School of Mathematics and Statistics
MT4517 Rings & Fields
Exercises 2

Exercise 2.1. Let R be any ring. Use the Ring Axioms to prove that $0 * a = 0 = a * 0$ for any $a \in R$.

If R has a multiplicative identity 1, prove that $(-1) * (-1) = 1$.

Exercise 2.2. Let $\mathbb{R}$ denote the real numbers and R be the set of mappings $f : \mathbb{R} \to \mathbb{R}$. Define $+$ and $*$ on R by

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (f*g)(x) &= f(x) * g(x) \end{aligned}$$

for all $x \in \mathbb{R}$. Prove that R is a commutative ring with identity. Which elements of R have multiplicative inverses? Does R have zero divisors? Is R a ring when multiplication $*$ is defined to be composition of mappings?

Exercise 2.3. Let R be a set with operations $+$ and $*$ satisfying **A2** and **M2**. Prove that if $1 = 0$ in R, then R is a ring with one element.

Exercise 2.4. Let R be a ring satisfying **M2** and **M3**. Prove that R has no zero divisors.

Exercise 2.5. Let $R = \mathbb{R} \times \mathbb{R} = \{ (a, b) \ : \ a, b \in \mathbb{R} \}$. Define $+$ and $*$ by

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac, bd).$$

Prove that R is a commutative ring with identity. Find the elements of R that have a multiplicative inverse.

Exercise 2.6. Prove that the set $R = \{ a + b\sqrt{2} \ : \ a, b \in \mathbb{Z} \}$ is a ring under real addition and multiplication. Find the multiplicative inverses of the elements $1 + \sqrt{2}$ and $3 + 2\sqrt{2}$.

Show that $a + b\sqrt{2} \in R$ is a unit if and only if $a^2 - 2b^2 = 1$. Deduce that R is not a division ring and hence not a field.

Exercise 2.7. Let M be the set of matrices

$$\left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \ : \ a, b \in \mathbb{R} \right\}.$$

Show that M is a ring under the operations of matrix addition and multiplication. Is M a commutative ring? Does M have an identity?

Exercise 2.8. Let $\mathbb{H}$ denote the quaternions as defined in Example 2.6. From the multiplication table defined in Example 2.6, we deduce that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \tag{2.1}$$

Prove that it is possible to deduce the multiplication table from the equalities in (2.1).

Exercise 2.9. Let $x \in \mathbb{Z}/(n)$ with $x \neq 0$. Prove that x is a unit if and only if x is coprime to n. Deduce that $\mathbb{Z}/(n)$ is a division ring if and only if n is a prime.

Exercise 2.10. Let $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ be a non-zero element in the quaternions $\mathbb{H}$. Prove that

$$(a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k})\frac{1}{a^2 + b^2 + c^2 + d^2}$$

is the multiplicative inverse of $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. Deduce that $\mathbb{H}$ is a division ring.

Prove that $\mathbb{H}$ is not an integral domain.

Exercise 2.11. Prove the following statements:

(i) $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ are division rings;

(ii) $\mathbb{Z}$ is not a division ring;

(iii) the ring $M_2(\mathbb{R})$ is not a division ring.

Exercise 2.12. Let R be a ring and $a, b \in R \setminus \{0\}$ with $ab = 0$. Prove that neither a nor b is a unit.

Exercise 2.13. Prove that the set $C[0,1]$ of real-valued continuous functions on the interval $[0,1]$ is a ring under the operations $+$ and $*$. What are the one and zero in $C[0,1]$? Is $C[0,1]$ commutative, or an integral domain? Does $C[0,1]$ have zero divisors? What are the units in $C[0,1]$? Is $C[0,1]$ a division ring or a field?

Exercise 2.14. Let R be a ring and let $M_n(R)$ denote the set of $n \times n$ matrices with entries in R. Prove that $M_n(R)$ is a ring.

Exercise 2.15. Let R be a set with operations $+$ and $*$ satisfying the Ring Axioms except **A4** and including **M2**. Prove that **A4** holds. [Hint: Expand $(1 + 1) * (a + b)$ using **A2**, **A3** and **D** in two different ways.]

Exercise 2.16. Let R be a ring where $x^2 = x * x = x$ for all $x \in R$. Prove that R is a commutative ring where $2x = x + x = 0$ for all x.

Exercise 2.17. Is it true that $(x + y)^2 = x^2 + 2x * y + y^2$ in all rings?