

UNIVERSITY OF ST ANDREWS
MT5826 Finite Fields
Tutorial Sheet: Chapter 1

1. For the equivalence relation *congruence modulo n* on \mathbb{Z} , show that
- (a) the binary operation

$$[a] + [b] = [a + b]$$

is well-defined, i.e. does not depend on our choice of representatives;

- (b) the set $\{[0], [1], \dots, [n - 1]\}$ forms a group under this operation;
- (c) this group is cyclic with $[1]$ as a generator.
2. Let ϕ denote Euler's phi function: for $n \in \mathbb{N}$, $\phi(n)$ is the number of integers k with $1 \leq k \leq n$ and $\gcd(k, n) = 1$. Prove that
- (a) For $n > 1$, $\phi(n) = n - 1 \Leftrightarrow n$ is prime.
- (b) If p is prime and $k > 0$, then

$$\phi(p^k) = p^k - p^{k-1}.$$

- (c) $\phi(mn) = \phi(m)\phi(n)$ if $\gcd(m, n) = 1$.
- (d) If n has prime factorization $p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$, then

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_r}\right).$$

3. Let $G = \langle a \rangle$ be a group of finite order m . Prove that
- (a) for any positive divisor d of m , G contains one and only one subgroup of index d ;
- (b) for any positive divisor f of m , G contains one and only one subgroup of order f .

(Hint: this is part (iii) of Theorem 1.13 of the notes; you may wish to use parts (i) and (ii) in your proof.)

4. (a) Show that $(\mathbb{Z}, +, *)$ is an integral domain but not a field.
- (b) Show that the set of 2×2 matrices with entries from \mathbb{R} , with matrix addition and multiplication, form a non-commutative ring with an identity.
- (c) Show that \mathbb{Z}_n , with operations $[a] + [b] = [a + b]$ and $[a][b] = [ab]$, is a commutative ring with an identity.

5. For the following statements, decide whether each is True/False and give a proof or counterexample as appropriate: (a) \mathbb{Z} is a subring of \mathbb{Q} ; (b) \mathbb{Z} is an ideal of \mathbb{Q} .
6. Prove the assertion in Example 2.7 of the notes that \mathbb{Z} is a principal ideal domain. You may wish to use the following proof outline:
- Either $I = \{0\}$, or
 - $I \neq \{0\}$. In this case, I contains both positive and negative elements. Let m be the least positive element of I . Show that $I = (m)$. (Hint: Consider $a \in I$ and use the Division Algorithm to write $a = mq + r$.)
7. Show that $\mathbb{Z}/(n)$ is not a field when n is composite.
8. Write out the addition and multiplication tables for \mathbb{F}_7 .
9. Prove the fact (needed in the proof of Freshmen's Exponentiation) that p is a divisor of $\binom{p}{i}$, for any $i \in \mathbb{Z}$ with $0 < i < p$.
10. Find all irreducible polynomials over \mathbb{F}_2 of degree 4.
11. Each residue class $g + (f)$ of $F[x]/(f)$ contains at least one representative $r \in F[x]$ with $\deg r < \deg f$, namely the remainder when g is divided by f . Prove that this representative is unique.
12. (a) State (without considering elements) whether the following are fields: (i) $\mathbb{F}_3[x]/(x^3 - x - 1)$; (ii) $\mathbb{F}_5[x]/(x^3 - x^2 + x - 1)$.
 (b) Is $\mathbb{F}_3[x]/(x^3 - x - 1) \cong \mathbb{F}_3[x]/(x^3 - x + 1)$?
 (c) What is the multiplicative order of $x + (x^3 - x - 1)$ in $\mathbb{F}_3[x]/(x^3 - x - 1)$? (Hint: it must divide $3^3 - 1$.)
13. Write out addition and multiplication tables for
- (a) $\mathbb{F}_3[x]/(f)$ where $f(x) = x^2 + 1$;
 - (b) $\mathbb{F}_2[x]/(f)$ where $f(x) = x^3 + 1$.
- In each case decide whether or not it is a field, giving justification.
14. Let $a \in F$, where F is a field. Prove that
- (a) If a is a multiple root of $f \in F[x]$, then it is a root of both f and its derivative f' .
 - (b) If a is a root of both f and its derivative f' , then it must be a multiple root of f .