Algorithmic Generalisations of Small Cancellation Theory

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University of St Andrews

Computational Group Theory Workshop Oberwolfach 2011

All this is joint work with:

- Stephen Linton,
- Richard Parker,
- Colva Roney-Dougal, and

a Post-Doc we are about to hire.

The project is already ongoing for > 2 years.
However: no publications and no publishable software (yet).

This talk: Overview over the main ideas

Everybody else who wants to take part is welcome to do so!

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  - that an explicitly given rewrite system solves the word problem in linear time (Dehn's Algorithm).

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- *or fails.*

Construction

For $1 \leq i \leq n$ let $\mathcal{O}_i$ be pairwise disjoint finite sets and H_i groups. Set

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$\mathcal{O}_1 = \{1\}$, $H_1 = C_3 = \langle s \mid s^3 = 1 \rangle$ and $\mathcal{O}_2 = \{2\}$, $H_2 = C_2 = \langle t \mid t^2 = 1 \rangle$ gives the modular group $\mathrm{PSL}_2(\mathbb{Z}) \cong C_3 * C_2$.

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$\Longrightarrow$ Actually covers all free products of finitely generated groups

Recall: $F = C_3 * C_2 = \langle S, R, T \mid SR = 1 = S^3 = T^2 \rangle$
RW-System: $SR \to \epsilon$, $RS \to \epsilon$, $TT \to \epsilon$, $SS \to R$, $RR \to S$.

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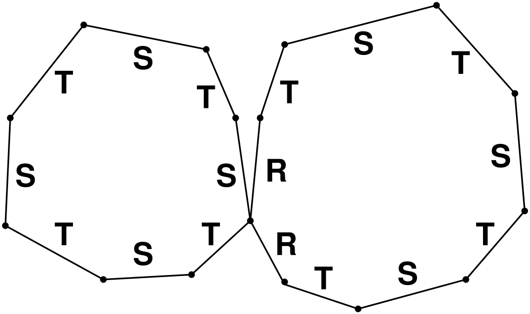

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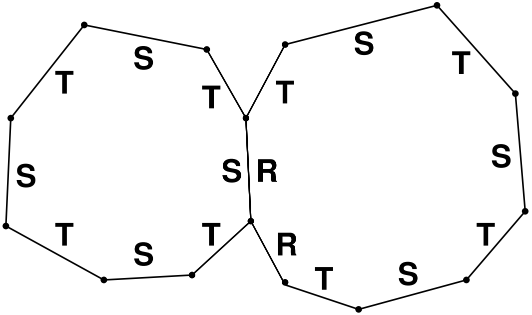

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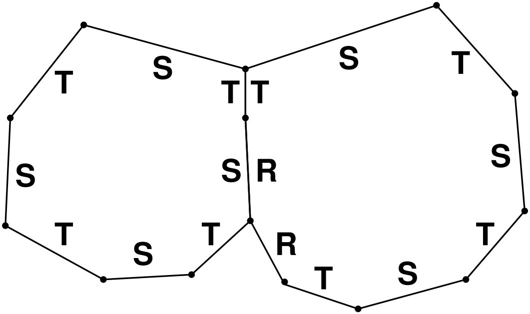

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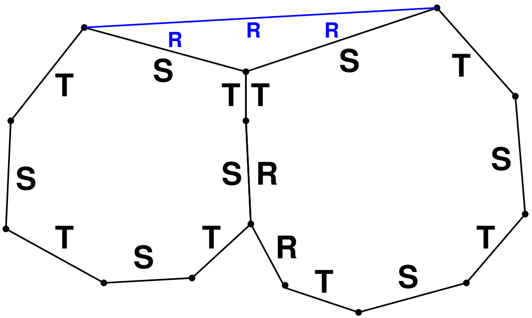

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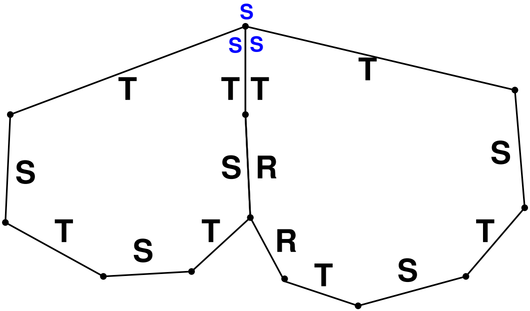

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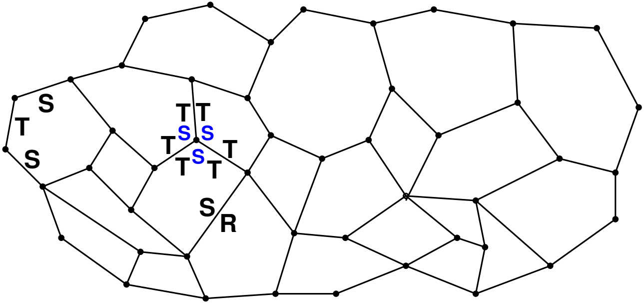

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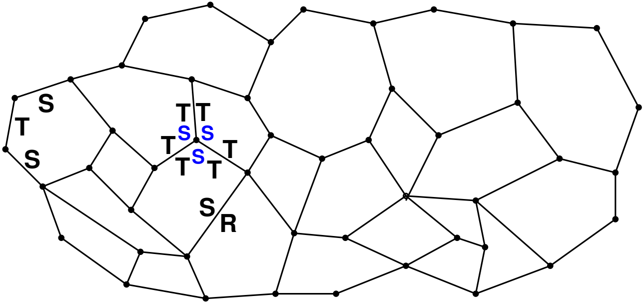

This is a generalisation of van Kampen diagrams.

Combinatorical curvature: A diagram is a planar graph. We endow

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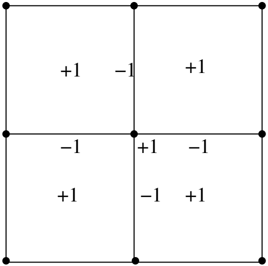

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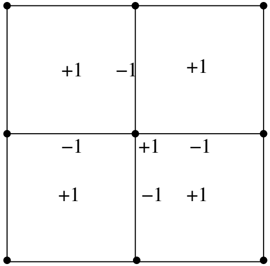

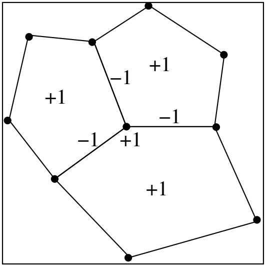

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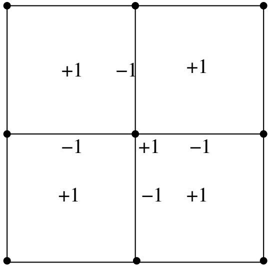

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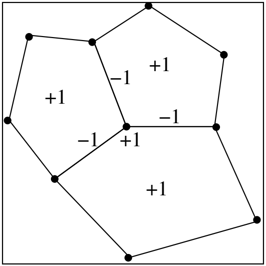

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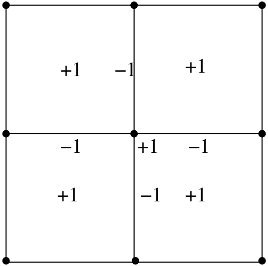

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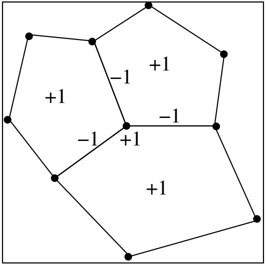

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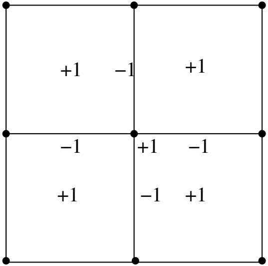

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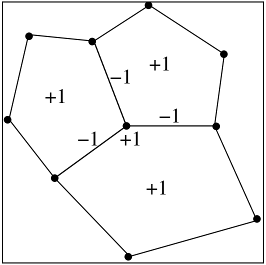

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Idea

Analyse curvature locally for all possible diagrams (“instantiation”).

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- there are no spheres, and
- we can derive an upper bound for the number of faces in terms of the boundary length.

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  - So we never need a proof containing the bigger half and
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- Change generating set to improve “small cancellation”.

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