School of Mathematics and Statistics
MT4517 Rings & Fields
Exercises 4

Exercise 4.1. Let R be the factor ring $\mathbb{F}_2[x]/(x^3+x+1)$. Find the multiplicative inverses of the elements $x, 1+x$, and $1+x^2$ in R.

Exercise 4.2. Let R be the factor ring $\mathbb{F}_3[x]/(x^2+1)$. Prove that R is a field.

Exercise 4.3. Let R be the factor ring $\mathbb{F}_2[x]/(x^2-1)$. Prove that R is not a field.

Exercise 4.4. Let R and S be rings and let $f : R \longrightarrow S$ be a homomorphism. Prove that the kernel of f is an ideal in R and that the image of f is a subring of S.

Exercise 4.5. Let $f(a+bi) = a-bi$ denote the conjugation map on the complex numbers $\mathbb{C}$. Prove that f is a homomorphism from $\mathbb{Z}[i]$ to $\mathbb{Z}[i]$.

Exercise 4.6. Let $\mathbb{R}^2$ denote the ring from Exercise 2.6 and let R denote the set of mappings $f : \{0,1\} \to \mathbb{R}$ with $+$ and $*$ as defined in Exercise 2.2. Prove that $\mathbb{R}^2$ and R are isomorphic.

Exercise 4.7. Prove Lemma 7.2 from lectures.

Exercise 4.8. Let $f : \mathbb{Z} \to M_2(\mathbb{R})$ be defined by

$$f(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Prove that f is a homomorphism.

Exercise 4.9. Prove that the map ϕ from $\mathbb{Z}/(12)$ to $\mathbb{Z}/(4)$ given by $n \mapsto n \mod 4$ for $n \in \mathbb{Z}/(12)$ is a ring homomorphism. What is its kernel? Is the map from $\mathbb{Z}/(14)$ to $\mathbb{Z}/(8)$ given by $n \mapsto n \mod 8$ for $n \in \mathbb{Z}/(14)$ a ring homomorphism?

Exercise 4.10. Which of the maps from $\mathbb{C}$ to $\mathbb{C}$ given by the following are ring homomorphisms

$$x+yi \mapsto x,\ x+yi \mapsto |x+yi|,\ x+yi \mapsto y+xi?$$

Exercise 4.11. Describe all the ring homomorphisms from $\mathbb{Z}/(12)$ onto $\mathbb{Z}/(4)$. Describe all the ring homomorphisms from $\mathbb{Z}/(12)$ to $\mathbb{Z}/(5)$. For which values of m, n is it possible to find a surjective ring homomorphism from $\mathbb{Z}/(m)$ onto $\mathbb{Z}/(n)$?

Exercise 4.12. Prove that up to isomorphism there are two rings of order 2.

Exercise 4.13. Recall that the *scalar product* or *dot product* of 2 vectors $\vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is defined by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and that the *vector product* or *cross product* is defined by

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Let $\phi : \mathbb{R}^3 \to \mathbb{H}$ be defined by

$$\phi(\vec{u}) = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k}.$$

Such quaternions with zero 'constant' are called *pure*. Show the relationship between the scalar and vector products in vector space $\mathbb{R}^3$ and the multiplication in the ring of quaternions $\mathbb{H}$.

Prove that the set $\mathbb{R}^3$ under vector addition and the vector product is not a ring.

Exercise 4.14. Let R denote the factor ring $\mathbb{Z}[i]/(1+3i)$. Show that $i - 3 \in (1+3i)$ and that $i + (1+3i) = 3 + (1+3i)$ in R. Use this to prove that $10 + (1+3i) = (1+3i)$ in R and that $a + bi + (1+3i) = a + 3b + (1+3i)$ where $a, b \in \mathbb{Z}$.

Exercise 4.15. Let R be a commutative ring and let I and J be ideals in R with $I \subseteq J$.

(i) Show that $\phi : R/I \longrightarrow R/J$ given by $\phi(x + I) = x + J$ is a well-defined surjective homomorphism.

(ii) Let $R = \mathbb{Z}[i]$, let $n \in \mathbb{Z} \setminus \{0\}$ be arbitrary, and let $I = Rn = \{\, rn : r \in R \,\}$. Prove that I is an ideal and that $a + bi \in I$ if and only if $n \mid a$ and $n \mid b$. Prove that R/I is a finite ring.

(iii) Let $J \neq \{0\}$ be an ideal in R. Prove that $J \cap \mathbb{Z}$ is an ideal in $\mathbb{Z}$ and that $J \cap \mathbb{Z} \neq \{0\}$.

(iv) Show that R/J is a finite ring.

Exercise 4.16. Prove that every ideal in the quotient ring R/I of a principal ideal domain R is principal. Give an example of a ring which is not a domain but for which every ideal is a principal ideal.