UNIVERSITY OF ST ANDREWS
MT5827 Lie Algebras
Tutorial Sheet 4: Chapters 5 and 6

1. Let L be a finite-dimensional complex Lie algebra. Show that L is soluble if and only if $[L, L]$ is nilpotent.

2. Let L be a Lie algebra over the field $\mathbb{F}$. Let $M_1, \ldots, M_k$ and $N_1, \ldots, N_l$ be finite-dimensional irreducible modules of L, such that

$$M_1 \oplus \cdots \oplus M_k \cong N_1 \oplus \cdots \oplus N_l.$$

Show that $k = l$ and that for each isomorphism type of irreducible module, the multiplicity with which it occurs is the same on both sides.

3. Let $\mathbb{F}$ be a field and let $S \in \mathbb{F}^{n\times n}$ be an arbitrary matrix. Define

$$\mathrm{gl}_S(\mathbb{F}^{1\times n}) := \left\{ x \in \mathrm{gl}(\mathbb{F}^{1\times n}) \mid -xS = Sx^t \right\}.$$

- Show that $\mathrm{gl}_S(\mathbb{F}^{1\times n})$ is a Lie subalgebra of $\mathrm{gl}(\mathbb{F}^{1\times n})$.
- Find $\mathrm{gl}_S(\mathbb{R}^{1\times 2})$ if $S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
- Does there exist a matrix S such that $\mathrm{gl}_S(\mathbb{R}^{1\times 2})$ is equal to the set of all diagonal matrices in $\mathrm{gl}(\mathbb{R}^{1\times 2})$?

4. Find the Jordan normal form of the matrix

$$M := \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}.$$

What is the Jordan decomposition of M?