Finding normal subgroups of even order

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- We are looking for a randomised algorithm.
- Assume we can generate uniformly distributed random elements in *G*.
- "High probability" means for the moment "higher than $1/[G : N]$".

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Since we can compute normal closures, our initial problem is exactly, what we need to do.

Finding even order normal subgroups

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In particular, $C \cap N$ contains an involution.

That is, we can replace (N, G) with $(C \cap N, C)$ and use the statement again, provided we find another non-central involution.

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We find involutions by powering up random elements.

Computing involution centralisers

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What if D is large?

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1. Consider $c := [x, y] := x^{-1}y^{-1}xy$, if $c \neq 1$, we take $z := c$.

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   - If some $c_i := [x, y_i] \neq 1$, then take $z := c_i$ as in 1.
   - Otherwise $x \in C_G(Y)$ but $x \notin Z(G)$, thus $Y \neq G$, we take $z := y$.

Combining Algorithms 1 and 3

Algorithm 4: FINDELMOFEVENNORMALSUBGROUP

Let $G = \langle g_1, \ldots, g_k \rangle \leq \mathrm{GL}(d, q)$.

1. Use Algorithm INVOLUTIONDESCENT to produce candidate elements.
   (If there are too many central involutions, select some randomly.)
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Algorithm 4: FindElmOfEvenNormalSubgroup

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- One non-trivial group element is returned.
- The algorithm is Monte Carlo and could return a wrong result.

Examples

This approach works well in many important cases:

G	N	time
$A_{20} \wr A_{30}$	$A_5^{\times 30}$	120
$\mathrm{SL}(3,3) \wr A_{10} < \mathrm{GL}(30,3)$	$\mathrm{SL}(3,3)^{\times 10}$	724
$\mathrm{Sp}(6,3) \otimes 2.\mathrm{O}(7,3) < \mathrm{GL}(48,3)$ (computing projectively)	$\mathrm{Sp}(6,3) \otimes 1$ or $1 \otimes 2.\mathrm{O}(7,3)$	645
$6.\mathrm{Suz} < \mathrm{GL}(12,25)$	central 2	227
S_{100}	A_{100}	165
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$\mathrm{PSL}(10,5)$	—	1248
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(here we have averaged over 10 runs, times in ms)

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The success rate was 100% in all cases (using 200 runs).

Reductions for imprimitive matrix groups

Situation

Let $G \leq \mathrm{GL}_n(\mathbb{F}_q)$ acting linearly on $V := \mathbb{F}_q^{1\times n}$, such that V is absolutely irreducible. Assume there is N with $Z(G) < N \lhd G$ such that

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We use Algorithm FindElmOfEvenNormalSubgroup, for the result x, do:

- compute the normal closure $M := \langle x^G \rangle$,
- use the MeatAxe to check whether $V|_M$ is reducible,
- if $x \in N$, we find a reduction.

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- There could be no non-central involutions.
- There could be extremely many central involutions.
- We could get an involution centraliser wrong.
- We might not find all non-central involutions.
- G might not have an even order normal subgroup.

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