School of Mathematics and Statistics
MT4517 Rings & Fields
Exercises 5

Exercise 5.1. Prove that for all n every ideal of the ring $\mathbb{Z}/(n)$ is principal. Is $\mathbb{Z}/(n)$ a principal ideal domain? How would you determine the number of ideals of $\mathbb{Z}/(n)$?

Exercise 5.2. Let R be an integral domain and let $r \in R$. Prove that r is a unit if and only if $(r) = R$.

Exercise 5.3. Let R be an integral domain. Prove that $0 \in R$ is not an irreducible.
Also prove that if $u \in R$ is a unit and $r \in R$ is irreducible, then au is irreducible.

Exercise 5.4. Let $n \in \mathbb{Z}$ such that $n \neq m^2$ for all $m \in \mathbb{Z}$. Prove that the function $N : \mathbb{Z}[\sqrt{n}] \longrightarrow \mathbb{N}$ defined by

$$N(a + b\sqrt{n}) = a^2 - nb^2$$

satisfies

$$N(a + b\sqrt{n})N(c + d\sqrt{n}) = N((a + b\sqrt{n})(c + d\sqrt{n})).$$

Exercise 5.5. Let R be an integral domain and $R' = R \setminus \{0\}$. Is it true or false that:

(a) if $|$ is an equivalence relation on R', then R is a field;

(b) if $x \sim y$ for all $x, y \in R'$, then R is a field;

(c) if $a \sim b$ and $c \sim d$, then $ac \sim bd$;

(d) if $a \sim b$ and $c \sim d$, then $(a + c) \sim (b + d)$;

(e) if every element of R' is a unit or a prime, then R is a field?

Give a proof if the statement is true and a counter-example if it is false.

Exercise 5.6. Determine whether 17 is irreducible in each of

$$\mathbb{Z}, \mathbb{Z}[x], \mathbb{Z}[i], \mathbb{Z}[\sqrt{10}].$$

Exercise 5.7. Find two different factorisations of 10 in $\mathbb{Z}[\sqrt{-6}]$. Find an irreducible element of $\mathbb{Z}[\sqrt{-6}]$ that is not prime.

Exercise 5.8. Show that 8 can be written as both the product of 2 irreducibles and as the product of 3 irreducibles in $\mathbb{Z}[\sqrt{-7}]$.

Exercise 5.9. Let R be a ring and I an ideal of R such that R/I is an integral domain. Prove that I is a prime ideal (that is, prove the direct implication of Theorem 9.2).

Exercise 5.10. Let R be the ring with elements $\{\, a/b \in \mathbb{Q} \;:\; b \equiv 1$ or $2 \pmod{3} \,\}$ and the usual operations $+$ and $*$ in $\mathbb{Q}$ and let $I = \{\, a/b \in R \;:\; a \equiv 0 \pmod{3} \,\}$. Then

(i) prove that R is a subring of $\mathbb{Q}$;

(ii) prove that I is an ideal in R;

(iii) prove that R/I is a field.

Exercise 5.11. Prove that $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{3}]$ are not isomorphic rings.

Exercise 5.12. Let $\alpha \in \mathbb{Z}[i]$ be prime. Prove that precisely one of the following holds:

(i) $\alpha \sim 1 + i$;

(ii) $\alpha \sim p$ where $p \in \mathbb{Z}$ is prime and $p \equiv 3 \pmod{4}$;

(iii) α divides a prime $p \in \mathbb{Z}$ with $p \equiv 1 \pmod{4}$.

Exercise 5.13. Let R be a principal ideal domain and let I be an ideal of R. Prove that R/I is a principal ideal domain.

Find an example of a ring that is not an integral domain but where every ideal is principal.

Exercise 5.14. Let I and J be ideals and K be a prime ideal of a ring R. Prove that if $IJ \leqslant K$, then $I \leqslant K$ or $J \leqslant K$.

Exercise 5.15. Let R and S be rings and let $f : R \longrightarrow S$ be a homomorphism. Prove that $\ker(f)$ is a prime ideal if S is an integral domain. Prove that $\ker(f)$ is a maximal ideal if S is a field and f is surjective.