Chapter 5

Jordan decomposition and Killing form

11 Jordan decomposition

We recall some definitions and results from linear algebra:

Definition/Proposition 11.1 (Jordan normal form)
Let V be an n-dimensional vector space over $\mathbb{C}$ and $T \in \mathrm{End}(V)$. Then V has a basis $\mathcal{B}$ such that the matrix corresponding to T with respect to $\mathcal{B}$ is of the block matrix form

$$\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_k \end{bmatrix}$$

and each J_i is of the form

$$\begin{bmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 1 & \lambda_i & \ddots & 0 & \vdots \\ 0 & 1 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_i \end{bmatrix},$$

for some $\lambda_i \in \mathbb{C}$. The J_i are called **Jordan blocks**, we say that such a matrix is **in Jordan normal form**. The number of Jordan blocks with a given diagonal entry λ and a given size is equal for all choices of such a basis $\mathcal{B}$. An endomorphism T is called **diagonalisable**, if all Jordan blocks in its Jordan normal form have size (1×1), that is, the Jordan normal form is a diagonal matrix. Obviously, T is nilpotent if and only if all diagonal entries in all Jordan blocks are equal to 0.

From this result we immediately get:

Definition/Proposition 11.2 (Jordan decomposition)
Let $T \in \mathrm{End}(V)$ for a finite-dimensional $\mathbb{C}$-vector space V. The Jordan decomposition of T is an expression of T as $T = D + N$ with $D, N \in \mathrm{End}(V)$, such that D is diagonalisable, N is nilpotent and $DN = ND$. Both endomorphisms D and N are uniquely defined by these conditions. There is a polynomial $p \in \mathbb{C}[X]$ with $D = p(T)$.

Proof. We only give a rough idea here:
Choose a basis $\mathcal{B}$ of V such that the matrix of T with respect to $\mathcal{B}$ is in Jordan normal form. The matrix of D with respect to $\mathcal{B}$ is the diagonal matrix containing only the diagonal entries of the

Jordan blocks, such that $N := T - D$ is nilpotent. The endomorphisms D and N commute since for every Jordan block the two matrices

$$\begin{bmatrix} \lambda_i & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_i & \ddots & 0 & \vdots \\ 0 & 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \lambda_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \ddots & 0 & \vdots \\ 0 & 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

commute. One proves next the existence of the polynomial p, which we skip here.
We need to prove the uniqueness. Let $T = D + N = \tilde{D} + \tilde{N}$ be two Jordan decompositions of T. Since D and $\tilde{D}$ are polynomials in T, they commute with each other and thus can be diagonalised simultaneously. But then since $D + N = \tilde{D} + \tilde{N}$ we get $D - \tilde{D} = \tilde{N} - N$ is nilpotent which can only be if $D = \tilde{D}$. ■

Proposition 11.3 (Solubility implies zero traces)
Let L be a soluble subalgebra of $\mathrm{gl}(V)$ where V is a finite-dimensional $\mathbb{C}$-vector space. Then for all $x \in L$ and all $y \in [L, L]$ we have $\mathrm{Tr}(xy) = 0$.

Proof. We use Lie's Theorem 10.4: There is a basis $\mathcal{B}$ of V such that the every element $x \in L$ corresponds to a lower triangular matrix with respect to $\mathcal{B}$. Since $y \in [L, L]$ is a sum of commutators, the diagonal entries of its matrix with respect to $\mathcal{B}$ are all zero. But then all diagonal entries of the matrix of xy are zero and thus the trace of xy is zero. ■

For the other direction, we need a slightly stronger hypothesis:

Proposition 11.4 (Zero traces imply solubility)
Let V be a finite-dimensional $\mathbb{C}$-vector space and L a Lie subalgebra of $\mathrm{gl}(V)$. Suppose that $\mathrm{Tr}(xy) = 0$ for all $x, y \in L$. Then L is soluble.

Proof. Not extremely difficult, but left out of these notes for the sake of brevity. ■

Surprisingly, these two can be put together for this result:

Theorem 11.5 (Criterion for solubility)
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $\mathrm{Tr}(x^{\mathrm{ad}} y^{\mathrm{ad}}) = 0$ for all $x \in L$ and $y \in [L, L]$.

Proof. Assume that L is soluble. Then L^{ad} is a soluble subalgebra of $\mathrm{gl}(L)$ by Theorem 5.7 and because ad is a homomorphism of Lie algebras. The statement of the theorem now follows immediately from Proposition 11.3 since $[u, v]^{\mathrm{ad}} = [u^{\mathrm{ad}}, v^{\mathrm{ad}}]$ by the Jacobi identity.
Assume conversely that $\mathrm{Tr}(x^{\mathrm{ad}} y^{\mathrm{ad}}) = 0$ for all $x \in L$ and all $y \in [L, L]$. Then Proposition 11.4 implies that $[L, L]^{\mathrm{ad}} = [L^{\mathrm{ad}}, L^{\mathrm{ad}}]$ is soluble (using our hypothesis only for $x, y \in [L, L]$. Thus L^{ad} itself is soluble since $[L^{\mathrm{ad}}, L^{\mathrm{ad}}] = (L^{\mathrm{ad}})^{(1)}$. But since $L^{\mathrm{ad}} \cong L/Z(L)$ it follows using Theorem 5.7.(ii) that L itself is soluble as $Z(L)$ is abelian. ■

12 The Killing form

Definition/Proposition 12.1 (The Killing form)
Let L be a Lie algebra over a field $\mathbb{F}$. Then the mapping

$$\begin{array}{rccc} \kappa: & L \times L & \to & \mathbb{F} \\ & (x, y) & \mapsto & \mathrm{Tr}(x^{\mathrm{ad}} y^{\mathrm{ad}}) \end{array}$$

is **bilinear**, that is, $\kappa(x + \lambda\tilde{x}, y) = \kappa(x, y) + \lambda\kappa(\tilde{x}, y)$ and $\kappa(x, y + \lambda\tilde{y}) = \kappa(x, y) + \lambda\kappa(x, \tilde{y})$ for all $x, \tilde{x}, y, \tilde{y} \in L$ and all $\lambda \in \mathbb{F}$. The map κ is called the **Killing form**. It is **symmetric**, that is,

$$\kappa(x, y) = \kappa(y, x) \qquad \text{for all } x, y \in L.$$

The Killing form is **associative**, that is,

$$\kappa([x, y], z) = \kappa(x, [y, z]) \qquad \text{for all } x, y, z \in L.$$

The latter property comes from the fact that $\mathrm{Tr}((uv - vu)w) = \mathrm{Tr}(u(vw - wv))$ for all endomorphisms $u, v, w \in \mathrm{End}(V)$ for any vector space V.

We can now restate Theorem 11.5 using this language:

Theorem 12.2 (Cartan's First Criterion)
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L is soluble if and only if $\kappa(x, y) = 0$ for all $x \in L$ and $y \in [L, L]$.

The Killing form can not only "'detect solubility"', but also semisimplicity. We need a few more definitions.

Definition 12.3 (Perpendicular space, non-degeneracy)
Let V be a vector space over a field $\mathbb{F}$ and $\tau : V \times V \to \mathbb{F}$ a symmetric bilinear form. For any subspace $W \leq V$ we define

$$W^{\perp} := \{v \in V \mid \tau(v, w) = 0 \text{ for all } w \in W\}$$

and call it the **perpendicular space** of W. It is a subspace of V. We call τ **non-degenerate**, if $V^{\perp} = \{0\}$, that is, there is no $0 \neq u \in V$ with $\tau(u, v) = 0$ for all $v \in V$. Otherwise, we call τ **degenerate**. If τ is non-degenerate, then

$$\dim_{\mathbb{F}}(V) = \dim_{\mathbb{F}}(W) + \dim_{\mathbb{F}}(W^{\perp})$$

for all subspaces $W \leq V$.

Lemma 12.4 (Perpendicular space of ideals with respect to the Killing form)
Let L be a Lie algebra, K be an ideal of L and κ the Killing form of L. Then $K^{\perp}$ (with respect to κ) is an ideal of L as well.

Proof. This uses the associativity of the Killing form: Let $x \in K^{\perp}$, that is, $\kappa(x, z) = 0$ for all $z \in K$. We have $\kappa([x, y], z) = \kappa(x, [y, z]) = 0$ for all $y \in L$ and all $z \in K$ because $[y, z] \in K$. ∎

Theorem 12.5 (Cartan's Second Criterion)
Let L be a finite-dimensional Lie algebra over $\mathbb{C}$. Then L is semisimple if and only if κ is non-degenerate.

Proof. Suppose that L is semisimple. By Lemma 12.4, the space $L^{\perp}$ (with respect to κ) is an ideal of L, such that $\kappa(x, y) = 0$ for all $x \in L^{\perp}$ and all $y \in [L^{\perp}, L^{\perp}]$ (indeed, even for all $y \in L$). Thus, by Theorem 12.2, the ideal $L^{\perp}$ is soluble. However, because we assumed that L is semisimple, it has no soluble ideals except $\{0\}$ and thus $L^{\perp} = 0$ and thus κ is non-degenerate.
Suppose that L is not semisimple. By Exercise 6 on Tutorial Sheet 2 it then has a non-zero abelian ideal A. Let $a \in A$ be a non-zero element. For every $x \in L$, the map $a^{\mathrm{ad}}x^{\mathrm{ad}}a^{\mathrm{ad}}$ sends all of L to 0, since $[[z, a], x] \in A$ and thus $[[[z, a], x], a] = 0$ for every $z \in L$. Thus $(a^{\mathrm{ad}}x^{\mathrm{ad}})^2 = 0$ and therefore $a^{\mathrm{ad}}x^{\mathrm{ad}}$ is a nilpotent endomorphism. However, nilpotent endomorphisms have trace 0, so a is a non-zero element of $L^{\perp}$ and κ is shown to be degenerate. ∎

Lemma 12.6 (Killing form on ideal)
Let I be an ideal in a finite-dimensional Lie algebra over $\mathbb{C}$. Then I is in particular a subalgebra and thus a Lie algebra on its own. The Killing form of I is then the restriction of the Killing from of L to I:

$$\kappa_I(x, y) = \kappa(x, y) \qquad \text{for all } x, y \in I.$$

Proof. Choose a basis of I and extend it to a basis of L. Then write matrices of x^{ad} for elements $x \in I$ with respect to this basis. The result follows. ■

Lemma 12.7 (Ideals in semisimple Lie algebras)
Let I be a non-trivial proper ideal in a complex semisimple Lie algebra L, then $L = I \oplus I^{\perp}$. The ideal I is a semisimple Lie algebra in its own right.

Proof. Let κ denote the Killing form on L, it is non-degenerate by Cartan's Second Criterion 12.5 since L is semisimple. The restriction of κ to $I \cap I^{\perp}$ is identically 0, so by Cartan's First Criterion 12.2 we get $I \cap I^{\perp} = 0$ because L does not have a non-zero soluble ideal. Counting dimensions now gives $L = I \oplus I^{\perp}$.
We need to show that I is a semisimple Lie algebra. Suppose not, then its Killing form is degenerate (using Cartan's Second Criterion 12.5). Thus, there is an $0 \neq a \in I$ such that $\kappa_I(a, x) = 0$ for all $x \in I$, where κ_I is the Killing form of I. By Lemma 12.6 this means that $\kappa(a, x) = 0$ for all $x \in I$. But then $a \in L^{\perp}$ since $L = I \oplus I^{\perp}$ contradicting that L is semisimple. ■

Using Lemma 12.7 it is now relatively easy to prove Theorem 5.12:

Theorem 12.8 (Characterisation of semisimple Lie algebras)
A finite-dimensional Lie algebra L over $\mathbb{C}$ is semisimple if and only if it is the finite direct sum of minimal ideals which are simple Lie algebras.

Proof. We only give the idea for the "only if" part: Use induction by the dimension, for the induction step choose a minimal non-zero ideal I and use Lemma 12.7 to write $L = I \oplus I^{\perp}$ and to show that $I^{\perp}$ is again semisimple of lower dimension. The ideal I is a simple Lie algebra because it was chosen minimal. ■

13 Abstract Jordan decomposition

Can we have a Jordan decomposition in an abstract Lie algebra?
If L is a one-dimensional Lie algebra, then every linear map $\varphi : L \to \mathrm{gl}(V)$ is a representation. So in general, an element $x \in L$ can be mapped to an arbitrary endomorphism of V. However, for complex semisimple Lie algebras, we can do better:

Theorem 13.1 (Abstract Jordan decomposition)
Let L be a finite-dimensional semisimple Lie algebra. Each $x \in L$ can be written uniquely as $x = d + n$, where $d, n \in L$ are such that d^{ad} is diagonalisable, n^{ad} is nilpotent, and $[d, n] = 0$. Furthermore, if $[x, y] = 0$ for some $y \in L$, then $[d, y] = 0 = [n, y]$.
The decomposition $x = d + n$ as above is called **abstract Jordan decomposition** of x.

Proof. Omitted. ■

This in fact covers all representations of L:

Theorem 13.2 (Jordan decompositions)
Let L be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$ and let $\varphi : L \to \mathrm{gl}(V)$ by any representation. Let $x = d + n$ be the abstract Jordan decomposition of x. Then the Jordan decomposition of $x\varphi \in \mathrm{gl}(V)$ is $x\varphi = d\varphi + n\varphi$.

Proof. Omitted. ■