Computing Minimal Polynomials

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All of this is joint work with Cheryl Praeger

and is based on earlier ideas of

Peter Neumann and Cheryl Praeger.

The Problem

An example

Baby Monster group $B = \langle a, b \rangle$ with $a, b \in \mathbb{F}_2^{4370 \times 4370}$

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(times in GAP, other systems behave similarly).

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Is this a typical example?

Order polynomials

Definition (Order polynomial)

$\mathbb{F}$ field, $\mathcal{A}$ f.d. $\mathbb{F}$-algebra, $V \in$ mod-$\mathcal{A}$, $v \in V$, $M \in \mathcal{A}$.
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Lemma (Generator of annihilator)

The order polynomial $\mathrm{ord}_M(v)$ divides every polynomial $q \in \mathbb{F}[x]$ with $v \cdot q(M) = 0$.

The standard approach

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The characteristic polynomial

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$$v_1 M^{d_1} = \sum_{i=0}^{d_1-1} a_i v_1 M^i \quad \text{then} \quad \mathrm{ord}_M(v_1) = x^{d_1} - \sum_{i=0}^{d_1-1} a_i x^i.$$

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Choose some $v_2 \in V \setminus \langle v_1 \rangle_M$ and find smallest $d_2 \in \mathbb{N}$, such that $(v_1, v_1 M, \ldots, v_1 M^{d_1 - 1}, v_2, v_2 M, \ldots, v_2 M^{d_2})$ is linearly dependent.

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$$v_2 M^{d_2} = \sum_{i=0}^{d_1 - 1} b_i v_1 M^i + \sum_{i=0}^{d_2 - 1} c_i v_2 M^i \quad \text{then}$$

$$\mathrm{ord}_M(v + \langle v_1 \rangle_M) = x^{d_2} - \sum_{i=0} c_i x^i.$$

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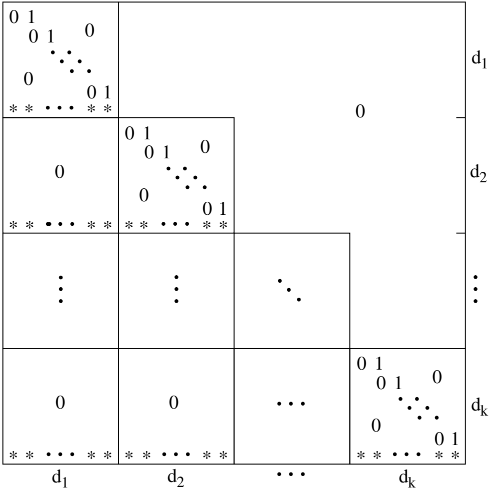
Going on like this we find an $\mathbb{F}$-basis Y of V:

$$Y := (v_1, v_1 M, \ldots, v_1^{d_1-1}, \ldots, v_k, v_k M, \ldots, v_k M_k^{d_k-1}).$$

The matrix $Y \cdot M \cdot Y^{-1}$



- Block lower-triangular
- with companion matrices along diagonal
- some sparse garbage below the diagonal

The minimal polynomial

$\longrightarrow$ compute the absolute order polynomials $\mathrm{ord}_M(v_i)$ instead the relative ones $\mathrm{ord}_M(v_i + \langle v_1, \ldots, v_{i-1} \rangle)_M$.

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Lemma (Minimal polynomial)

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Problem:

- $\dim_{\mathbb{F}}(V_i) - \dim_{\mathbb{F}}(V_{i-1})$ might be small
- even if $\dim_{\mathbb{F}}(V_i)$ is big.

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Characteristic polynomial: asymptotically $\leq 5n^3$ field ops.

Minimal polynomial: asymptotically $\sim n^4$ field ops.

(both worst case analysis)

A Monte Carlo approach

What can we do about it?

Two lemmas

Lemma (Order polynomials in cyclic spaces)

Let $W := \langle v \rangle_M < V$ be a cyclic subspace and $p := \mathrm{ord}_M(v)$ be the order polynomial of v. Let $w = v \cdot q(M) \in W$ with $\deg(q) < \deg(p)$. Then

$$\mathrm{ord}_M(w) = \frac{p}{\gcd(p, q)}.$$

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Lemma (Relative and absolute order polynomials)

Let $W < V$ be M-invariant and $v \in V$. If $q := \mathrm{ord}_M(v + W)$ is the relative order polynomial of v, then $v \cdot q(M) \in W$ and

$$\mathrm{ord}_M(v) = q \cdot \mathrm{ord}_M(v \cdot q(M)).$$

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Needs $\leq (j+8) \cdot D^2 + j \cdot D$ field ops. where $D := \dim_{\mathbb{F}}(V_j)$.

A Monte Carlo algorithm

Proposition

Let $\mathbb{F} = \mathbb{F}_q$, randomise $v_1, \ldots, v_u \in V$ independently and uniformly distributed, $\chi_M = \prod_{i=1}^{t} q_i^{e_i}$. Then:

$$\mathrm{Prob}\left(\mathrm{lcm}(\mathrm{ord}_M(v_1), \ldots, \mathrm{ord}_M(v_u)) = \mu_M\right)$$

is at least $$\prod_{i=1}^{t}\left(1 - q^{-u\deg(q_i)}\right).$$

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Algorithm: Input M, $0 < \epsilon < 1/2$

- Compute χ_M, Y, $\text{ord}_M(v_i + V_{i-1})$ for $1 \leq i \leq k$
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Needs asymptotically $\le 5n^3 + \text{FACTORISATION}(n)$ field ops.

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Irreducible factors of χ_M:

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- typical behaviour for such matrices,
- most matrices are not of this type,
- however, such matrices might occur in applications.