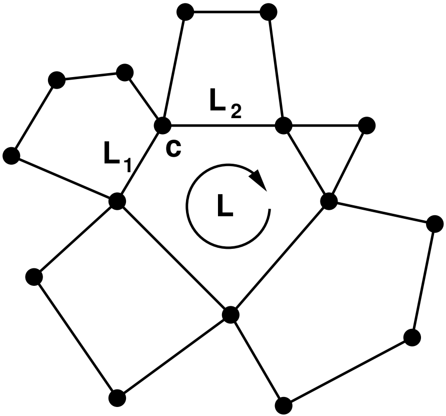
i	1	2	3	4	5	6	7
a_i	2	-3	4	1	-5	3	2
$s_{1,i}$	2	-1	3	4	-1	2	4
$s_{6,i}$	3	5	7	4	8	9	4

Corollary

Assume that there are $k \in \mathbb{N}$ and $\varepsilon \geq 0$ such that for all $j \in L$ there is an $i \leq k$ with $s_{j,i} < -\varepsilon$, then $S < -\varepsilon \cdot \ell / k$.

Sunflower

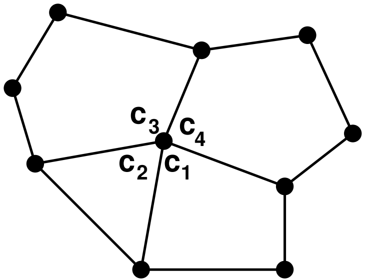
To show that every internal face has curvature $< -\varepsilon$:



Use Goes Up and Stays Up on $\frac{L_1+L_2}{2L} - c$.

Poppy

To show that every internal vertex has curvature ≤ 0:



Use Goes Up and Stays Up on $c + \frac{1-v/2}{v} = c + \frac{2-v}{v}$.

Do valency $v = 3$ first, if nothing found, increase v.

This terminates: higher valencies tend to be negatively curved anyway.

Overview over Tom analysis

What have we achieved?

If we did not find any bad sunflower or poppy, we have

- determined an explicit ε,
- proved hyperbolicity, and
- can in principle solve the diagram boundary problem.

If we did find bad sunflowers or poppy, we can still

- improve our choices for the corner values (leads to difficult optimisation/linear program problems),
- forbid more diagrams (if possible) (need to show that every boundary is proved by a permitted one),
- or switch to a more powerful officer (with further sight or redistribution), ...

and try again. If $\langle S, X; R\rangle$ is not hyperbolic, this will not work.