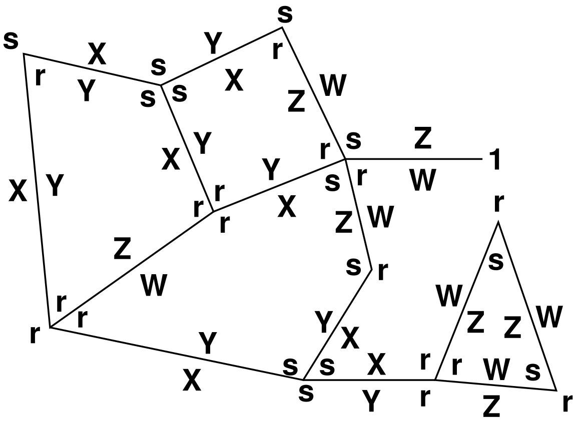
Algorithmic Generalisations of Small Cancellation Theory

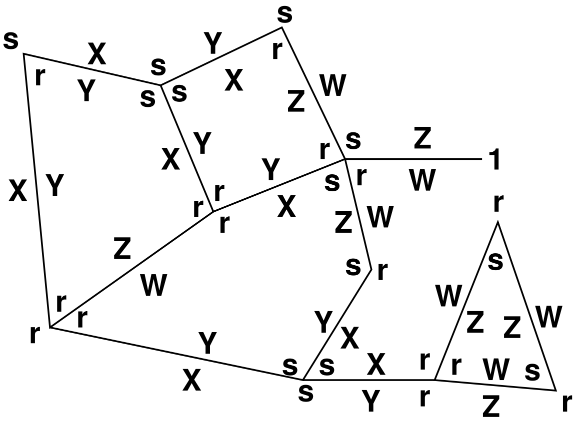
Max Neunhöffer

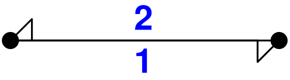
joint work with Stephen Linton,

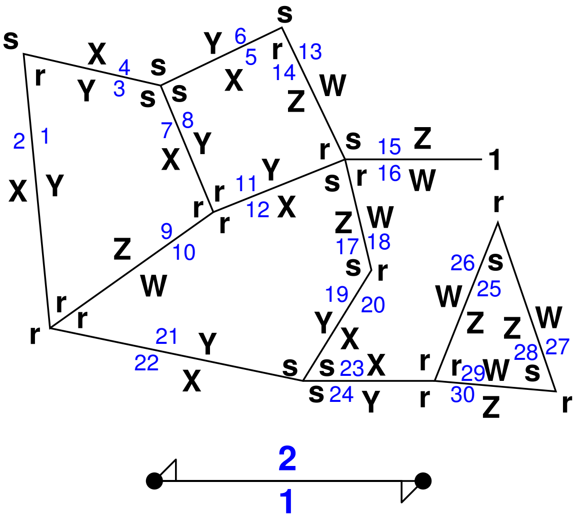
Richard Parker and Colva Roney-Dougal

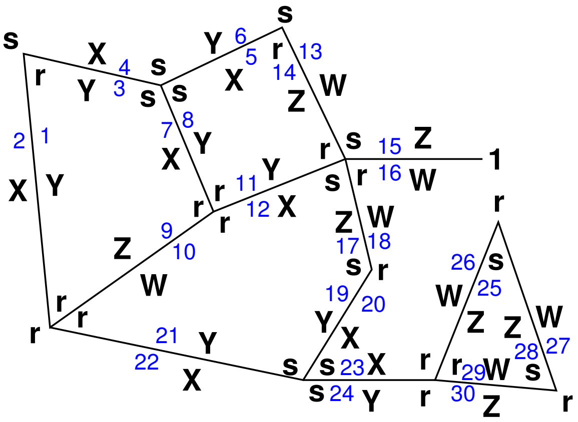
Nikolauskonferenz, Aachen, 10 December 2011

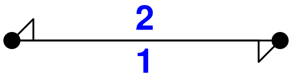

s
s Y
X r
s
r W
Y s s X
Z
Y r s Z
X 1
Y
s r W
X Y r
r
r X W
r Z
Z
s
s r
W W
r
r W
Y Z Z
r
X
Y
s s X r r W s
X
s
r
Y r
Z



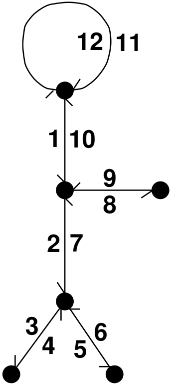

2
1


s
X
4
s
Y
6
s
r
5
13
r
Y
3
s
s
X
14
W
Z
2
1
7
8
X
Y
s
15
Z
1
X
Y
r
Y
r
11
s
r
16
W
r
r
12
X
Z
W
9
r
Z
10
17
18
s
r
26
s
W
r
19
20
W
25
Y
W
r
r
X
Z
Z
27
21
Y
28
22
s
s
23
X
r
r
29
W
s
X
s
24
Y
r
30
Z
r
2
1

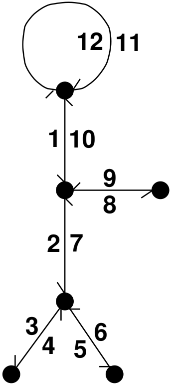




Edges are pairs of directed edges which are labelled by 2 letters each.

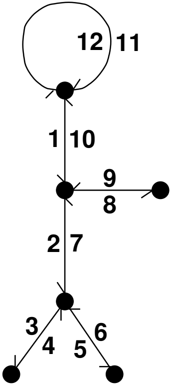


Pt	E	F	V
1	10	2	12
2	7	3	10
3	4	4	7
4	3	5	4
5	6	6	3
6	5	7	6
7	2	8	5
8	9	9	2
9	8	10	9
10	1	11	8
11	12	1	1
12	11	12	11



Pt	E	F	V
1	10	2	12
2	7	3	10
3	4	4	7
4	3	5	4
5	6	6	3
6	5	7	6
7	2	8	5
8	9	9	2
9	8	10	9
10	1	11	8
11	12	1	1
12	11	12	11

$\{$connected planar maps$\}/\sim$ is in bijection with



Pt	E	F	V
1	10	2	12
2	7	3	10
3	4	4	7
4	3	5	4
5	6	6	3
6	5	7	6
7	2	8	5
8	9	9	2
9	8	10	9
10	1	11	8
11	12	1	1
12	11	12	11

$\{$connected planar maps$\}/\sim$ is in bijection with

$\{(E, F, V) \in S_n^3 \quad | \quad EFV = 1, \langle E, F \rangle$ is transitive,
$\#$cycles of E, F and V is $n+2$,
E is a fixed-point free involution$\}/\sim$

Definition (Pongos)

A pongo is a set A with a subset $A_+ \subset A$, such that $A_0 := A\dot{\cup}\{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in A_+ \text{ for } x, y \in A, \text{ then } yx \in A_+.$$

The elements in A_+ are called acceptors.

Definition (Pongos)

A pongo is a set A with a subset $A_+ \subset A$, such that $A_0 := A \dot{\cup} \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in A_+ \text{ for } x, y \in A, \text{ then } yx \in A_+.$$

The elements in A_+ are called acceptors.

Lemma (Cyclicity)

Let A be a pongo, if $a_1 a_2 \cdots a_k \in A_+$, then all rotations $a_i a_{i+1} \cdots a_k a_1 a_2 \cdots a_{i-1} \in A_+$.

Definition (Pongos)

A pongo is a set A with a subset $A_+ \subset A$, such that $A_0 := A \dot\cup \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in A_+ \text{ for } x, y \in A, \text{ then } yx \in A_+.$$

The elements in A_+ are called acceptors.

Lemma (Cyclicity)

Let A be a pongo, if $a_1 a_2 \cdots a_k \in A_+$, then all rotations $a_i a_{i+1} \cdots a_k a_1 a_2 \cdots a_{i-1} \in A_+$.

The V-cycle (E-cycle) labels are from a pongo A (resp. B).

Definition (Pongos)

A pongo is a set A with a subset $A_+ \subset A$, such that $A_0 := A \dot{\cup} \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in A_+ \text{ for } x, y \in A, \text{ then } yx \in A_+.$$

The elements in A_+ are called acceptors.

Lemma (Cyclicity)

Let A be a pongo, if $a_1 a_2 \cdots a_k \in A_+$, then all rotations $a_i a_{i+1} \cdots a_k a_1 a_2 \cdots a_{i-1} \in A_+$.

The V-cycle (E-cycle) labels are from a pongo A (resp. B).
A V-cycle (E-cycle) is valid, if the product of its labels is an acceptor.

Definition (Pongos)

A pongo is a set A with a subset $A_+ \subset A$, such that $A_0 := A \dot{\cup} \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in A_+ \text{ for } x, y \in A, \text{ then } yx \in A_+.$$

The elements in A_+ are called acceptors.

Lemma (Cyclicity)

Let A be a pongo, if $a_1 a_2 \cdots a_k \in A_+$, then all rotations $a_i a_{i+1} \cdots a_k a_1 a_2 \cdots a_{i-1} \in A_+$.

The V-cycle (E-cycle) labels are from a pongo A (resp. B).
A V-cycle (E-cycle) is valid, if the product of its labels is an acceptor.

Definition (Japanese)

A pongo A is called Japanese, if for every $a \in A$ there is a unique $b \in A$ with $ab \in A_+$.

Let A and B be two pongos.

Definition (Set of relators)

A set of relators R is a finite set of cyclic words in $A \times B$.

Let A and B be two pongos and R a set of relators.

Definition (Set of relators)

A set of relators R is a finite set of cyclic words in $A \times B$.

Problem (Diagram boundary problem)

Devise (algorithmically) a procedure that decides for any cyclic word w in $A \times B$, whether or not there is a diagram such that

- *every internal F-cycle is labelled by a relator, and*
- *the external F-cycle is labelled by w.*

Such a diagram is called a diagram proving w.

Let A and B be two pongos and R a set of relators.

Definition (Set of relators)

A set of relators R is a finite set of cyclic words in $A \times B$.

Problem (Diagram boundary problem)

Devise (algorithmically) a procedure that decides for any cyclic word w in $A \times B$, whether or not there is a diagram such that

- *every internal F-cycle is labelled by a relator, and*
- *the external F-cycle is labelled by w.*

Such a diagram is called a diagram proving w.

Problem (Isoperimetric inequality)

Find and prove (algorithmically) a function $f : \mathbb{N} \to \mathbb{N}$, such that for every cyclic word $w \in A \times B$ of length ℓ that is proved by a diagram, there is one proving w with at most $f(\ell)$ internal F-cycles.

Let A and B be two pongos and R a set of relators.

Definition (Set of relators)

A set of relators R is a finite set of cyclic words in $A \times B$.

Problem (Diagram boundary problem)

Devise (algorithmically) a procedure that decides for any cyclic word w in $A \times B$, whether or not there is a diagram such that

- *every internal F-cycle is labelled by a relator, and*
- *the external F-cycle is labelled by w.*

Such a diagram is called a diagram proving w.

Problem (Isoperimetric inequality)

Find and prove (algorithmically) a function $f : \mathbb{N} \to \mathbb{N}$, such that for every cyclic word $w \in A \times B$ of length ℓ that is proved by a diagram, there is one proving w with at most $f(\ell)$ internal F-cycles.

If f is linear, we call (A, B, R) hyperbolic.

This can be applied

This can be applied

- to solve the word problem in quotients of the free group,

This can be applied

- to solve the word problem in quotients of the free group,
- to solve the word problem in quotients of the modular group,

This can be applied

- to solve the word problem in quotients of the free group,
- to solve the word problem in quotients of the modular group,
- to decide if w can be rewritten to w' using a given rewrite system,

This can be applied

- to solve the word problem in quotients of the free group,
- to solve the word problem in quotients of the modular group,
- to decide if w can be rewritten to w' using a given rewrite system,
- to solve the word problem in cancellative monoids

This can be applied

- to solve the word problem in quotients of the free group,
- to solve the word problem in quotients of the modular group,
- to decide if w can be rewritten to w' using a given rewrite system,
- to solve the word problem in cancellative monoids
- to solve jigsaw-puzzles in which you can use arbitrarily many copies of each piece,

This can be applied

- to solve the word problem in quotients of the free group,
- to solve the word problem in quotients of the modular group,
- to decide if w can be rewritten to w' using a given rewrite system,
- to solve the word problem in cancellative monoids
- to solve jigsaw-puzzles in which you can use arbitrarily many copies of each piece,
- etc. ???

This can be applied

- to solve the word problem in quotients of the free group,
- to solve the word problem in quotients of the modular group,
- to decide if w can be rewritten to w' using a given rewrite system,
- to solve the word problem in cancellative monoids
- to solve jigsaw-puzzles in which you can use arbitrarily many copies of each piece,
- etc. ???

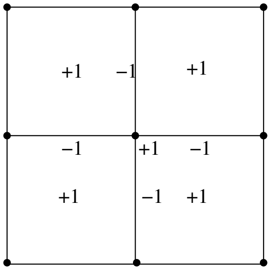
You only have to chose the right pongo!

Combinatorical curvature: We endow

- each V-cycle with $+1$ unit of curvature,
- each E-cycle with -1 unit of curvature and
- each F-cycle with $+1$ unit of curvature.

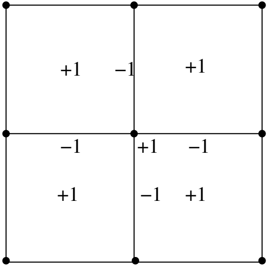
Combinatorical curvature: We endow

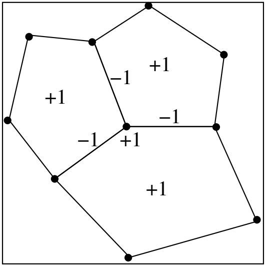
- each V-cycle with $+1$ unit of curvature,
- each E-cycle with -1 unit of curvature and
- each F-cycle with $+1$ unit of curvature.



Combinatorical curvature: We endow

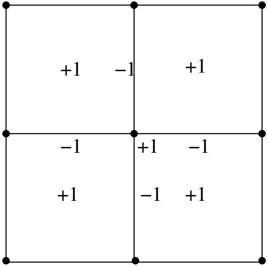
- each V-cycle with $+1$ unit of curvature,
- each E-cycle with -1 unit of curvature and
- each F-cycle with $+1$ unit of curvature.



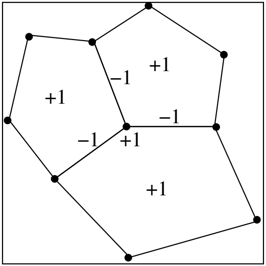


Combinatorical curvature: We endow

- each V-cycle with $+1$ unit of curvature,
- each E-cycle with -1 unit of curvature and
- each F-cycle with $+1$ unit of curvature.



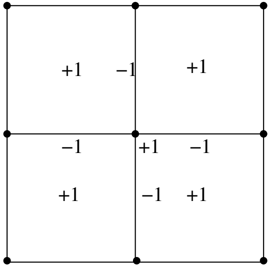
$$1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0$$



$$1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}$$

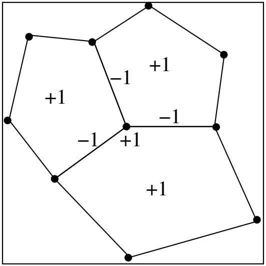
Combinatorical curvature: We endow

- each V-cycle with $+1$ unit of curvature,
- each E-cycle with -1 unit of curvature and
- each F-cycle with $+1$ unit of curvature.



$1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0$

locally flat

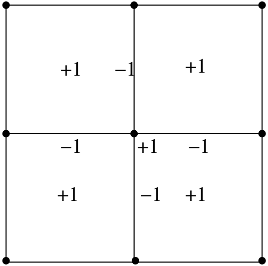


$1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}$

locally positively curved

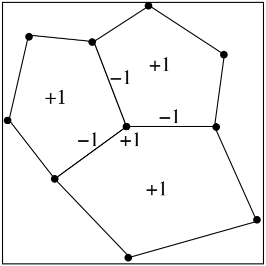
Combinatorical curvature: We endow

- each V-cycle with $+1$ unit of curvature,
- each E-cycle with -1 unit of curvature and
- each F-cycle with $+1$ unit of curvature.



$$1 - 4 \cdot \frac{1}{2} + 4 \cdot \frac{1}{4} = 0$$

locally flat



$$1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{5} = \frac{1}{10}$$

locally positively curved

Idea

Analyse curvature locally for all possible diagrams (“instantiation”).

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar map we have: $\#V + \#E + \#F = n + 2$ *and thus*
$\#V - \#E + \#F = 2$
(number of V-, E- and F-cycles, including the external one).

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar map we have: $\#V + \#E + \#F = n + 2$ *and thus* $\#V - \#E + \#F = 2$
(number of V-, E- and F-cycles, including the external one).
Thus: The total sum of our combinatorial curvature is always $+2$.

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar map we have: $\#V + \#E + \#F = n + 2$ *and thus*
$\#V - \#E + \#F = 2$
(number of V-, E- and F-cycles, including the external one).
Thus: The total sum of our combinatorial curvature is always $+2$*.*

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar map we have: $\#V + \#E + \#F = n + 2$ *and thus*
$\#V - \#E + \#F = 2$
(number of V*-,* E*- and* F*-cycles, including the external one).*
Thus: The total sum of our combinatorial curvature is always $+2$*.*

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,
- we can disjoin positively curved cases of boundary regions,

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar map we have: $\#V + \#E + \#F = n + 2$ *and thus*
$\#V - \#E + \#F = 2$
(number of V-, E- and F-cycles, including the external one).
Thus: The total sum of our combinatorial curvature is always $+2$.

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,
- we can disjoin positively curved cases of boundary regions,
- there are no spheres, and

Idea (Curvature redistribution)

We redistribute the curvature locally in a conservative way.
Purpose: To smear it out locally.

Lemma (Euler's formula)

In a planar map we have: $\#V + \#E + \#F = n + 2$ *and thus*
$\#V - \#E + \#F = 2$
(number of V-, E- and F-cycles, including the external one).
Thus: The total sum of our combinatorial curvature is always $+2$.

If the local curvature (after redistribution) is negative in the interior,

- there must be some positively curved region near the boundary,
- we can disjoin positively curved cases of boundary regions,
- there are no spheres, and
- we can derive an upper bound for the number of F-cycles in terms of the boundary length.