Computing the 2-modular characters of Fi_{23}

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Modular representations

G finite group, F field, $p := \mathrm{char}(F) \mid |G|$, V an F-vector space

- A modular representation of G on V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$.
- G acts on V via ρ (V is an FG-module).
- ρ is called irreducible if there is no proper G-invariant subspace $0 \neq U \subsetneq V$.

Aim: Classification of the irreducible modular representations of the sporadic simple groups.

Example ($Fi_{23} \mod 2$)

$G = Fi_{23}$ and $p = 2$, $|G| = 4.089.470.473.293.004.800$
(joint work with G. Hiß and F. Noeske).

Finding a composition series

Let $0 \neq U \subsetneq V$ be G-invariant.

- We get two new representations:

$$\rho_U : G \to \mathrm{GL}(U), \quad \rho_{V/U} : G \to \mathrm{GL}(V/U).$$

- Iteration gives as „atoms“ $\rho_{S_i} : G \to \mathrm{GL}(S_i)$ on the composition factors S_i of V.
- Package MeatAxe [Parker, Thackray 1978,...] computes a composition series automatically (`chop`).

Example ($Fi_{23} \mod 2$)

Permutation module $1_{2.Fi_{22}}^{Fi_{23}}$ of dimension 31.671 contains composition factors $1a, 782a, 1.494a, 3.588a, 19.940a$.
about 4 days of CPU time in 8 GB main memory.

Construction of modular representations

How can we construct new representations from given ones?

Theorem (Burnside-Brauer)

G simple, V non-trivial irreducible FG-module. For every irreducible FG-module W there is an $m \in \mathbb{N}$ such that

$$W \text{ is a composition factor of } V^{\otimes m}.$$

OK, then we are done!? NO!

Example (Fi_{23} mod 2)

$\dim_F 19.940a \otimes 19.940a = 367.603.600$
One GF(2)-matrix $\approx$ 18.403.938 GB $\approx 17,5$ PB.

Condensation in theory. . . [Green 1980]

Let $e = e^2 \in FG$ an idempotent. Consider $V = Ve \oplus V(1-e)$.

Definition (Schur functor)

$$\begin{array}{rcl} \mathcal{F} : \mathrm{mod}-FG & \rightarrow & \mathrm{mod}-eFGe \\ V & \mapsto & Ve \\ \phi \in \mathrm{Hom}_{FG}(V,W) & \mapsto & \phi|_{Ve} \in \mathrm{Hom}_{eFGe}(Ve,We) \end{array}$$

- $\mathcal{F}$ is exact.
- If V is an irreducible FG-Modul, then Ve is irreducible or $Ve = 0$.

I.e. $\mathcal{F}$ maps a composition series onto a composition series.

- If $Ve \neq 0$ for all irreducible FG-modules V, then $eFGe$ and FG are Morita equivalent.

... and practice [Thackray 1981]

Let $K \leq G$ such that p does not divide $|K|$. We choose

$$e := \frac{1}{|K|} \sum_{k \in K} k \in FK \leq FG.$$

Task: Given $g \in G$, determine action of ege on $(V \otimes W)e$.

Without explicit computation of $V \otimes W$!

Theorem (Lux, Wiegelmann 1997)

This can be done!

... and practice (2)

Example (Fi_{23} mod 2)

- $K \leq G$, $|K| = 3^9 = 19.683$.
- $eFGe$ and FG are Morita equivalent.
- $\dim_F(19.940a \otimes 19.940a)e = 25.542$.

  One GF(2) matrix $\approx$ 77,8 MB.

  About 1 week of CPU time to compute the operation of one element ege on $(19.940a \otimes 19.940a)e$.

But now we are done, aren't we? Unfortunately not.

The Generation Problem

Remember: We investigate Ve by giving matrices for generators of $eFGe$.

Question (The Generation Problem)

How can eFGe be generated by "a few" elements?

If $\mathcal{E} \subseteq FG$ with $\langle \mathcal{E} \rangle = FG$. Then $\langle e\mathcal{E}e \rangle = eFGe$ does not follow!

- Let $\mathcal{C} := \langle e\mathcal{E}e \rangle \leq eFGe$.
  Instead of Ve we consider the $\mathcal{C}$-module $Ve|_{\mathcal{C}}$.
- Contrary to Ve we can not directly conclude things from $Ve|_{\mathcal{C}}$ to V.

Generation

Let $K \trianglelefteq N \leq G$.

Theorem (F. Noeske, 2005)

If $\mathcal{T}$ is a set of double coset representatives of $N \backslash G / N$ and $\mathcal{N}$ a set of generators of N, then we have

$$eFGe = \langle e\mathcal{N}e, e\mathcal{T}e \rangle .$$

Example (Fi_{23} mod 2)

- N the 7th maximal subgroup, $[G : N] = 1.252.451.200$
- $|\mathcal{T}| = 36$ and $|\mathcal{N}| = 3$, i.e. 38 generators for $eFGe$.
- Computation of $\mathcal{T}$: see second half of this talk.

Brauer Characters

Let $G_{p'}$ be the set of p-regular elements of G.

- To each modular representation on V we can assign a class function β_V on $G_{p'}$ (Brauer character).

Aim

Determine the irreducible Brauer characters $\varphi_1, \ldots, \varphi_\ell$.

- β_V is a $\mathbb{Z}_{\geq 0}$-linear combination of the $\varphi_1, \ldots, \varphi_\ell$. This decomposition corresponds to the decomposition of V into its composition factors.
- $\beta_{V \otimes W}(g) = \beta_V(g) \cdot \beta_W(g)$ for all $g \in G_{p'}$.
- We know explicitly Brauer characters $\vartheta_1, \ldots, \vartheta_\ell$ such that

$$\langle \vartheta_1, \ldots, \vartheta_\ell \rangle_{\mathbb{Z}} = \langle \varphi_1, \ldots, \varphi_\ell \rangle_{\mathbb{Z}} .$$

Equations in Brauer characters

Question: How to determine $\varphi_1, \ldots, \varphi_\ell$ from the $\vartheta_1, \ldots, \vartheta_\ell$?

- We have

$$\vartheta_i = \sum_{j=1}^{\ell} a_{ij}\,\varphi_j, \qquad a_{ij} \in \mathbb{Z}_{\geq 0}.$$

- Because of $\langle \vartheta_1, \ldots, \vartheta_\ell \rangle_{\mathbb{Z}} = \langle \varphi_1, \ldots, \varphi_\ell \rangle_{\mathbb{Z}}$ it follows that

$$\begin{aligned} \beta_{V \otimes W} &= \sum_{i=1}^{\ell} t_i\,\vartheta_i, \quad t_i \in \mathbb{Z} && \text{(GAP)} \\ &= \sum_{j=1}^{\ell} s_j\,\varphi_j, \quad s_j \in \mathbb{Z}_{\geq 0} && \text{(condens.\&MEATAXE)} \end{aligned}$$

The Result

Solution of

System of Equations

$$\sum_{i=1}^{\ell} a_{ij}\, t_i = s_j, \qquad j = 1, \ldots, \ell$$

with GAP gives the 2-modular characters.

Example (Degrees in the principal block of Fi_{23} mod 2)

1,	782,	1.494,	3.588,	19.940,
57.408,	94.588,	94.588,	79.442,	583.440,
1.951.872,	724.776,	979.132,	1.997.872,	1.997.872,
7.821.240,	8.280.208,	5.812.860,	17.276.520,	34.744.192.

Problem

$G := \mathrm{Fi}_{23} = \langle a, b\rangle$ with $|G| = 4.089.470.473.293.004.800$,

$N = \langle n_1, n_2, n_3\rangle \leq G$ with $|N| = 3.265.173.504$,
where the n_i are given as words in a and b.

Known: $G = Ng_1N \,\dot\cup\, Ng_2N \,\dot\cup\, \cdots \,\dot\cup\, Ng_{36}N$,
where $NgN = \{n \cdot g \cdot n' \mid n, n' \in N\}$.

Problem: Find $\{g_1, \ldots, g_{36}\}$ as words in a and b.

Application: Let $K \lhd N$ and $F := GF(2)$ and $2 \not| \, |K|$.

Then we have for $e^2 = e := \frac{1}{|K|}\sum_{k\in K} k \in FG$:

$$eFGe = \langle eg_1e, \ldots, eg_{36}e, en_1e, en_2e, en_3e\rangle_{F-\mathrm{Alg.}}$$

(F. Noeske, 2005)

Double cosets and suborbits

$$G = Ng_1N \,\dot\cup\, Ng_2N \,\dot\cup\, \cdots \,\dot\cup\, Ng_{36}N$$

G acts on the set $N\backslash G := \{Ng \mid g \in G\}$

$$N\backslash G = Ng_1 \cdot N \,\dot\cup\, \cdots \,\dot\cup\, Ng_{36} \cdot N$$

Thus: double cosets $\leftrightarrow$ suborbits

Problems:

- $|N\backslash G| = 1.252.451.200 \approx 1.25 \cdot 10^9$
- Permutations for a, b, n_1, n_2, n_3 would need about 5 GB
- Not easy to determine
- G is given as a permutation group on 31.671 points
- To determine elements g_i would take too long

Realisation of the action on $N\backslash G$

Linear representation of G on $V := F^{1\times 1494}$ ($F = GF(2)$):

$$\text{group homomorphism} \quad \rho : G \to \mathrm{GL}_{1494}(F)$$

Find vector $v_1 \in V$ such that $v_1 \cdot \rho(N) = \{v_1\}$.

$\Longrightarrow$ The orbit $v_1 \cdot \rho(G)$ is isomorphic to $N\backslash G$ as G-set.

Thus we can:

- Store and compare points of $N\backslash G$ as vectors in V
- Act with group elements (words in a, b) on them

Still too large:

- Each vector needs about 200 bytes ($\approx 1494/8$)
- Altogether about 250 GB (main memory!)
- It takes uncomfortably long to enumerate all vectors!

A Trick

Let $U < N$ with $|U| = 6561$.

Idea: do things "by U-orbits":

- N-orbits are unions of U-orbits
- enumerate U-orbits

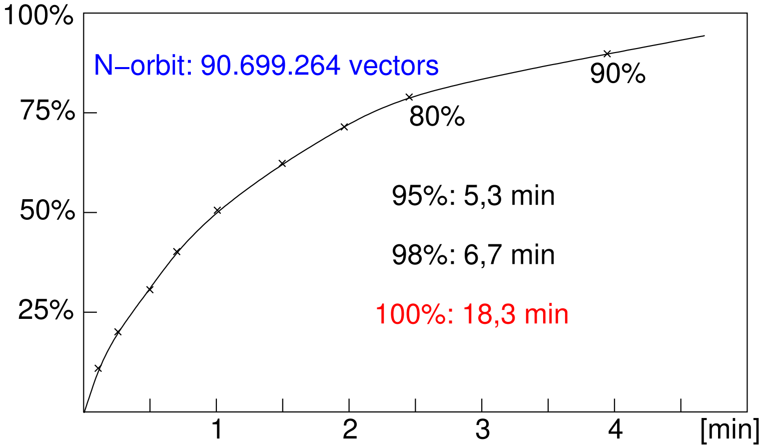
To this end:

- choose in each U-orbit B a subset $\min(B) \subseteq B$ ("U-minimal" vectors), such that
- we have an algorithm, that computes, given an U-orbit B and a $v \in B$, a $u(v) \in U$ such that $v \cdot \rho(u(v)) \in \min(B)$.
- store B by storing $\min(B)$

$\Longrightarrow$ Save about a factor of 250 of memory and time!

Still tedious

Progression of an N-orbit enumeration by U-orbits:



Halves of orbits

We enumerate only one half of each N-orbit!

Suppose we know $H \subseteq v \cdot \rho(N)$ with $|H| > |v \cdot \rho(N)|/2$.

Question: Does $w \in V$ lie in the orbit $v \cdot \rho(N)$?

Answer:
Apply random elements $\{m_1, \ldots, m_{40}\}$ of N to w and test whether $\{w \cdot \rho(m_i) \mid 1 \leq i \leq 40\} \cap H \neq \emptyset$.

If yes: w lies in $v \cdot \rho(N)$ with certainty

If no: w probably does not lie in $v \cdot \rho(N)$

$\Longrightarrow$ Enough to find different N-orbits in $v_1 \cdot \rho(G)$.

The first 35 are now doable

N-suborbit lengths in $v_1 \cdot \rho(G)$:

1	10.077.696	20.155.392	3.888
22.674.816	90.699.264	5.038.848	78.732
34.012.224	10.077.696	3.359.232	19.683
272.097.792	68.024.448	5.038.848	186.624
90.699.264	136.048.896	7.558.272	62.208
90.699.264	30.233.088	3.779.136	124.416
272.097.792	30.233.088	3.779.136	15.552
10.077.696	944.784	1.679.616	15.552
944.784	30.233.088	1.679.616	768

Apply orbit enumeration to $v_1 \cdot \rho(G)$, do it by N-orbits.

We find all N-orbits, except for the one with 768 vectors.

Not yet proved, that these 35 orbits are pairwise disjoint!

Criminal search

We are looking for one of 768 vectors v with the following properties:

- $v \in v_1 \cdot \rho(G)$
- $S := Stab_N(v) < N$ has index 768 in N

Approach:

- guess $S < N$ with $[N : S] = 768$ (not unique!)
- compute candidates $C := \{v \in V \mid v \cdot \rho(S) = \{v\}\}$
- check for all $v \in C$ whether $v \cdot \rho(a)$ lies in one of the 35 (half) known N-orbits (remember: $G = \langle a, b \rangle$)
- if yes, it follows that $v \in v_1 \cdot \rho(G)$ (proven!).

  If v itself is not in one of the 35 orbits, then we are ready.
- $\Longrightarrow$ produces in fact vector v_{36}

The Last Representative

We still need a word g_{36} in a and b, that maps v_1 to v_{36}!

Do the following:

- enumerate vectors in $v_1 \cdot \rho(G)$ with a “breadth first” search by applying a and b until the memory is full
- search backwards starting with v_{36} for a “known” vector using a “depth first” search (apply a^{-1} and b^{-1})
- put forward and backward search together
- $\Longrightarrow$ finds word within a few minutes
- possible improvement: “by U-orbits”
  (here not necessary)

Verification

For candidates for $g_1, \ldots, g_{36}$:

Enumerate all N-suborbits $v_1 \cdot \rho(g_i) \cdot \rho(N)$ completely.

Needs about 3 h on a machine with 4 GB main memory.

$\Longrightarrow$ Proven:
All N-orbits lie in $v_1 \cdot \rho(G)$ and are pairwise disjoint.

Remark: There is still a lot of potential for improvements here:

- One only has to compare orbits of equal length.
- To test whether two orbits are disjoint, only one has to be enumerated.
- So only one orbit has to fit into main memory at a time.

Overview and 2-modular character table of Fi_{23}

We have

- constructed a permutation representation of Fi_{23} on 1.252.451.200 points,
- enumerated all 36 N-suborbits and determined their lengths,
- found N-N-double coset representatives $g_1, \ldots, g_{36}$ as words in a, b,
- fulfilled the requirements for the condensation computations and
- computed the 2-modular character table of Fi_{23} (joint work with Gerhard Hiß and Felix Noeske).

Outlook

These enumeration methods can be applied if

- there is an appropriate (small) linear representation,
- there is an appropriate vector v_1 (or something similar),
- one (or a few) appropriate helper subgroups can be found,
- one needs large orbits or double coset representatives.

The condensation methods can be applied if

- there is an appropriate condensation subgroup K,
- things are “small enough” with resp. to memory and time,
- the generation problem can be solved by using Noeske’s Theorem and determining double coset representatives.

All this is implemented in GAP and will be published as a package soon.