Algorithmic Generalisations of Small Cancellation Theory

Max Neunhöffer

joint work with Jeffrey Burdges, Stephen Linton,

Richard Parker and Colva Roney-Dougal

Pure Colloquium, St Andrews, 22 March 2012

Question

What can you tell me about the finitely presented group

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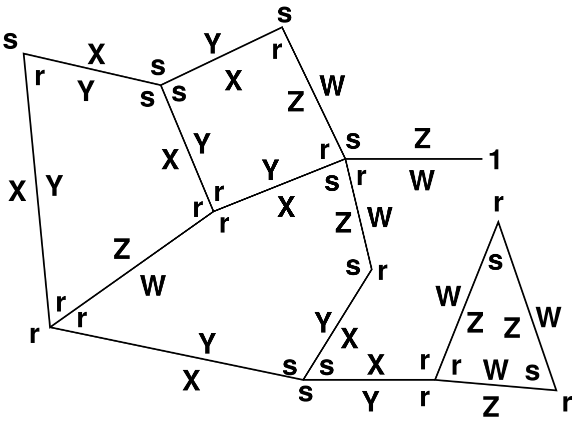
Can we solve the word problem?

We draw connected finite graphs in the plane and label them:



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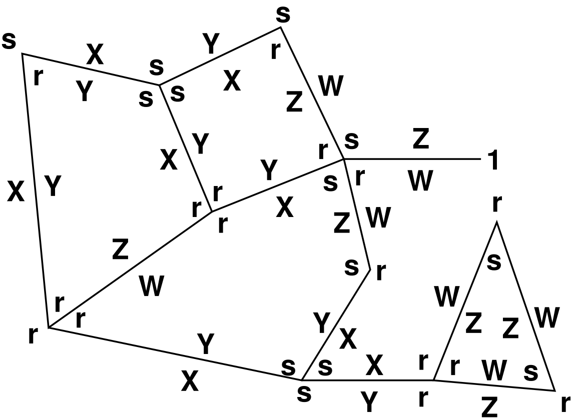

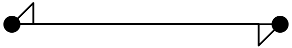

Edges are pairs of directed edges which are labelled by 2 letters each.

Diagram problems

Let R be a finite set of cyclic words, called relators.

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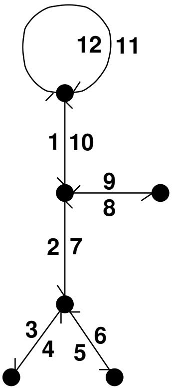
12
11
1
10
9
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2
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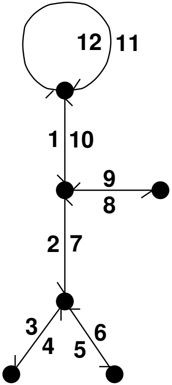
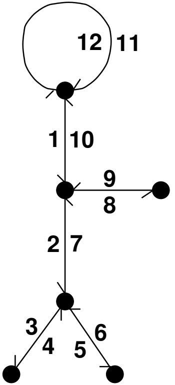
4
5
6



#	E	F	V
1	10	2	12
2	7	3	10
3	4	4	7
4	3	5	4
5	6	6	3
6	5	7	6
7	2	8	5
8	9	9	2
9	8	10	9
10	1	11	8
11	12	1	1
12	11	12	11



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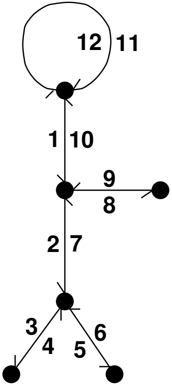

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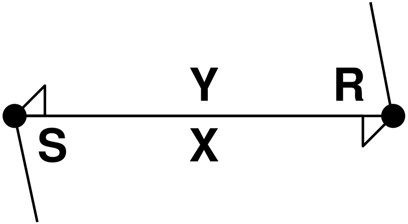
$$\{(E,F,V) \in S_n^3 \;\mid\; n \in \mathbb{N}, EFV = 1, \langle E,F\rangle \text{ is transitive, } \#\text{cycles of } E, F \text{ and } V \text{ is } n+2, E \text{ is a fixed-point free involution}\}/\sim$$

Rules for the labels

We label every half-edge with two symbols,

- one for the corner to the right of where it starts, and
- one for the right hand side of it:



We now need rules for the corner labels and the edge labels.

Definition (Pongos)

A pongo is a set P with a subset $P_+ \subset P$, such that $P_0 := P \dot{\cup} \{0\}$ is a semigroup with 0 and:

$$\text{if } xy \in P_+ \text{ for } x, y \in P, \text{ then } yx \in P_+.$$

The elements in P_+ are called acceptors.

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Lemma (Cyclicity)

Let P be a pongo, if $p_1 p_2 \cdots p_k \in P_+$, then all rotations $p_i p_{i+1} \cdots p_k p_1 p_2 \cdots p_{i-1} \in P_+$.

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Using a finite pongo is equivalent to using a finite state automaton.

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(For the experts:
This is a generalisation of the rules of van Kampen diagrams.)

Definition (Valid diagram)

Let P be a pongo and A be an edge alphabet. A valid diagram is: an $n \in \mathbb{N}$ and three permutations $E, F, V \in S_{\{1,2,\ldots,n\}}$ and a labelling function $\ell : \{1, \ldots, n\} \to P \times A, x \mapsto (\ell_P(x), \ell_A(x))$, such that

- $EFV = 1$,
- E is a fixed point free involution,
- $\langle E, F \rangle$ is a transitive subgroup of S_n,
- the total number of cycles in E, F and V is $n + 2$,
- $\ell_P(x) \cdot \ell_P(xV) \cdot \ell_P(xV^2) \cdot \cdots \in P_+$ for every V-cycle $x \langle V \rangle$, and
- $\ell_A(xE) = \overline{\ell_A(x)}$ for all E-cycles (x, xE).

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If f can be chosen linear, we call (P, A, R) hyperbolic.

$G := \langle S, R, T \mid SR, T^2, S^3, (ST)^7, (STS^2T)^{13} \rangle$ can be studied by:

$$\begin{aligned}
P &= \{S, R, 1\} \text{ with } P_+ = \{1\} \text{ and } SR = RS = 1, SS = R, RR = S \\
A &= \{T\} \text{ with } \overline{T} = T \\
R &= \{((S,T),(S,T),(S,T),(S,T),(S,T),(S,T),(S,T)), \\
&\quad ((R,T),(R,T),(R,T),(R,T),(R,T),(R,T),(R,T)), \\
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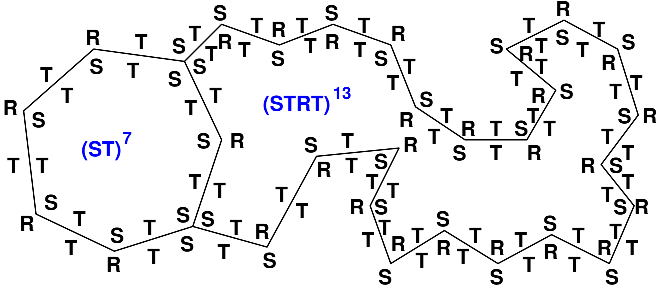

$$(ST)^7(TRTS)^{13} = (ST)^5(STST)(TRTS)(TRTS)^{12} = (ST)^5R(TRTS)^{12}$$

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You only have to chose the right pongo and edge alphabet!

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Theorem (Euler's formula)

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Observation

The total sum of our combinatorial curvature is always $+1$.

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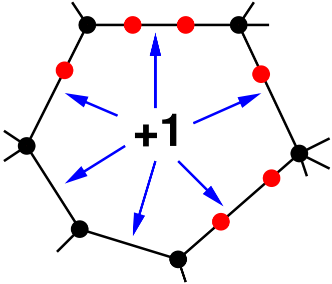

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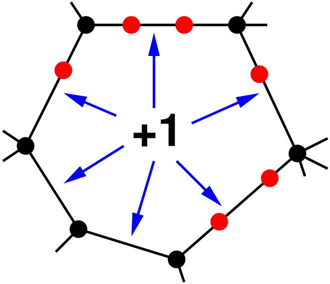

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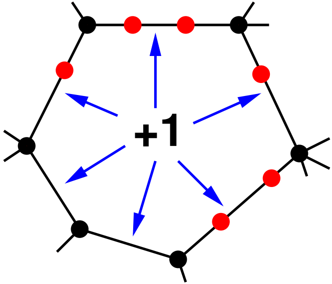

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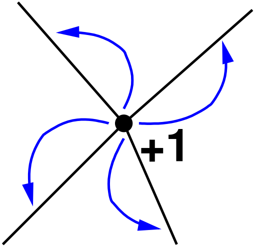

evenly

- Edges have different length (number of mini-edges).
  Both half-edges in an edge get an equal amount.
- Vertices have different valency. Only outgoing half-edge receives.

All curvature is now on the half-edges.

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$$(C_1, C_2, \ldots, C_d) \in \{E, F, F^{-1}\}^d \qquad \text{(e.g. “EFEFE”).}$$

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Definition of the pubcrawl $(C_1, C_2, \ldots, C_d)$

Let $Y := X \times D$ and define $\Delta : Y \to Y, (x, i) \mapsto (xC_i, \pi_D(i + 1))$.

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A pubcrawler crawls around (locally) from half-edge to half-edge and collects curvature. He deposits it on his orbit.

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Δ describes a step of the crawler, we sum curvature over $\langle\Delta\rangle$-orbits.

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If $S \geq 0$ then there is a $j \in L$ such that for all $i \in \mathbb{N}$ the partial sum

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i	1	2	3	4	5	6	7
a_i	2	-3	4	1	-5	3	2
$s_{1,i}$	2	-1	3	4	-1	2	4
$s_{6,i}$	3	5	7	4	8	9	4

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Corollary

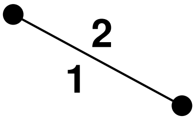
Assume that there are $k \in \mathbb{N}$ and $\varepsilon \leq 0$ such that for all $j \in L$ there is an $i \leq k$ with $s_{j,i} < \varepsilon$, then $S < \varepsilon \cdot \ell / k$.

Search for bad orbits of pubcrawl “EFEFE”

Data structure in computer

Id	E	F	F^{-1}	Rel
1	2			*
2	1			*
				*
				*
				*
				*

Illustration



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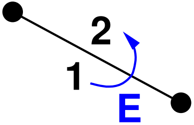

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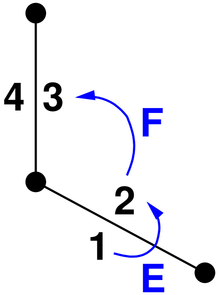

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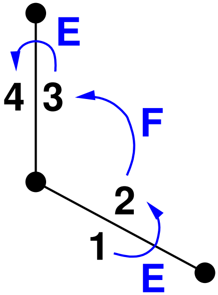

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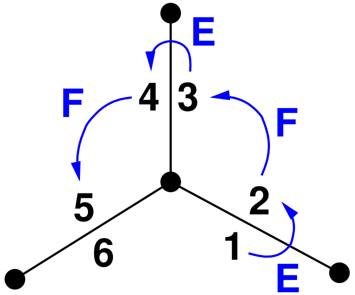

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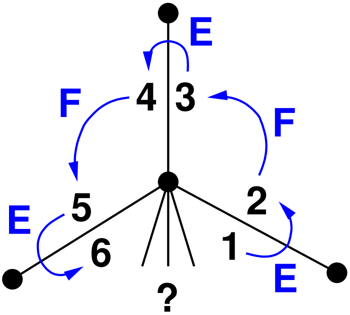

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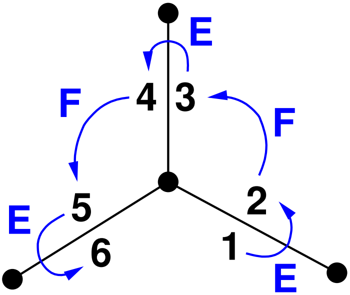

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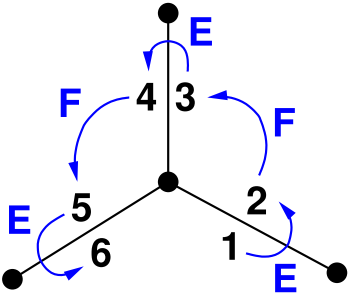

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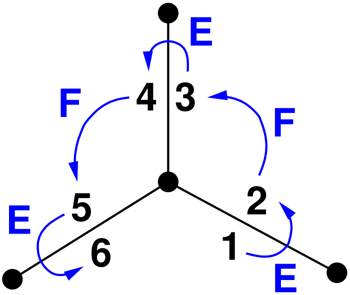

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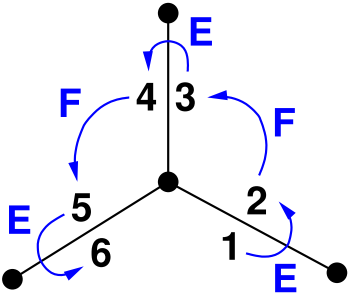

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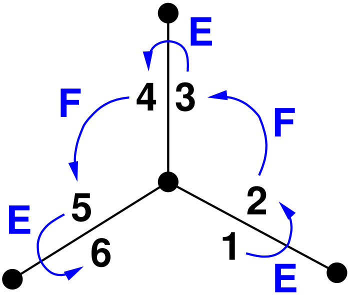

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Note that we use lower bounds for the vertex valencies!

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Since the amount of positive curvature close to the boundary can be bounded from above by an expression in the boundary length, we get a

$\boxed{\text{linear isoperimetric inequality}}$

and thus have proved hyperbolicity. ∎

Outlook

We want to

- tune our program.
- investigate lots of groups.
- do algorithmic analysis to solve the word problem in practice.
- prove that for every hyperbolic group presentation there is a successful pubcrawl.
- investigate applications to monoids and rewrite systems.
- find more interesting pongos — what do they do?
- use this technology to tackle relative hyperbolicity computationally.
- write everything up and publish the theory.
- publish the software as open source.