What is mathematics? — Some answers by way of example

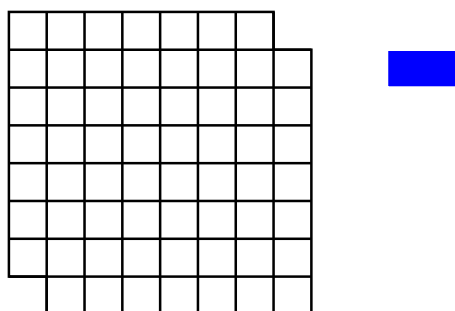Max Neunhöffer

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Contents

1 What is Mathematics?

Introduce myself. Ask audience.
I cannot answer this satisfactorily. First answer: "Fun". But I am a mathematician, so what I do, is maths. I show you some.
Give short overview, explain participation, try to "experience maths" first hand.

2 Playing Domino

Imagine we have an 8×8-grid with square fields in which two opposite corners are removed. This has 62 fields. Assume we have 31 dominoes, each exactly as big as two of the square fields.

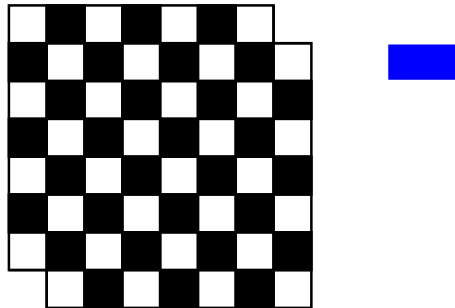**Is it possible to put all of them on the board to cover it completely?**

Show a try, get stuck.

Distribute square paper, let them try, about 3 minutes

How can we answer this question? *Short discussion, suggest trying, computers, one possible answer would be a solution.*

What if it is not possible?

Proof: Color every second of the fields in black:

There are 32 white fields and 30 black fields. However, every domino will always occupy a black field and a white field, regardless on how it lies on this grid. Therefore, 31 dominoes occupying the complete grid would fill 31 white and 31 black fields. This is a contradiction, so there is no way to completely fill the board with the 31 dominoes.

This was Mathematics. What did we do?

- We had a problem, it was clearly posed.

- We did not know the answer or any method of finding it.

- We tried things . . .

- Eventually we **got an idea**!

- With this idea, we got a completely satisfactory answer settling the question **once and for all**. This is because we **proved the answer "No"**.

3   Weighing Balls

3.1   Nine balls, one heavier

Assume we are given 9 balls of the same size, and a balance:
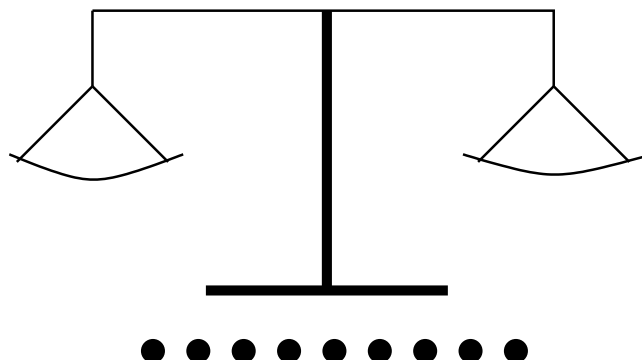
Furthermore, assume that all balls have the same weight, except for one of them, which is slightly heavier.

Using the balance, how many weighings do we need to find out which one is heavier?

Let them try (3–5 minutes), maybe discussion.

Answer: 2 weighings.

Proof: Number the balls from 1 to 9. First weigh $\{1, 2, 3\}$ against $\{4, 5, 6\}$, if one side is heavier (say $\{1, 2, 3\}$), we know the heavier one is amongst $\{1, 2, 3\}$, we weigh for example 1 against 2 and either find the heavier one or conclude it is 3. If $\{1, 2, 3\}$ weigh the same as $\{4, 5, 6\}$ we know that the heavier one is amongst $\{7, 8, 9\}$ and proceed in the same way.

Could we have handled 10 **balls with** 2 **weighings?**

Try discussion.

Answer/proof: No! With 2 weighings we only see 9 different outcomes, so we cannot distinguish 10 different cases!

3.2 Twelve balls, one heavier or lighter

Assume we are now given 12 balls, the balance, and one of them is either a bit heavier or a bit lighter.

How many weighings do we need to find out which one it is and whether it is heavier or lighter?

There are 24 possibilities to distinguish. Can you do it in 3 weighings? (which would give you $3^3 = 27$ different outcomes)

Let them try 3–5 minutes.

Number the balls from 1 to 12. Is weighing $\{1, 2, 3\}$ against $\{4, 5, 6\}$ any good? No, since if it is even, then there remain 12 cases which cannot be distinguished with the remaining 2 weighings.

Likewise, 5 balls against 5 is no good (10 remaining possibilities if one side is heavier).

So we start by doing $\{1, 2, 3, 4\}$ against $\{5, 6, 7, 8\}$.

If it is even, then we know that balls 1 to 8 are normal. We then check $\{1, 2, 3\}$ against $\{9, 10, 11\}$. If it is even, then 12 is the odd one out, we compare it to 1. If $\{1, 2, 3\}$ is heavier, then one of $\{9, 10, 11\}$ is lighter. We find it by comparing 9 with 10. Similarly, if $\{1, 2, 3\}$ is heavier.

If the balance in the first weighing is not even, then one of $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ is heavier. Assume the first. If $\{1, 2, 3, 4\}$ is heavier than $\{5, 6, 7, 8\}$, then either one of $\{1, 2, 3, 4\}$ is heavier or one of $\{5, 6, 7, 8\}$ is lighter, in particular all of $\{9, 10, 11, 12\}$ are normal. Then move $\{1, 2, 3\}$ to the other side and weigh $\{9, 10, 11, 4\}$ against $\{1, 2, 3, 8\}$. If the first is heavier, then 4 must be heavier or 8 lighter, we find out with one more try. If $\{1, 2, 3, 8\}$ is heavier, then one of $\{1, 2, 3\}$ is heavier, we find it with one more try. If $\{9, 10, 11, 4\}$ and $\{1, 2, 3, 8\}$ weigh the same, then all balls 1 to 4 and 8 to 12 are normal, so one of $\{5, 6, 7\}$ must be lighter, we find out which one with one more weighing.

This was already quite complicated, and a similar argument as above shows, that we cannot do 13 balls with 3 weighings, although these are only 26 possibilities.
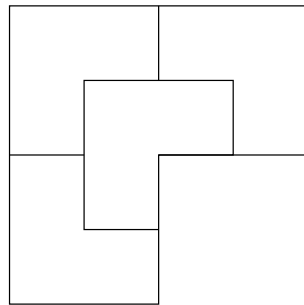
4 Subdividing Shapes

Definition: Two shapes in the plane are called *congruent* if one fits exactly on top of the other.
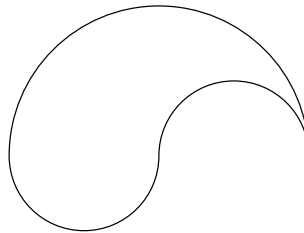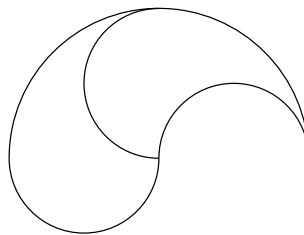
4.1 L
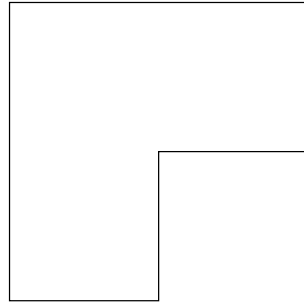
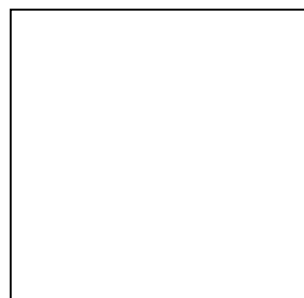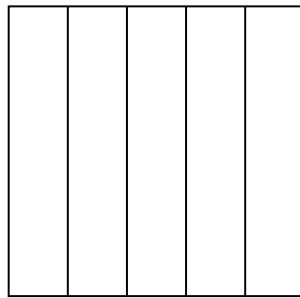Can you subdivide this shape into 4 pairwise congruent shapes?

4.2 Drop

Can you subdivide this shape into 2 pairwise congruent shapes?

4.3 Square

Can you subdivide this shape into 5 pairwise congruent shapes?

5 Rewriting Words, the MIU language

Here we play with words, the only letters allowed are M, I and U.

5.1 Rules and first problem

In the beginning we have only the word MI.
We are allowed to form new words from the words we already have by the following simple rules:

1. If a word ends with an I, then we may append a U to get a new word.

   Example: MI → MIU.

2. If a word starts with an M, then we may double everything after it.

   Example: MIU → MIUIU.

3. Any subword III may be replaced by U.

   Example: MIIII → MUI

4. Any subword UU may be removed.

   Example: MUUII → MII

Question: Given only MI in the beginning, can we make MU by a sequence of these steps?
Let them try for 5 minutes

5.2 Solution and philosophy

We are looking for "invariants", that is, properties of the words which are kept as we make new words.
Observation 1: Every word we ever make starts with an M.
Proof: MI which is given to us does. Whenever a word starts with an M and we apply any of the rules 1. to 4., then the result will also start with an M.

Observation 2: The number of times the letter I occurs in any of our words is never divisible by 3.
Proof: MI has one I, and 1 is not divisible by 3. Now consider our rules:
Applying rule 1. or 4. does not change the number of times it contains the letter I.
If a word Mx contains k times the letter I, then after applying rule 2., the word Mxx contains $2k$ times the letter I. If k is not divisible by 3, then $2k$ is also not divisible by 3.
If we apply rule 3., we change three Is to one U, so if the number of Is was not divisible by 3, then it will not be divisible by 3 in the new word.
However, MU does contain 0 times the letter I, which is divisible by 3. Therefore, it cannot be manufactured from MI.
Some philosophical remarks:

Imagine the language of mathematics would be the language of all words consisting of the letters M, I and U. We would have "true" statements and "false" statements. If we **assume** that MI is a true statement and that "proving" one statement from another would be applying the rules 1. to 4., then the set of words we can build is the set of statements that we can prove under the assumption that MI is a true statement. So this game is a very much simplified model of mathematics. In particular it shows what we do in mathematics (assume some statements to be true, and logically conclude other statements to be true by some fixed set of rules, i.e. proving other statements). It also shows the possibility of true statements which cannot be proved from the assumptions we make!

6   Now, was this Mathematics?

All of this is maths. I hope it was fun for you.
Let me highlight some features of what we did:

- Mathematics is a language.

- Numbers are only a tiny part of it!

- We make certain assumptions, that is, we assume certain statements to be true (our axioms).

- We clearly state and formulate problems and questions.

- We usually do not know the answers to these questions in the beginning, neither do we have a simple way to find these answers.

- We try things, thoughts, methods, tricks . . .

- We get an **idea**.

- We try to find out whether or not it works, most of the time it does not work, sometimes it does.

- We prove statements to answer and settle our questions once and for all without leaving any doubt.

This is Mathematics and this is what we teach here.