School of Mathematics and Statistics
MT4517 Rings & Fields
Exercises 6

Exercise 6.1. Prove that the ring $\mathbb{Z}[\sqrt{n}]$ is a factorization domain for all $n \in \mathbb{Z}$ with $n \neq m^2$ for all $m \in \mathbb{Z}$.

Exercise 6.2. Prove that $\mathbb{Z}[\sqrt{6}]$ is a unique factorization domain. Why does $\sqrt{6}\sqrt{6} = 2 * 3$ not violate unique factorization in $\mathbb{Z}[\sqrt{6}]$?

Exercise 6.3. Show that $\mathbb{Z}[\sqrt{10}]$ is not a unique factorization domain.

Exercise 6.4. Express as products of irreducibles:

(a) $4 + 7\sqrt{2}$ in $\mathbb{Z}[\sqrt{2}]$;

(b) $4 - \sqrt{-3}$ in $\mathbb{Z}[\sqrt{-3}]$;

(c) $5 + 3\sqrt{-7}$ in $\mathbb{Z}[\sqrt{-7}]$.

Exercise 6.5. In the ring $\mathbb{Z}[\sqrt{-5}]$, prove that

(i) the units are 1 and -1;

(ii) $3, 2 + \sqrt{-5}, 2 - \sqrt{-5}$ are irreducible;

(iii) 9 has two factorizations into a product of irreducibles;

(iv) the ideals $(3, 2 + \sqrt{-5})$ and $(3, 2 - \sqrt{-5})$ are prime.

Exercise 6.6. Is the r in the definition of a Euclidean function unique?

Exercise 6.7. Find four different $q, r \in \mathbb{Z}[i]$ such that $2 + i = (1 + i)q + r$ and $N(r) < N(1 + i)$.

Find $a+bi, c+di \in \mathbb{Z}[i]$ such that there exist only three, and then two, and then one, different $q, r \in \mathbb{Z}[i]$ such that $a + bi = (c + di)q + r$ and $N(r) < N(c + di)$.

Exercise 6.8. Let k be an arbitrary positive integer. Then, using Exercise 5.8 or otherwise, show that there is an element in $\mathbb{Z}[\sqrt{-7}]$ that can be written as the product of $2k, 2k + 1, \ldots, 3k$ irreducibles.

Exercise 6.9. Let $f : \mathbb{Z}[i] \longrightarrow \mathbb{Z}[i]$ be defined by $f(a + bi) = a - bi$. Prove that f is a homomorphism and that $f(a + bi)$ is prime in $\mathbb{Z}[i]$ if $a + bi$ is prime in $\mathbb{Z}[i]$.

Exercise 6.10. Let $\mathbb{Z}[\omega] = \{\, x + \omega y \; : \; x, y \in \mathbb{Z} \,\}$ where $\omega^2 + \omega + 1 = 0$, let $\alpha = x + \omega y \in \mathbb{Z}[\omega]$, and let $\bar{\alpha}$ denote the complex conjugate of α. Define $N(\alpha) = \alpha\bar{\alpha}$.

(i) Prove that $N(\alpha) = x^2 - xy + y^2$ and that $N(\alpha)N(\beta) = N(\alpha\beta)$ for all $\alpha, \beta \in \mathbb{Z}[\omega]$.

(ii) Prove that $\alpha \in \mathbb{Z}[\omega]$ is irreducible if $N(\alpha)$ is a prime number.

(iii) Prove that $\mathbb{Z}[\omega]$ is a euclidean ring.

(iv) Prove that $1 - \omega$ is a prime element in $\mathbb{Z}[\omega]$.

Exercise 6.11. Is $\mathbb{Z}[\sqrt{-3}]$ a Euclidean ring?

Exercise 6.12. Let $R = \mathbb{Z}[\sqrt{n}]$ where n is square-free. Then prove the following:

(i) if $n < -1$, then the units of R are -1 and 1;

(ii) if $n > 1$ and $|R^*| > 2$, then R^* is infinite;

(iii) if $n = 2$, then $R^* = \{\, \pm(1 \pm \sqrt{2})^k \; : \; k \geq 1 \,\}$.

Exercise 6.13. Let R be a euclidean ring with euclidean function N and let $a, b \in R$. Prove that if $a \mid b$ and $N(a) = N(b)$, then $a \sim b$.

Exercise 6.14. Prove that $(2, \sqrt{10})$, $(3, 4 + \sqrt{10})$, and $(3, 4 - \sqrt{10})$ are prime ideals in $\mathbb{Z}[\sqrt{10}]$.

Exercise 6.15. Let R be the ring of 2×2 matrices with entries in a field F. Verify that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}$$

Find other similar expressions and deduce that the two-sided ideal generated by a single matrix is either $\{0\}$ or the whole ring.