q-Schur algebras, Wedderburn decomposition and James' conjecture

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All this is joint work with

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(Paris)

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$\varphi_e : A \to \mathbb{Q}(\zeta_e), v \mapsto \zeta_e$ and $\varphi_\ell : A \to \mathbb{F}_\ell, v \mapsto u$ are called specialisations.

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Write for $\underline{a} = (\lambda, w, \mu) \in M(n, r)$:

$$\mathrm{ro}(\underline{a}) := \lambda \quad \text{and} \quad \mathrm{co}(\underline{a}) := \mu \quad \text{and} \quad \sigma(\underline{a}) := z,$$

where z is the longest element in $W_\lambda w W_\mu$.

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The $p_{y,w}$ are the famous Kazhdan-Lusztig polynomials and $(C_w)_{w \in W}$ the Kazhdan-Lusztig basis.

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What are these interesting properties?

Cells and cell modules I

Back to $\mathcal{H}$, let $(g_{x,y,z})_{x,y,z\in W}$ be the structure constants:

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For a left cell Λ and $z \in \Lambda$, define

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Analogously: $z \leq_R x$ if there is $y \in W$ with $g_{x,y,z} \neq 0$.

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Define $\sim_L$, left cells, $\mathcal{S}_{\leq\Lambda}$, $\mathcal{S}_{<\Lambda}$ and $\mathrm{LC}^{(\Lambda)}$ exactly as for Hecke-algebras (and R-version as well).

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Theorem (Kazhdan-Lusztig, Du)

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Theorem (Dipper-James)

$\mathcal{H}_W(K,v)$ and $K\mathcal{S}_q(n,r)$ are semisimple.

Simple cell modules

One of the wonders of the Kazhdan-Lusztig basis is:

Theorem (Kazhdan-Lusztig, Du)

For the field $K := \mathbb{Q}(v)$ and the extensions of scalars $K\mathcal{H}_W(A, v) = \mathcal{H}_W(K, v)$ and $K\mathcal{S}_q(n, r)$ we have:

$$K\mathrm{LC}^{(\Lambda)} \text{ is a simple module for every left cell } \Lambda.$$

This gives filtrations of $K\mathcal{H}$ and $K\mathcal{S}$ by simple modules.

Theorem (Dipper-James)

$\mathcal{H}_W(K, v)$ and $K\mathcal{S}_q(n, r)$ are semisimple.
In fact, $\mathcal{H}_W(\mathbb{F}, u)$ is semisimple unless u is an e-th root of unity with $e \leq r$ (and likewise for $\mathcal{S}_q(n, r)$).

Trace forms and dual bases

Let

$$\tau(h) := \sum_{\chi \in \mathrm{Irr}(\mathcal{H}_W(K,v))} \frac{\chi(h)}{c_\chi}$$

for some elements $0 \neq c_\chi \in K$.

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If $h = \sum_{\underline{a} \in M(n,r)} \beta_{\underline{a}} \theta_{\underline{a}}$ then $\beta_{\underline{b}} = \tau(h \cdot \theta_{\underline{b}}^\vee)$ for all $\underline{b} \in M(n,r)$,

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and thus $f_{\underline{a},\underline{b},\underline{c}} = \tau(\theta_{\underline{a}}\, \theta_{\underline{b}}\, \theta_{\underline{c}}^\vee)$ for all $\underline{a}, \underline{b}, \underline{c} \in M(n,r)$.

The asymptotic algebra

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- $\mathcal{D}(n, r) := \{\underline{a} \in M(n, r) \mid \mathrm{ro}(\underline{a}) = \mathrm{co}(\underline{a}) \text{ and } \sigma(\underline{a}) \in \mathcal{D}\}$,

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Lusztig's conjectures **P**1 to **P**15

Lusztig formulates 15 "conjectures" **P**1 to **P**15:

- **P**2 If $\gamma_{x,y,d^{-1}} \neq 0$ with $d \in \mathcal{D}$, then $x = y^{-1}$.
- **P**3 For $y \in W$ exists a unique $d \in \mathcal{D}$ with $\gamma_{y^{-1},y,d^{-1}} \neq 0$.
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- **P**9 If $x \leq_L y$ and $\mathbf{a}(x) = \mathbf{a}(y)$, then $x \sim_L y$.
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These are proved for $\mathcal{H}_W(A, v)$ if

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For other Iwahori-Hecke algebras they are conjectures.

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We prove for $\mathcal{S}_q(n, r)$ statements **Q**1 to **Q**15: Setting

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Proofs use **P**1 to **P**15 and some additional q-Schur algebra arguments.

An explicit Wedderburn basis

Let Λ be a left cell such that $\mathrm{LC}^{(\Lambda)}$ has character ψ and

$$\tau(h) := \sum_{\chi \in \mathrm{Irr}(K\mathcal{S}_q(n,r))} \frac{\chi(h)}{c_\chi}$$

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Then the representing matrix of $h \in \mathcal{S}_q(n,r)$ on $\mathrm{LC}^{(\Lambda)}$ is

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Use Frobenius-Schur relations for $K\mathcal{S}_q(n,r)$: Fix $\underline{a}, \underline{b} \in \Lambda$,

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acts on $\mathrm{LC}^{(\Lambda)}$ as a matrix with one entry 1 and 0 elsewhere.

An explicit Wedderburn basis II

Theorem (Wedderburn basis (Brunat, N., 2008/2010))

The set

$$\mathcal{B} := \left\{ c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee} \mid \underline{c} \in M(n,r), \underline{d} \in \mathcal{D}(n,r), \underline{c} \sim_L \underline{d} \right\}$$

is a Wedderburn basis *of* $K\mathcal{S}_q(n,r)$.
For $c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee}$ *and* $c_{\underline{d}'}^{-1}\theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee}$ *in* $\mathcal{B}$ *we have:*

$$(c_{\underline{d}}^{-1}\theta_{\underline{c}}\theta_{\underline{d}}^{\vee}) \cdot (c_{\underline{d}'}^{-1}\theta_{\underline{c}'}\theta_{\underline{d}'}^{\vee}) = \begin{cases} 0 & \textit{if } \mathrm{LC}^{(\underline{d})} \ncong \mathrm{LC}^{(\underline{d}')} \\ 0 & \textit{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}')} \textit{ and } \underline{d} \not\sim_R \underline{c}' \\ c_{\underline{d}'}^{-1}\theta_{\underline{c}''}\theta_{\underline{d}'}^{\vee} & \textit{if } \mathrm{LC}^{(\underline{d})} \cong \mathrm{LC}^{(\underline{d}')} \textit{ and } \underline{d} \sim_R \underline{c}' \end{cases}$$

$\underline{c}''$ *is the* unique element *with* $\underline{c}'' \sim_L \underline{d}'$ *and* $\underline{c}'' \sim_R \underline{c}$ *and such a* $\underline{c}''$ *in fact exists.*

The dual basis of $\mathcal{B}$

These relations immediately imply: The dual basis $\mathcal{B}^\vee$ of

$$\mathcal{B} = \left\{ c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \mid \underline{c} \in M(n,r), \underline{d} \in \mathcal{D}(n,r), \underline{c} \sim_L \underline{d} \right\}$$

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In fact: $\left(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \right)^\vee = \theta_{\underline{c}^t} \theta_{\underline{d}'}^\vee$ where $\underline{c}^t \sim_L \underline{d}' \in \mathcal{D}(n,r)$.

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Note:

$$\left\langle (\theta_{\underline{a}})_{\underline{a} \in M(n,r)} \right\rangle_A = \mathcal{S}_q(n,r) \subseteq \langle \mathcal{B} \rangle_A$$

and

$$\left\langle \mathcal{B}^\vee \right\rangle_A \subseteq \left\langle (\theta_{\underline{a}}^\vee)_{\underline{a} \in M(n,r)} \right\rangle_A$$

not depending on the choice of τ!

Preimages of the $t_{\underline{a}}$

Since the formula for the Du-Lusztig homomorphism is

$$\Phi(h) = \sum_{\substack{\underline{b} \in M(n,r) \\ \underline{d'} \in \mathcal{D}(n,r) \\ \underline{d'} \sim_L \underline{b}}} \tau(h \cdot \theta_{\underline{d'}} \theta^{\vee}_{\underline{b}}) \cdot t_{\underline{b}} \quad \text{for } h \in K\mathcal{S}_q(n,r)$$

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we can use $\mathbf{Q}1$ to $\mathbf{Q}15$ and our theorem to show:

Theorem (Preimages of the $t_{\underline{c}}$ (Brunat, N., 2008/2010))

Let τ be an arbitrary non-degenerate symmetrising trace form on $K\mathcal{S}_q(n,r)$, then

$$\Phi(c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee) = t_{\underline{c}} \quad \textit{for all } \underline{c} \in M(n,r),$$

where $c_{\underline{d}}^{-1} \theta_{\underline{c}} \theta_{\underline{d}}^\vee \in \mathcal{B}$, that is $\underline{c} \sim_L \underline{d} \in \mathcal{D}(n,r)$.

Lusztig's homomorphism revisited

In view of our Wedderburn basis, we have for $A = \mathbb{Z}[v, v^{-1}]$ and $K = \mathbb{Q}(v)$ and $\mathcal{J}(n,r)_K := K\mathcal{J}(n,r)_A$:

$$\begin{tikzcd}
\mathcal{S}_q(n,r) \arrow[r] \arrow[d, equal] & \langle \mathcal{B} \rangle_A \arrow[r] \arrow[d, "\Phi"', "\cong"] & K\mathcal{S}_q(n,r) \arrow[d, "\Phi"', "\cong"] \\
\mathcal{S}_q(n,r) \arrow[r, "\Phi"] & \mathcal{J}(n,r)_A \arrow[r] & \mathcal{J}(n,r)_K
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Furthermore, we get that $\mathcal{J}(n,r)_A \cong \langle\mathcal{B}\rangle_A$ is isomorphic to a direct sum of full matrix rings over A.

James' conjecture ...

Let $s := |M(n, r)|$ and $M = (m_{\underline{a},\underline{b}}) \in A^{s \times s}$ be:

$$\Phi(\theta_{\underline{a}}) = \sum_{\underline{b} \in M(n,r)} m_{\underline{a},\underline{b}} \cdot t_{\underline{b}}$$

for all $\underline{b} \in M(n, r)$.

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Let $\mathbb{F}_\ell$ be a finite prime field, $u \in \mathbb{F}_\ell$ of order $2e$, and

$$\begin{tikzcd} A = \mathbb{Z}[v, v^{-1}] \arrow[rr, "\varphi_e"] \arrow[dr, "\varphi_\ell"'] & & \mathbb{Z}[\zeta_{2e}] \arrow[dl, "\varphi_\ell^e"] \\ & \mathbb{F}_\ell & \end{tikzcd}$$

be a commutative diagram of ring homomorphisms $(\zeta_{2e} \in \sqrt[2e]{1} \subseteq \mathbb{C}$ primitive) with $\varphi_e(v) = \zeta_{2e}$ and $\varphi_\ell(v) = u$.

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We want to compare the representation theory of

$$K\mathcal{S}_q(n,r) \quad \text{and} \quad \mathbb{Q}(\zeta_{2e})\mathcal{S}_q(n,r) \quad \text{and} \quad \mathbb{F}_\ell\mathcal{S}_q(n,r).$$

. . . in a reformulation by Geck

Let $\varphi_\ell = \varphi_\ell^e \circ \varphi_e$ as above.

Theorem (Geck)

James' conjecture for q-Schur algebras is equivalent to the fact that for $\ell > r$ we have

$$rank_{\mathbb{Q}(\zeta_{2e})}(\varphi_e(M)) = rank_{\mathbb{F}_\ell}(\varphi_\ell(M)),$$

where $\varphi_e(M) = (\varphi_e(m_{\underline{a},\underline{b}}))$ and $\varphi_\ell(M) = (\varphi_\ell(m_{\underline{a},\underline{b}}))$.

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That is, the rank of the matrix M when specialised to $\mathbb{Q}(\zeta_{2e})$ is the same as when specialised to $\mathbb{F}_\ell$.

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That is, the rank of the matrix M when specialised to $\mathbb{Q}(\zeta_{2e})$ is the same as when specialised to $\mathbb{F}_\ell$.

By our results, M is the base change matrix between

$$(\theta_{\underline{a}})_{\underline{a}\in M(n,r)} \quad \text{and} \quad \mathcal{B} = \{c_{\underline{d}}^{-1}\theta_{\underline{c}}\,\theta_{\underline{d}}^{\vee}\},$$

all within $K\mathcal{S}_q(n,r)$!

The End

A potential attack?

Let $Q_\tau = (q_{\underline{a},\underline{b}})$ be the base change from $(\theta_{\underline{a}}^\vee)$ to $(\theta_{\underline{a}})$

$$\theta_{\underline{a}}^\vee = \sum_{\underline{b}\in M(n,r)} q_{\underline{a},\underline{b}} \cdot \theta_{\underline{b}} \quad \text{for all } \underline{a} \in M(n,r),$$

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and D be the one from $\mathcal{B}^\vee$ to $\mathcal{B}$, which is monomial, then:

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$$D = M^t \cdot Q_\tau \cdot M,$$

since M^t is the base change from $\mathcal{B}^\vee$ to $(\theta_{\underline{a}}^\vee)$.

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since M^t is the base change from $\mathcal{B}^\vee$ to $(\theta^\vee_{\underline{a}})$.

If, for some φ_e and φ_ℓ, we could find a nice τ, such that

- the elements c_χ all lie in A,
- $Q_\tau \in A^{s \times s}$, and
- the number of c_χ that vanish under φ_e is equal to the number of c_χ that vanish under φ_ℓ,

then James' conjecture would follow for φ_e and φ_ℓ.