UNIVERSITY OF ST ANDREWS
MT5827 Lie Algebras
Tutorial Sheet 5: Chapters 5 and 6

1. Let $A, B \in \mathbb{F}^{n\times n}$ be square matrices over the field $\mathbb{F}$. Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ where Tr denotes the trace, which is the sum of the diagonal entries.

2. Let V be a finite-dimensional vector space over the field $\mathbb{F}$ and $\varphi \in \mathrm{End}(V)$ an endomorphism. The trace $\mathrm{Tr}(\varphi)$ of φ is defined in the following way: Choose a basis $\mathcal{B}$ of V and express φ as a matrix with respect to $\mathcal{B}$. Define $\mathrm{Tr}(\varphi)$ as the sum of the diagonal entries of this matrix. Show that this definition does not depend on the choice of $\mathcal{B}$.

3. Compute the Killing form of $\mathrm{sl}_2(\mathbb{C})$. Since $\mathrm{sl}_2(\mathbb{C})$ is simple and thus semisimple you should get a symmetric, non-degenerate 3×3-matrix. Compute the Killing form of $\mathrm{gl}_2(\mathbb{C})$. Is it non-degenerate?

4. Take $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \in \mathrm{sl}_2(\mathbb{C})$ and compute its abstract Jordan decomposition. Do the same for $\begin{bmatrix} -3 & -1 \\ 4 & 1 \end{bmatrix} \in \mathrm{sl}_2(\mathbb{C})$.