q-Schur algebras, Wedderburn decomposition and James' conjecture

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All this is joint work with

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Iwahori-Hecke-Algebras of type A

Let $W := S_r$ and S its Coxeter generators.
Let R be a commutative ring, and $v \in R^\times$.

The Iwahori-Hecke algebra $\mathcal{H}_W(R, v)$ is the R-free R-algebra with R-basis $(T_w)_{w \in W}$ satisfying

$$\begin{array}{ll} T_w T_{w'} = T_{ww'} & \text{if } l(ww') = l(w) + l(w'), \\ (T_s - v)(T_s + v^{-1}) & \text{for } a \in S, \end{array}$$

where l is the length function on W.

A ring homomorphism $\varphi : R \to R'$ induces another one:

$$\mathcal{H}_W(R, v) \to \mathcal{H}_W(R', \varphi(v))$$

Set $A := \mathbb{Z}[v, v^{-1}]$:
$\mathcal{H}_W(A, v)$ is called the generic Hecke algebra.
$\varphi : A \to \mathbb{F}_\ell$ is called a specialisation.

q-Schur algebras

Let $\Lambda(n, r) := \{$compositions of r with at most n parts$\}$.

For $\lambda \in \Lambda(n, r)$ let W_λ be the parabolic subgroup.

We set $q := v^2$ and

$$\mathcal{S}_q(n, r) := \mathrm{End}_{\mathcal{H}} \left(\bigoplus_{\lambda \in \Lambda(n,r)} x_\lambda \mathcal{H} \right),$$

where $x_\lambda = \sum_{w \in W_\lambda} v^{l(w)} T_w \in \mathcal{H}$.

For $\lambda, \mu \in \Lambda(n, r)$ let $D_{\lambda,\mu}$ be the set of distinguished W_λ-W_μ-double coset representatives.

Let $M(n, r) := \left\{(\lambda, w, \mu) \mid \lambda, \mu \in \Lambda(n, r) \text{ and } w \in D_{\lambda,\mu}\right\}$.

Write for $\underline{a} = (\lambda, w, \mu) \in M(n, r)$:

$$\mathrm{ro}(\underline{a}) := \lambda \quad \text{and} \quad \mathrm{co}(\underline{a}) := \mu \quad \text{and} \quad \sigma(\underline{a}) := z,$$

where z is the longest element in $W_\lambda w W_\mu$.

Bases of the Iwahori-Hecke algebra $\mathcal{H}$

$(T_w)_{w\in W}$ is an A-basis of $\mathcal{H} := \mathcal{H}_W(A, v)$.

In 1979 Kazhdan and Lusztig defined a basis $(C_w)_{w\in W}$:

$$\overline{C_w} = C_w \quad \text{and} \quad C_w = \sum_{y\le w} p_{y,w} T_w \text{ for } w \in W$$

where $p_{y,w} \in \mathbb{Z}[v^{-1}]$ and $p_{w,w} = 1$ and $\le$ is the Bruhat-Chevalley order and $\overline{} : \mathcal{H} \to \mathcal{H}$ is the involution

$$\overline{v} := v^{-1} \quad \text{and} \quad \overline{\sum_{w\in W} a_w T_w} := \sum_{w\in W} \overline{a_w} T_{w^{-1}}^{-1}.$$

The $p_{y,w}$ are the famous Kazhdan-Lusztig polynomials and $(C_w)_{w\in W}$ the Kazhdan-Lusztig basis.

Bases of the q-Schur algebra $\mathcal{S}$

$\mathcal{S}_q(n, r)$ has a standard basis $(\phi^w_{\lambda,\mu})_{(\lambda,w,\mu)\in M(n,r)}$.

Recall

$$\mathcal{S}_q(n, r) := \mathrm{End}_{\mathcal{H}}\left(\bigoplus_{\lambda\in\Lambda(n,r)} x_\lambda \mathcal{H}\right),$$

we have $\phi^w_{\lambda,\mu} \in \mathrm{Hom}_{\mathcal{H}}(x_\lambda \mathcal{H}, x_\mu \mathcal{H})$.

Using $(C_w)_{w\in W}$, Du defined a basis $(\theta_{\underline{a}})_{\underline{a}\in M(n,r)}$ with similar properties.

We call it the Du-Kazhdan-Lusztig basis of $\mathcal{S}_q(n, r)$.

What are these interesting properties?

Cells and cell modules I

Back to $\mathcal{H}$, let $(g_{x,y,z})_{x,y,z\in W}$ be the structure constants:

$$C_x \cdot C_y = \sum_{z\in W} g_{x,y,z} C_z$$

We have $g_{x,y,z} \in \mathbb{Z}[v, v^{-1}]$, the coefficients are ≥ 0!

Define $z \leq_L y$ if there is $x \in W$ with $g_{x,y,z} \neq 0$, that is: C_z occurs in some $C_x \cdot C_y$ as above.

$\leq_L$ is a preorder, this defines an equivalence relation $\sim_L$, the equivalence classes are called left cells.

For a left cell Λ and $z \in \Lambda$, define

$$\mathcal{H}_{\leq\Lambda} := \langle C_w \mid w \leq_L z\rangle_A \text{ and } \mathcal{H}_{<\Lambda} := \langle C_w \mid w <_L z\rangle_A$$

and set $\mathrm{LC}^{(\Lambda)} := \mathcal{H}_{\leq\Lambda}/\mathcal{H}_{<\Lambda}$, the left cell module of Λ with basis $(C_w + \mathcal{H}_{<\Lambda})_{w\in\Lambda}$.

Analogously: $z \leq_R x$ if there is $y \in W$ with $g_{x,y,z} \neq 0$.

Cells and cell modules II

Again $\mathcal{S}$, let $(f_{\underline{a},\underline{b},\underline{c}})_{\underline{a},\underline{b},\underline{c} \in M(n,r)}$ be the structure constants:

$$\theta_{\underline{a}} \cdot \theta_{\underline{b}} = \sum_{\underline{c} \in M(n,r)} f_{\underline{a},\underline{b},\underline{c}} \cdot \theta_{\underline{c}}$$

Lemma

We have $f_{\underline{a},\underline{b},\underline{c}} = 0$ unless $\mathrm{co}(\underline{a}) = \mathrm{ro}(\underline{b})$ and $\mathrm{ro}(\underline{c}) = \mathrm{ro}(\underline{a})$ and $\mathrm{co}(\underline{c}) = \mathrm{co}(\underline{b})$, and

$$f_{\underline{a},\underline{b},\underline{c}} = h_\mu^{-1} \cdot g(\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c}))$$

in this case for some $0 \neq h_\mu \in \mathbb{Z}[v, v^{-1}]$.

Define $\underline{c} \leq_L \underline{b}$ if there is $\underline{a} \in M(n,r)$ with $f_{\underline{a},\underline{b},\underline{c}} \neq 0$.

Define $\sim_L$, left cells, $\mathcal{S}_{\leq \Lambda}$, $\mathcal{S}_{< \Lambda}$ and $\mathrm{LC}^{(\Lambda)}$ exactly as for Hecke-algebras.

Simple cell modules

One of the wonders of the Kazhdan-Lusztig basis is:

Theorem (Kazhdan-Lusztig, Du)

For the field $K := \mathbb{Q}(v)$ and the extensions of scalars $K\mathcal{H}_W(A, v) = \mathcal{H}_W(K, v)$ and $K\mathcal{S}_q(n, r)$ we have:

$\mathrm{KLC}^{(\Lambda)}$ is a simple module for every left cell Λ.

This gives filtrations of $K\mathcal{H}$ and $K\mathcal{S}$ by simple modules.

Theorem (Dipper-James)

$\mathcal{H}_W(K, v)$ and $K\mathcal{S}_q(n, r)$ are semisimple.
In fact, $\mathcal{H}_W(\mathbb{F}, u)$ is semisimple unless u is an e-th root of unity with $e \leq r$ (and likewise for $\mathcal{S}_q(n, r)$).

Trace forms and dual bases

Let

$$\tau(h) := \sum_{\chi \in \mathrm{Irr}(\mathcal{H}_W(K,v))} \frac{\chi(h)}{c_\chi}$$

for some elements $0 \neq c_\chi \in K$.

Then τ is a symmetrising trace form on $\mathcal{H}_W(K, v)$, i.e.:

- $(h, h') \mapsto \tau(hh')$ is bilinear and
- non-degenerate and
- symmetric: $\tau(hh') = \tau(h'h)$ for all $h, h' \in \mathcal{H}$.

Thus, for every basis $(B_w)_{w \in W}$ there is a dual basis $(B_w^\vee)_{w \in W}$ with

$$\tau(B_v B_w^\vee) = \delta_{v,w}.$$

We do the same for $K\mathcal{S}_q(n, r)$ and use $(\theta_{\underline{a}}^\vee)_{\underline{a} \in M(n,r)}$, note:

If $h = \sum_{\underline{a} \in M(n,r)} \beta_{\underline{a}} \theta_{\underline{a}}$ then $\beta_{\underline{b}} = \tau(h \cdot \theta_{\underline{b}}^\vee)$ for all $\underline{b} \in M(n, r)$,

and thus $f_{\underline{a},\underline{b},\underline{c}} = \tau(\theta_{\underline{a}} \theta_{\underline{b}} \theta_{\underline{c}}^\vee)$ for all $\underline{a}, \underline{b}, \underline{c} \in M(n, r)$.

The asymptotic algebra

Let $\mathbf{a}(z)$ be the highest degree of a $g_{x,y,z}$ and
$\gamma_{x,y,z^{-1}}$ the coefficient of $g_{x,y,z}$ at $v^{\mathbf{a}(z)}$.

Using the $\gamma_{x,y,z^{-1}}$ Lusztig defined:

- a subset $\mathcal{D} \subseteq W$ of distinguished involutions,
- a semisimple A-algebra $\mathcal{J}_A$ (the asymptotic algebra)
- a homomorphism $\Phi : \mathcal{H}_W(\mathbb{Z}[v, v^{-1}], v) \to \mathcal{J}_A$. (the Lusztig homomorphism).

Du defined:

- $\mathcal{D}(n, r) := \left\{\underline{a} \in M(n, r) \mid \mathrm{ro}(\underline{a}) = \mathrm{co}(\underline{a}) \text{ and } \sigma(\underline{a}) \in \mathcal{D}\right\}$,
- $\mathcal{J}(n, r)_A$ with its standard basis $(t_{\underline{a}})_{\underline{a} \in M(n,r)}$, (the asymptotic algebra)
- with identity $\displaystyle\sum_{\underline{d} \in \mathcal{D}(n,r)} t_{\underline{d}}$, and
- the Du-Lusztig hom. $\Phi : \mathcal{S}_q(n, r) \to \mathcal{J}(n, r)_A$.

Lusztig's conjectures **P**1 to **P**15

Lusztig formulates 15 "conjectures" **P**1 to **P**15:

P2 If $\gamma_{x,y,d^{-1}} \neq 0$ with $d \in \mathcal{D}$, then $x = y^{-1}$.

P3 For $y \in W$ exists a unique $d \in \mathcal{D}$ with $\gamma_{y^{-1},y,d^{-1}} \neq 0$.

P6 For $d \in \mathcal{D}$ we have $d = d^{-1}$.

P9 If $x \leq_L y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_L y$.

P10 If $x \leq_R y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_R y$.

P13 Every left cell contains a unique element $d \in \mathcal{D}$.

These are proved for $\mathcal{H}_W(A, v)$ if

- W is a finite Weyl group,
- W is an affine Weyl group,
- W is an infinite dihedral group.

For other Iwahori-Hecke algebras they are conjectures.

Statements **Q**1 to **Q**15

We prove for $\mathcal{S}_q(n, r)$ statements **Q**1 to **Q**15: Setting

$$\gamma_{\underline{a},\underline{b},\underline{c}^t} := \begin{cases} \gamma(\sigma(\underline{a}), \sigma(\underline{b}), \sigma(\underline{c})^{-1}) & \text{if } f_{\underline{a},\underline{b},\underline{c}} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\underline{a}^t := (\mu, w^{-1}, \lambda)$ for $\underline{a} = (\lambda, w, \mu)$, we get:

Q2 If $\gamma_{\underline{a},\underline{b},\underline{d}^t} \neq 0$ with $\underline{d} \in \mathcal{D}(n, r)$, then $\underline{a} = \underline{b}^t$.

Q3 $\forall\ \underline{a} \in M(n, r)\ \exists$ a unique $\underline{d} \in \mathcal{D}(n, r)$ with $\gamma_{\underline{a}^t,\underline{a},\underline{d}^t} \neq 0$.

Q6 For $\underline{d} \in \mathcal{D}(n, r)$ we have $\underline{d} = \underline{d}^t$.

Q9 If $\underline{a} \leq_L \underline{b}$ and $\mathbf{a}(\sigma(\underline{a})) = \mathbf{a}(\sigma(\underline{b}))$, then $\underline{a} \sim_L \underline{b}$.

Q10 If $\underline{a} \leq_R \underline{b}$ and $\mathbf{a}(\sigma(\underline{a})) = \mathbf{a}(\sigma(\underline{b}))$, then $\underline{a} \sim_R \underline{b}$.

Q13 Every left cell contains a unique element $\underline{d} \in \mathcal{D}(n, r)$.

Proofs use **P**1 to **P**15 and some additional q-Schur algebra arguments.

An explicit Wedderburn basis

Let Λ be a left cell such that $\mathrm{LC}^{(\Lambda)}$ has character ψ and

$$\tau(h) := \sum_{\chi \in \mathrm{Irr}(\mathcal{H}_W(K,v))} \frac{\chi(h)}{c_\chi}$$

for some elements $0 \neq c_\chi \in K$.
Then the representing matrix of $h \in \mathcal{S}_q(n,r)$ on $\mathrm{LC}^{(\Lambda)}$ is

$$D^{(\Lambda)}(h) = \left(\tau(\theta_{\underline{a}}^\vee \cdot h \cdot \theta_{\underline{b}})\right)_{\underline{a},\underline{b} \in \Lambda}.$$

Use Frobenius-Schur relations for $K\mathcal{S}_q(n,r)$: Fix $\underline{a}, \underline{b} \in \Lambda$,

$$\sum_{\underline{c} \in M(n,r)} \tau(\theta_{\underline{a}}^\vee \cdot \theta_{\underline{c}}^\vee \cdot \theta_{\underline{b}}) \cdot \theta_{\underline{c}} = \theta_{\underline{b}} \theta_{\underline{a}}^\vee$$

acts on $\mathrm{LC}^{(\Lambda)}$ as a matrix with one entry 1 and 0 elsewhere.

An explicit Wedderburn basis II

Theorem (Wedderburn basis (Brunat, N., 2008))

The set

$$\mathcal{B} := \left\{ c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee} \mid \underline{c} \in M(n,r),\ \underline{d} \in \mathcal{D}(n,r),\ \underline{c} \sim_L \underline{d} \right\}$$

is a Wedderburn basis *of* $K\mathcal{S}_q(n,r)$*.*
For $c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}$ *and* $c_{\underline{d'}}^{-1}\theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee}$ *in* $\mathcal{B}$ *we have:*

$$(c_{\underline{d}}^{-1}\,\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}) \cdot (c_{\underline{d'}}^{-1}\theta_{\underline{c'}}\theta_{\underline{d'}}^{\vee}) = \begin{cases} 0 & \text{if } \mathrm{LC}^{(\underline{d})} \not\cong \mathrm{LC}^{(\underline{d'})} \\ 0 & \text{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d'})} \text{ and } \underline{d} \not\sim_R \underline{c'} \\ c_{\underline{d'}}^{-1}\theta_{\underline{c''}}\theta_{\underline{d'}}^{\vee} & \text{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d'})} \text{ and } \underline{d} \sim_R \underline{c'} \end{cases}$$

$\underline{c''}$ *is the* unique element *with* $\underline{c''} \sim_L \underline{d'}$ *and* $\underline{c''} \sim_R \underline{c}$ *and such a* $\underline{c''}$ *in fact exists.*

The dual basis of $\mathcal{B}$

These relations immediately imply: The dual basis $\mathcal{B}^\vee$ of

$$\mathcal{B} = \left\{ c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \mid \underline{c} \in M(n,r), \underline{d} \in \mathcal{D}(n,r), \underline{c} \sim_L \underline{d} \right\}$$

is

$$\mathcal{B}^\vee = \left\{ \theta_{\underline{c}} \theta_{\underline{d}}^\vee \mid \underline{c} \in M(n,r), \underline{d} \in \mathcal{D}(n,r), \underline{c} \sim_L \underline{d} \right\}$$

In fact: $\left(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \right)^\vee = \theta_{\underline{c}^t} \theta_{\underline{d}'}^\vee$ where $\underline{c}^t \sim_L \underline{d}' \in \mathcal{D}(n,r)$.

Note:

$$\left\langle (\theta_{\underline{a}})_{\underline{a} \in M(n,r)} \right\rangle_A = \mathcal{S}_q(n,r) \subseteq \langle \mathcal{B} \rangle_A$$

and

$$\left\langle \mathcal{B}^\vee \right\rangle_A \subseteq \left\langle (\theta_{\underline{a}}^\vee)_{\underline{a} \in M(n,r)} \right\rangle_A$$

not depending on the choice of τ!

Preimages of the $t_{\underline{a}}$

Since the formula for the Du-Lusztig homomorphism is

$$\Phi(h) = \sum_{\substack{\underline{b} \in M(n,r) \\ \underline{d}' \in \mathcal{D}(n,r) \\ \underline{d}' \sim_L \underline{b}}} \tau(h \cdot \theta_{\underline{d}'} \theta_{\underline{b}}^{\vee}) \cdot t_{\underline{b}} \quad \text{for } h \in K\mathcal{S}_q(n,r)$$

we can use **Q**1 to **Q**15 and our theorem to show:

Theorem (Preimages of the $t_{\underline{c}}$ (Brunat, N., 2008))

Let τ be an arbitrary non-degenerate symmetrising trace form on $K\mathcal{S}_q(n,r)$, then

$$\Phi(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^{\vee}) = t_{\underline{c}} \quad \text{for all } \underline{c} \in M(n,r),$$

where $c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^{\vee} \in \mathcal{B}$, that is $\underline{c} \sim_L \underline{d} \in \mathcal{D}(n,r)$.

Lusztig's homomorphism revisited

In view of our Wedderburn basis, we have for $A = \mathbb{Z}[v, v^{-1}]$ and $K = \mathbb{Q}(v)$ and $\mathcal{J}(n, r)_K := K\mathcal{J}(n, r)_A$:

$$\begin{tikzcd}
\mathcal{S}_q(n, r) \arrow[r] \arrow[d, equal] & \langle \mathcal{B} \rangle_A \arrow[r] \arrow[d, "\Phi"', "\cong"] & K\mathcal{S}_q(n, r) \arrow[d, "\Phi"', "\cong"] \\
\mathcal{S}_q(n, r) \arrow[r, "\Phi"] & \mathcal{J}(n, r)_A \arrow[r] & \mathcal{J}(n, r)_K
\end{tikzcd}$$

since Du has shown that Φ is an isomorphism after extension of scalars to K.

Thus, the Du-Lusztig homomorphism is the same as the inclusion

$$\mathcal{S}_q(n, r) \subseteq \langle \mathcal{B} \rangle_A.$$

Furthermore, we get that $\langle \mathcal{B} \rangle_A$ is isomorphic to a direct sum of full matrix rings over A.

James' conjecture ...

Let $s := |M(n,r)|$ and $M = (m_{\underline{a},\underline{b}}) \in A^{s \times s}$ be:

$$\Phi(\theta_{\underline{a}}) = \sum_{\underline{b} \in M(n,r)} m_{\underline{a},\underline{b}} \cdot t_{\underline{b}}$$

for all $\underline{b} \in M(n,r)$.

Let $\mathbb{F}_\ell$ be a finite prime field, $u \in \mathbb{F}_\ell$ of order $2e$, and

$$\begin{tikzcd} A = \mathbb{Z}[v, v^{-1}] \arrow[rr, "\varphi_e"] \arrow[dr, "\varphi_\ell"'] & & \mathbb{Z}[\zeta_{2e}] \arrow[dl, "\varphi_\ell^e"] \\ & \mathbb{F}_\ell & \end{tikzcd}$$

be a commutative diagram of ring homomorphisms ($\zeta_{2e} \in \sqrt[2e]{1} \subseteq \mathbb{C}$ primitive) with $\varphi_e(v) = \zeta_{2e}$ and $\varphi_\ell(v) = u$.

We want to compare the representation theory of

$$K\mathcal{S}_q(n,r) \quad \text{and} \quad \mathbb{Q}(\zeta_{2e})\mathcal{S}_q(n,r) \quad \text{and} \quad \mathbb{F}_\ell\mathcal{S}_q(n,r).$$

... in a reformulation by Geck

Let $\varphi_\ell = \varphi_\ell^e \circ \varphi_e$ as above.

Theorem (Geck)

James' conjecture for q-Schur algebras is equivalent to the fact that for $\ell > r$ we have

$$\mathrm{rank}_{\mathbb{Q}(\zeta_{2e})}(\varphi_e(M)) = \mathrm{rank}_{\mathbb{F}_\ell}(\varphi_\ell(M)),$$

where $\varphi_e(M) = (\varphi_e(m_{\underline{a},\underline{b}}))$ and $\varphi_\ell = (\varphi_\ell(m_{\underline{a},\underline{b}}))$.

That is, the rank of the matrix M when specialised to $\mathbb{Q}(\zeta_{2e})$ is the same as when specialised to $\mathbb{F}_\ell$.

By our results, M is the base change matrix between

$$(\theta_{\underline{a}})_{\underline{a} \in M(n,r)} \quad \text{and} \quad \mathcal{B} = \{c_{\underline{d}}^{-1} \theta_{\underline{c}}\, \theta_{\underline{d}}^\vee\},$$

all within $K\mathcal{S}_q(n, r)$!

A potential attack?

Let $Q_\tau = (q_{\underline{a},\underline{b}})$ be the base change from $(\theta_{\underline{a}}^\vee)$ to $(\theta_{\underline{a}})$

$$\theta_{\underline{a}}^\vee = \sum_{\underline{b} \in M(n,r)} q_{\underline{a},\underline{b}} \cdot \theta_{\underline{b}} \quad \text{for all } \underline{a} \in M(n,r),$$

and D be the one from $\mathcal{B}^\vee$ to $\mathcal{B}$, which is monomial, then:

$$D = M^t \cdot Q_\tau \cdot M,$$

since M^t is the base change from $\mathcal{B}^\vee$ to $(\theta_{\underline{a}}^\vee)$.

If, for some φ_e and φ_ℓ, we could find a nice τ, such that

- the elements c_χ all lie in A,
- $Q_\tau \in A^{s \times s}$, and
- the number of c_χ that vanish under φ_e is equal to the number of c_χ that vanish under φ_ℓ,

then James' conjecture would follow for φ_e and φ_ℓ.