ON THE SOCLE OF AN ENDOMORPHISM ALGEBRA

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Abstract. The socle of an endomorphism algebra of a finite dimensional module of a finite dimensional algebra is described. The results are applied to the modular Hecke algebra of a finite group with a cyclic Sylow subgroup.

1. Introduction

The interplay between a module of a finite dimensional algebra **A** over an algebraically closed field k and the endomorphism ring **E** of this module has always been of special interest. The corresponding Hom-functor relates the module categories of **A** and **E**. For example, this functor realizes the Fitting correspondence between the PIMs of **A** and those of **E**.

Our main interest is in the case where **E** is a modular Hecke algebra. By this we mean an algebra of the form $\mathbf{E} = \mathrm{End}_{kG}(\mathrm{Ind}_P^G(k))$, where G is a finite group, k is an algebraically closed field of characteristic p, and P is a Sylow p-subgroup of G. The successful use of the modular Hecke algebra in connection with Alperin's Weight Conjecture by Cabanes in [3] is a strong motivation for further study. In the experimental part of [9, 10], special focus was laid on the structural meaning of the socle of a modular Hecke algebra **E**. It is the purpose of this paper to throw light on some of the experimental results observed in [9, 10]. In particular, we can now explain the outcome of the experiments in case of a p-modular Hecke algebra of a finite group with a cyclic Sylow p-subgroup (see Theorem 1.4 below). The main device for achieving this is a convenient description of the socle of a (general) endomorphism algebra.

Before stating our theorems, let us fix some notation and assumptions.

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Hypothesis 1.1. Let k be an algebraically closed field and $\mathbf{A}$ a finite dimensional k-algebra. Furthermore, let Y be a finitely generated right $\mathbf{A}$-module with a decomposition

$$Y = \bigoplus_{j=1}^{n} Y_j$$

into indecomposable direct summands. Throughout the whole paper we assume that

(a) $Y_i \cong Y_j$ if and only if $i = j$,
(b) The head and socle of Y have the same composition factors (up to isomorphism, disregarding multiplicities).

We denote the endomorphism ring $\mathrm{End}_{\mathbf{A}}(Y)$ of Y by $\mathbf{E}$, and the covariant Hom-functor $\mathrm{Hom}_{\mathbf{A}}(Y, -)$ by F.

Condition (a) of the above hypothesis is only introduced for convenience. Without this "multiplicity freeness" one would obtain a Morita equivalent endomorphism ring.

In the first part of this paper we describe the socle of the right regular $\mathbf{E}$-module in terms of the structure of Y. Since the Hom-functor F is left exact, we may view $F(S)$ as a submodule of $\mathbf{E}_{\mathbf{E}}$ for every $S \leq Y$. It is easy to see that every simple submodule of $\mathbf{E}_{\mathbf{E}}$ is isomorphic to a submodule of $F(S)$ for some simple $S \leq Y$. Of particular interest are thus the socles of the modules $F(S)$ for simple $\mathbf{A}$-modules S.

Theorem 1.2. *Let the notation and assumptions be as in Hypothesis 1.1. Fix a simple $\mathbf{A}$-submodule $S \leq Y_j$ for some $1 \leq j \leq n$. Two non-zero homomorphisms of the form $Y_i \to S$ for some $1 \leq i \leq n$ are called equivalent, if they differ by an automorphism of Y_i. Let $\mathcal{K}_{S,j}$ denote the set of equivalence classes of such homomorphisms.*

Then there is a partial order $\leq$ on $\mathcal{K}_{S,j}$, such that the maximal elements of $\mathcal{K}_{S,j}$ with respect to $\leq$ correspond to the simple submodules of $F(S)$.

A proof of this theorem will be given in Section 2. Under favorable conditions one can determine the maximal elements, leading to the following corollary.

Corollary 1.3. *Let the notation and assumptions be as in Hypothesis 1.1. Suppose in addition that the head of each Y_j is simple, and for each simple module S in the head of Y, there is at most one non-projective direct summand Y_j of Y with head S.*

Then the map $S \mapsto \mathrm{soc}(F(S))$ yields a bijection between the isomorphism classes of the simple $\mathbf{A}$-modules in the head of Y and the isomorphism classes of the simple submodules of $\mathbf{E}_{\mathbf{E}}$.

The results obtained here can be understood as a generalization of the main results of Green in [6]. In this reference, Green assumes Hypothesis 1.1(b), and in addition that $\mathbf{E}$ is self-injective. Green shows that this latter condition forces the heads and socles of the Y_j to be simple. Moreover, the Y_j are determined by their heads and socles up to isomorpism. Green then obtains the stronger conclusion that $S \mapsto F(S)$ yields a bijection between the isomorphism classes of the simple $\mathbf{A}$-modules in the head of Y and the isomorphism classes of all simple $\mathbf{E}$-modules.

In the second part of this paper, we apply the previous theorem to special cases, in particular to the modular Hecke algebra of groups with a cyclic Sylow subgroup. Our main result is contained in the following theorem.

Theorem 1.4. *Let kG be the group ring over k for a finite group G and let P be a Sylow p-subgroup of G. Let $\mathbf{A}$ be a sum of blocks of kG with cyclic defect groups. Assume one of the following conditions.*

(a) *G is p-solvable;*

(b) $|P| = p$;

(c) $\mathbf{A}$ *is the principal block.*

(Thus P is cyclic in Cases (b) and (c)).

Let Y be the $\mathbf{A}$-component of the permutation module $\mathrm{Ind}_P^G(k)$ and put $\mathbf{E} = \mathrm{End}_{\mathbf{A}}(Y)$. Then each non-projective indecomposable direct summand of Y is uniserial. Moreover, in Cases (b) and (c), the hypothesis of Corollary 1.3 is satisfied, the PIMs of $\mathbf{E}$ have simple socles, and for each simple $\mathbf{E}$-module T, there are at most two non-isomorphic PIMs of $\mathbf{E}$ with T as socle.

We conclude our paper with some examples demonstrating the relevance of the hypotheses of Theorem 1.4 (Subsection 3.3.6).

2. The Socle of $\mathbf{E}_{\mathbf{E}}$

Throughout this section we assume the hypothesis and notations from 1.1. We aim at describing the socle of the right regular $\mathbf{E}$-module. Let us begin with some general considerations, thereby introducing further notation.

Assumption (a) of Hypothesis 1.1 implies that $\mathbf{E}$ is a basic algebra, i.e., each simple $\mathbf{E}$-module is one-dimensional. Homomorphisms are written and composed from the left, i.e., $\varphi(y)$ denotes the image of $y \in Y$ under $\varphi \in \mathbf{E}$, and $\varphi\psi(y) = \varphi(\psi(y))$ for $y \in Y$. All $\mathbf{A}$-modules are assumed to be right modules and finitely generated, unless explicitly stated otherwise. Recall that F denotes the covariant Hom-functor

$\mathrm{Hom}_{\mathbf{A}}(Y, -)$ from the category of finitely generated right $\mathbf{A}$-modules to the category of finitely generated right $\mathbf{E}$-modules.

By Fitting's correspondence, there is a decomposition of the right regular $\mathbf{E}$-module $\mathbf{E}_{\mathbf{E}}$ into PIMs of the form $\mathbf{E}_{\mathbf{E}} = \bigoplus_{j=1}^{n} \mathbf{E}_j$ with $\mathbf{E}_j := \mathrm{Hom}_{\mathbf{A}}(Y, Y_j)$ for $1 \leq j \leq n$.

For $1 \leq j \leq n$ we write $\varepsilon_j \in \mathbf{E}$ for the projection of Y onto Y_j. Then $\sum_{j=1}^{n} \varepsilon_j = \mathrm{id}_Y$ is a decomposition of id_Y into pairwise orthogonal primitive idempotents, and we have

$$\psi = \sum_{i,j=1}^{n} \varepsilon_i \psi \varepsilon_j. \tag{1}$$

for every $\psi \in \mathbf{E}$. We identify $\mathbf{E}_j = \mathrm{Hom}_{\mathbf{A}}(Y, Y_j)$ with $\varepsilon_j \mathbf{E}$ for $1 \leq j \leq n$, and $\mathrm{Hom}_{\mathbf{A}}(Y_i, Y_j)$ with $\varepsilon_j \mathbf{E} \varepsilon_i$ for $1 \leq i, j \leq n$.

2.1. **The general case.** The socle of $\mathbf{E}_{\mathbf{E}}$ equals the direct sum of the socles of the $\mathbf{E}_j$. We may therefore restrict our attention to the socles of the latter. The following lemma provides a further reduction.

Lemma 2.1. *Fix $1 \leq j \leq n$ and let $\varphi \in \mathbf{E}_j$ be such that $\langle \varphi \rangle_k$ is a simple submodule of $\mathbf{E}_j$. Then there is some i, $1 \leq i \leq n$, such that $\varphi = \varepsilon_j \varphi \varepsilon_i$.*

Proof. Indeed, we have $\varphi = \sum_{i=1}^{n} \varphi \varepsilon_i$. Each of these summands lies in $\langle \varphi \rangle_k$ and the set of non-zero summands is linearly independent, so that $\langle \varphi \rangle_k = \langle \varphi \varepsilon_i \rangle_k$ for some i by the simplicity of $\langle \varphi \rangle_k$. By assumption, $\varphi = \varepsilon_j \varphi$. □

Definition 2.2. (1) For any $1 \leq j \leq n$ we define an equivalence relation $\sim$ on $\bigcup_{i=1}^{n} \varepsilon_j \mathbf{E} \varepsilon_i$ as follows. Let $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$, $\varphi' \in \varepsilon_j \mathbf{E} \varepsilon_{i'}$. Then $\varphi \sim \varphi'$, if and only if $i = i'$ and there is an automorphism $\psi \in \varepsilon_i \mathbf{E} \varepsilon_i$ such that $\varphi' = \varphi \psi$. The equivalence class of $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$ is denoted by $[\varphi]$.

(2) Fix some $1 \leq j \leq n$ and an $\mathbf{A}$-submodule $0 \neq S \leq Y_j$. Put

$$\mathcal{K}_{S,j} := \{[\varphi] \in \bigcup_{i=1}^{n} \varepsilon_j \mathbf{E} \varepsilon_i \mid \varphi \neq 0 \text{ and } \varphi(Y) \leq S\}.$$

For $[\varphi], [\varphi'] \in \mathcal{K}_{S,j}$ with $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$, $\varphi' \in \varepsilon_j \mathbf{E} \varepsilon_{i'}$ we write $[\varphi] \leq [\varphi']$, if and only if there is $\psi \in \varepsilon_i \mathbf{E} \varepsilon_{i'}$ such that $\varphi' = \varphi \psi$.

Lemma 2.3. *Let $0 \neq \varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$ and $\psi \in \varepsilon_i \mathbf{E} \varepsilon_i$ be such that $[\varphi] = [\varphi \psi]$. Then ψ is an automorphism.*

Proof. The assumption implies $\varphi = \varphi(\psi \mu)$ for some automorphism μ. Hence $\varphi = \varphi(\psi \mu)^l$ for all positive integers l. Since φ is non-zero and

$\mathrm{rad}(\varepsilon_i\mathbf{E}\varepsilon_i)$ is nilpotent, this implies that $\psi\mu$ is an automorphism, hence the result. $\square$

Lemma 2.4. *The relation $\leq$ of Definition 2.2 is a partial order on $\mathcal{K}_{S,j}$.*

Proof. Clearly, $\leq$ is well defined, reflexive and transitive. Let $[\varphi], [\varphi'] \in \mathcal{K}_{S,j}$ with $[\varphi] \leq [\varphi']$ and $[\varphi'] \leq [\varphi]$. Using the notion of the definition, there are $\psi \in \varepsilon_i\mathbf{E}\varepsilon_{i'}$ and $\psi' \in \varepsilon_{i'}\mathbf{E}\varepsilon_i$ such that $\varphi' = \varphi\psi$ and $\varphi = \varphi'\psi'$. Thus $\varphi = \varphi\eta$ with $\eta = \psi\psi' \in \varepsilon_i\mathbf{E}\varepsilon_i$. Thus η is an automorphism by Lemma 2.3. But then ψ is an isomorphism, hence $i = i'$ and $[\varphi'] = [\varphi\psi] = [\varphi]$. $\square$

For the remainder of this section we consider the following configuration. We choose $0 \neq S \leq Y_j$ for some $1 \leq j \leq n$. Let $[\varphi] \in \mathcal{K}_{S,j}$ with $\varphi \in \varepsilon_j\mathbf{E}\varepsilon_i$ for some $1 \leq i \leq n$. The symbols S, φ, j and i will always have this meaning, unless explicitly stated otherwise.

The next result provides a characterization of the maximal elements of the sets $\mathcal{K}_{S,j}$.

Lemma 2.5. *The element $[\varphi]$ is maximal with respect to $\leq$, if and only if the following condition is satisfied:*

(*) *If $\psi \in \varepsilon_i\mathbf{E}\varepsilon_l$ for some $1 \leq l \leq n$ with $\varphi\psi \neq 0$, then ψ is an automorphism.*

Proof. Suppose that $[\varphi]$ is maximal with respect to $\leq$, and let $\psi \in \varepsilon_i\mathbf{E}\varepsilon_l$ such that $\varphi' := \varphi\psi \neq 0$. Then $[\varphi] \leq [\varphi']$ and thus $[\varphi] = [\varphi']$. In particular, $l = i$ and $\psi \in \varepsilon_i\mathbf{E}\varepsilon_i$. Lemma 2.3 now implies that ψ is an automorphism. Thus, Condition (*) holds.

Suppose that $[\varphi]$ satisfies Condition (*) and let $[\varphi'] \in \mathcal{K}_{S,j}$ with $[\varphi] \leq [\varphi']$. Thus there is $\psi \in \varepsilon_i\mathbf{E}\varepsilon_{i'}$ such that $\varphi' = \varphi\psi$. In particular, $\varphi\psi \neq 0$. By Condition (*), ψ is an automorphism, which implies $[\varphi] = [\varphi']$. Hence $[\varphi]$ is maximal with respect to $\leq$. $\square$

Although we won't need this fact in the following, one can easily show that each element of $\mathcal{K}_{S,j}$ lies below a maximal one.

Lemma 2.6. *There is some $1 \leq l \leq n$ and some $\psi \in \varepsilon_i\mathbf{E}\varepsilon_l$ such that $[\varphi\psi] \in \mathcal{K}_{S,j}$ is maximal.*

Proof. Observe that if $\psi \in \varepsilon_i\mathbf{E}\varepsilon_l$ is such that $[\varphi] \leq [\varphi\psi]$ but $[\varphi] \neq [\varphi\psi]$, then ψ is contained in the Jacobson radical of $\mathbf{E}$. Since the latter is nilpotent, every increasing chain of elements of $\mathcal{K}_{S,j}$ must become stationary. $\square$

If an $\mathbf{A}$-module can be embedded into two distinct direct summands Y_j and $Y_{j'}$ of Y, the configuration we consider is "independent" of the particular direct summand used to define it.

Lemma 2.7. *Assume that $S' \leq Y_{j'}$ for some $1 \leq j' \leq n$ and that $\iota : S \to S'$ is an isomorphism. Then*

$$\mathcal{K}_{S,j} \to \mathcal{K}_{S',j'}, \quad [\varphi] \mapsto [\iota\varphi],$$

is a bijection preserving maximal elements. If $[\varphi] \in \mathcal{K}_{S,j}$ spans a simple submodule of $\mathbf{E}_{\mathbf{E}}$, then so does $[\iota\varphi]$, and the two $\mathbf{E}$-modules are isomorphic.

Proof. This is obvious. □

The following lemma characterizes the elements of $\mathcal{K}_{S,j}$ which span simple submodules of $\mathbf{E}$.

Lemma 2.8. *The element $[\varphi] \in \mathcal{K}_{S,j}$ spans a simple socle constituent of $\mathrm{Hom}_{\mathbf{A}}(Y, S) \leq \mathbf{E}_j$, if and only if $[\varphi]$ is maximal.*

Proof. Suppose that $[\varphi]$ is maximal in $\mathcal{K}_{S,j}$. Since $\varphi \in \varepsilon_j\mathbf{E}\varepsilon_i$, it suffices to show that $\varphi\psi \in \langle\varphi\rangle_k$ for all $\psi \in \varepsilon_i\mathbf{E}\varepsilon_l$ and all $1 \leq l \leq n$. Let $\psi \in \varepsilon_i\mathbf{E}\varepsilon_l$ such that $\varphi\psi \neq 0$. Then ψ is an automorphism by Lemma 2.5. Write $\psi = a\varepsilon_i + \eta$ with $0 \neq a \in k$ and $\eta \in \mathrm{rad}(\varepsilon_i\mathbf{E}\varepsilon_i)$. If $\varphi\eta$ were non-zero, we would have $[\varphi] = [\varphi\eta]$ by the maximality of $[\varphi]$. This in turn would imply that η is an automorphism by Lemma 2.3, a contradiction. Thus $\varphi\eta = 0$ and hence $\varphi\psi = a\varphi$.

Now suppose that $[\varphi] \in \mathcal{K}_{S,j}$ spans a simple submodule of $\mathbf{E}_j$. Let $\varphi' \in \mathcal{K}_{S,j}$ with $[\varphi] \leq [\varphi']$. Thus $\varphi' = \varphi\psi$ with $\psi \in \varepsilon_i\mathbf{E}\varepsilon_{i'}$ for some $1 \leq i' \leq n$. Now $0 \neq \varphi' = \varphi\psi \in \langle\varphi\rangle_k$, which implies that $[\varphi] = [\varphi']$. Thus $[\varphi]$ is maximal. □

As a consequence of the lemma we note that $[\varphi] = \langle\varphi\rangle_k \setminus \{0\}$ if $[\varphi]$ is maximal. Clearly, Definition 2.2 and Lemma 2.8 prove Theorem 1.2.

Suppose that $S \leq S' \leq Y_j$. Then $\mathcal{K}_{S,j} \subseteq \mathcal{K}_{S',j}$, and the partial order on $\mathcal{K}_{S',j}$ restricts to that of $\mathcal{K}_{S,j}$. Moreover, $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, if and only if it is maximal in $\mathcal{K}_{S',j}$. In particular, if $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, it is also maximal in $\mathcal{K}_{\varphi(Y),j}$.

Next, we determine the isomorphism types corresponding to the maximal elements of $\mathcal{K}_{S,j}$.

Lemma 2.9. *Suppose that $[\varphi] \in \mathcal{K}_{S,j}$ is maximal. Then the simple socle constituent $\langle\varphi\rangle_k$ of $\mathbf{E}_j$ is isomorphic to the head constituent of $\mathbf{E}_i$ (recall that we assume $\varphi \in \varepsilon_j\mathbf{E}\varepsilon_i$).*

Proof. For any $1 \leq l \leq n$ we have

$$\mathrm{Hom}_{\mathbf{E}}\left(\varepsilon_l \mathbf{E}, \langle \varphi \rangle_k\right) \cong \langle \varphi \rangle_k \varepsilon_l \tag{2}$$

as k-vector spaces. Now $\langle \varphi \rangle \varepsilon_l$ is non-trivial, if and only if $l = i$. In this case, $\langle \varphi \rangle_k \varepsilon_i = \langle \varphi \rangle_k$. By Equation (2), the assertion is now obvious. $\square$

We collect the main results of this section in the following corollary.

Corollary 2.10. (a) *Fix some $1 \leq j \leq n$ and a submodule $S \leq Y_j$. Then there is a bijection between the simple submodules of $F(S)$ and the maximal elements of $\mathcal{K}_{S,j}$.*

(b) *Suppose that Y_j is projective and that $S = \mathrm{soc}(Y_j) \cong \mathrm{hd}(Y_j)$. Then $\mathcal{K}_{S,j}$ has a unique maximal element $[\varphi]$ with $\varphi \in \mathrm{Hom}_{\mathbf{A}}(Y_j, S) \leq F(S)$. Moreover, the head and the socle of $\mathbf{E}_j$ are simple and isomorphic to each other.*

Proof. (a) This is obvious by Lemmas 2.1 and 2.8.

(b) Let $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_j$ denote the epimorphism of Y_j onto S. Then $[\varphi]$ is the unique maximal element of $\mathcal{K}_{S,j}$ by our assumptions on Y_j. Let $\langle \psi \rangle_k \leq \mathbf{E}_j$ be a simple submodule. By Lemma 2.1, we may assume that $\psi \in \varepsilon_j \mathbf{E} \varepsilon_i$ for some $1 \leq i \leq n$. Let M be the image of ψ in Y_j. Note, that $M \leq Y_j$ has S as socle, so that there is a homomorphism $\tilde{\varphi} : Y_j \to S \leq M$.

By the projectivity of Y_j, there is a homomorphism $\eta : Y_j \to Y_i$, such that $\psi\eta = \tilde{\varphi}$. Since $\langle \psi \rangle_k$ is an $\mathbf{E}$-submodule we have $\langle \varphi \rangle_k = \langle \tilde{\varphi} \rangle_k = \langle \psi\eta \rangle_k = \langle \psi \rangle_k$. $\square$

The following lemma shows that, in order to describe the isomorphism types of the socle constituents of $\mathbf{E}$, it suffices to determine the socles of the $\mathbf{E}$-modules $F(S)$ for the simple submodules S of the Y_j.

Lemma 2.11. *Fix $1 \leq i, l \leq n$. Let $\psi \in \varepsilon_l \mathbf{E} \varepsilon_i$ be such that $\langle \psi \rangle_k$ is a simple submodule of $\mathbf{E}_l$. Then there is a simple submodule S of Y_j for some $1 \leq j \leq n$ and an element $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$, such that $\varphi(Y_i) = S$ and $\langle \psi \rangle_k \cong \langle \varphi \rangle_k$ as $\mathbf{E}$-modules.*

Proof. Put $M := \psi(Y_i)$ and let S' denote a simple quotient of M. Thus S' is isomorphic to a head constituent of Y, and hence there is some $1 \leq j \leq n$ and a simple submodule S of Y_j isomorphic to S'. Let ι denote a non-zero map $M \to S$ and put $\varphi := \iota\psi$. Then $\langle \varphi \rangle_k$ is a simple submodule of $\mathbf{E}_j$ isomorphic to $\langle \psi \rangle_k$. $\square$

Example 2.12. (Naehrig, [9]) An example for the situation in Lemma 2.11 can be found in the permutation module $\mathrm{Ind}_P^G(k)$ for $G = S_7$ and P a Sylow 2-subgroup of G, where k has characteristic 2.

The module $\mathrm{Ind}_P^G(k)$ has three indecomposable direct summands which belong to the non-principal kG-block. These modules are uniserial and of dimensions 6, 20, and 28, respectively. In what follows, we will denote modules by their dimension, e.g., $\mathbf{6}$ denotes a module of dimension 6. The ascending composition series are given as follows.

$$\begin{aligned} \mathbf{6}: &\quad \mathbf{6} \\ \mathbf{20}: &\quad \mathbf{6}, \mathbf{8}, \mathbf{6} \\ \mathbf{28}: &\quad \mathbf{8}, \mathbf{6}, \mathbf{6}, \mathbf{8}, \end{aligned}$$

where $\mathbf{6}$ and $\mathbf{8}$ are simple. Then (with a slightly simplified notation)

$$\mathcal{K}_{\mathbf{6}} = \{\varphi_1 : \mathbf{6} \to \mathbf{6}, \varphi_2 : \mathbf{20} \to \mathbf{6}\}$$

and

$$\mathcal{K}_{\mathbf{8}} = \{\psi : \mathbf{28} \to \mathbf{8} \leq \mathbf{28}\}.$$

We immediately see that $[\varphi_2]$ and $[\psi]$ are the maximal elements of $\mathcal{K}_{\mathbf{6}}$ and $\mathcal{K}_{\mathbf{8}}$, respectively. By Lemma 2.8, there are two isomorphism types of socle constituents of the corresponding $\mathbf{E}$-PIMs $F(\mathbf{6})$, $F(\mathbf{20})$, and $F(\mathbf{28})$. Our analysis shows that $\mathrm{soc}(F(\mathbf{6})) = \langle\varphi_2\rangle_k$ and $\mathrm{soc}(F(\mathbf{28})) = \langle\psi\rangle_k$ are simple, while $\mathrm{soc}(F(\mathbf{20})) \cong \langle\varphi_2, \psi\rangle_k$. In fact, the socle constituent of $F(\mathbf{20})$, which is isomorphic to $\langle\psi\rangle_k$, is spanned by a homomorphism $\eta : \mathbf{28} \to \mathbf{20}$ with image $\mathbf{14}$.

We finally discuss an equivalence relation on the PIMs $\mathbf{E}_j$ (and hence on the Y_j) introduced in the third author's PhD-thesis. Let us recall the definition.

Definition 2.13. ([10, Remark 4.1]) Let $\sim$ denote the transitive closure of the relation on $\{\mathbf{E}_1, \ldots, \mathbf{E}_n\}$ defined by $\mathbf{E}_i \sim \mathbf{E}_j$, if and only if $\mathrm{soc}(\mathbf{E}_i)$ and $\mathrm{soc}(\mathbf{E}_j)$ have a common irreducible constituent (up to isomorphism).

By Lemmas 2.9 and 2.8, $\mathbf{E}_i \sim \mathbf{E}_j$ if and only if there is an l, $1 \leq l \leq n$, submodules $S \leq Y_i$ and $T \leq Y_j$, and surjective maps $\varphi : Y_l \to S$, $\psi : Y_l \to T$, such that $[\varphi] \in \mathcal{K}_{S,i}$ and $[\psi] \in \mathcal{K}_{T,j}$ are maximal.

2.2. **Special cases.** To discuss some special cases of our main result, we introduce some more notation. Put

$$\mathcal{H} := \{S \mid S \text{ is a simple constituent of } \mathrm{hd}(Y)\}/\cong$$

and

$$\mathcal{S}_{\mathbf{E}} := \{T \mid T \text{ is a simple constituent of } \mathrm{soc}(\mathbf{E}_{\mathbf{E}})\}/\cong .$$

Corollary 2.14. *Assume that one of the following conditions is satisfied.*

(a) *For* $1 \le j \le n$*, the head of* Y_j *is simple, and for each* $S \in \mathcal{H}$*, there is at most one non-projective direct summand* Y_j *of* Y *with head* S*.*

(b) *The algebra* $\mathbf{A}$ *is symmetric, and for each simple module* $S \le \mathrm{hd}(Y)$*, the projective cover of* S *is isomorphic to one of the* Y_j*.*

Then there is a bijection $\mathcal{H} \to \mathcal{S}_{\mathbf{E}}$ *induced by the functor* $\mathrm{soc} \circ F$ *(i.e.,* $\mathrm{soc}(F(S))$ *is simple for* $S \in \mathcal{H}$*, determined by* S *up to isomorphism, and every socle constituent of* $\mathbf{E}_{\mathbf{E}}$ *is isomorphic to one of these).*

Proof. Let $S \in \mathcal{H}$. We may assume that $S \le Y_j$ for some $1 \le j \le n$. By assumption and Corollary 2.10(b), $\mathcal{K}_{S,j}$ contains a unique maximal element. By Part (a) of this corollary, the socle of $F(S)$ is simple. Suppose that $S' \le Y_{j'}$ is simple and $S \not\cong S'$. Then the maximal elements in $\mathcal{K}_{S,j}$ and $\mathcal{K}_{S',j'}$ arise from different direct summands of Y. Lemma 2.9 implies that $\mathrm{soc}(F(S)) \not\cong \mathrm{soc}(F(S'))$. Thus the given map is injective. It is also surjective, since by Lemma 2.11, every element of $\mathcal{S}_{\mathbf{E}}$ is isomorphic to $\mathrm{soc}(F(S))$ for some $S \in \mathcal{H}$. □

Note that Corollary 1.3 is just Part (a) of Corollary 2.14. We note one further consequence.

Corollary 2.15. *Suppose that* $\mathrm{hd}(Y_j)$ *is multiplicity free for all* $1 \le j \le n$*. Then* $\mathrm{soc}(F(S))$ *is multiplicity free for all simple submodules* $S \le Y$*.*

Proof. This is immediate by Corollary 2.10(a) and Lemma 2.9. □

Under the restrictive assumptions of Corollary 2.14(a), the relation $\le$ has another interpretation.

Lemma 2.16. *Assume that each indecomposable direct summand of* Y *has a simple head. Then there is a partial order* $\le$ *on the set* $\{Y_1, \ldots, Y_n\}$ *given by:*

$$Y_i \le Y_{i'} \text{ if and only if}$$
$$\text{there is a surjection } \psi : Y_{i'} \to Y_i.$$

Let $S \le Y_j$ *be simple and let* $[\varphi], [\varphi'] \in \mathcal{K}_{S,j}$ *with* $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$ *and* $\varphi' \in \varepsilon_j \mathbf{E} \varepsilon_{i'}$*. Then* $Y_i \le Y_{i'}$ *if and only if* $[\varphi] \le [\varphi']$*.*

3. Groups with a cyclic Sylow subgroup

In this second part of our paper we apply the previous results to the modular Hecke algebra $\mathbf{E}$ of a group with a cyclic Sylow p-subgroup. In particular, we are able to explain the computational results of [9] in this case. There, the third author found that in all computed examples, the PIMs of **E** had simple socles. Moreover, the number of PIMs with the same socle (up to isomorphism) was at most two.

The main difficulty here is to describe the indecomposable direct summands of the permutation module on the cosets of a Sylow subgroup. Using Green correspondence, one can reduce this problem to the centralizer of a subgroup of order p. This is a p-nilpotent group under our assumptions. We are thus lead to study this question for p-nilpotent groups. This is done in general, without assuming that a Sylow p-subgroup is cyclic, in Subsection 3.2. In particular, we give a description of the modular Hecke algebra in a general p-nilpotent group (see Proposition 3.2). To keep our exposition as elementary as possible, we avoid reference to the machinery of nilpotent blocks.

3.1. **Preliminaries.** Throughout this section, G denotes a finite group and k an algebraically closed field of characteristic p. All kG-modules are right kG-modules and of finite dimension, unless explicitly stated otherwise. As in Section 2, we write and compose maps from the left. Let V be a kG-module. We view $\mathrm{End}_k(V)$ as a kG-module in the usual way, i.e., $\varphi^g(v) := \varphi(vg^{-1})g$, for $\varphi \in \mathrm{End}_k(V)$, $v \in V$, $g \in G$. There is a natural isomorphism $\mathrm{End}_k(V) \cong V^* \otimes_k V$ as right kG-modules, where G acts diagonally on $V^* \otimes_k V$. This sends $\lambda \otimes v$ with $\lambda \in V^*$ and $v \in V$ to the map $\varphi_{\lambda,v}$ given by $\varphi_{\lambda,v}(w) = v\lambda(w)$, $w \in V$. For $a \in kG$ we write $\rho_a \in \mathrm{End}_k(V)$ for right multiplication with a on V. If V is a left kG-module, $\lambda_a \in \mathrm{End}_k(V)$ denotes the left multiplication with $a \in kG$.

If A is a G-algebra over k, we write, as usual, A^G for the set of G-fixed points of A.

For $a = \sum_{g\in G} s_g g \in kG$, we write $a' := \sum_{g\in G} s_g g^{-1}$. Thus $' : kG \to kG$, $a \mapsto a$ is an anti-isomorphism of kG.

3.2. **The p-nilpotent case.** We begin by investigating the modular Hecke algebra of a p-nilpotent group, not necessarily with a cyclic Sylow p-subgroup. Thus assume that G is p-nilpotent, i.e., $G = PM$ with $M = O_{p'}(G) \trianglelefteq G$ being a normal p-complement. Let ε be a centrally primitive idempotent of kM. If ε is invariant under the conjugation action of P on kM, then so is the homogeneous component $\varepsilon kM = \varepsilon kM\varepsilon$, and εkG is the unique block of kG covering εkM. The action of P on εkM gives εkM the structure of a kG-module. Also, $\mathrm{End}_{kM}(\varepsilon kM)$ is invariant under the action of P on $\mathrm{End}_k(\varepsilon kM)$. Let $V \leq \varepsilon kM$ be a simple, P-invariant kM-submodule of εkM.

For further reference, we collect a few obvious and well-known, but helpful results.

Lemma 3.1. *Assume that ε is P-invariant. Then the following statements hold:*

(a)

$$\varepsilon kM\varepsilon \to \mathrm{End}_{kM}(\varepsilon kM), \quad a \mapsto \lambda_a,$$

is a P-equivariant k-algebra isomorphism. In particular

$$(\varepsilon kM\varepsilon)^P \cong \mathrm{End}_{kG}(\varepsilon kM).$$

(b) *The "twisted" structural homomorphism*

$$\Theta : \varepsilon kM\varepsilon \to \mathrm{End}_k(V), \quad a \to \rho_{a'}, \tag{3}$$

is a P-equivariant k-algebra isomorphism. In particular,

$$(\varepsilon kM\varepsilon)^P \cong \mathrm{End}_{kP}(V),$$

and thus

$$\mathrm{End}_{kG}(\varepsilon kM) \cong \mathrm{End}_{kP}(V).$$

(c) *As a kP-module, V is an endopermutation module.*

Proof. Parts (a) and (b) are clear. The surjectivity of Θ follows from the fact that V is an absolutely simple kM-module.

(c) The map Θ in (3) is an isomorphism of kP-modules, so that $V^* \otimes_k V \cong \varepsilon kM$ as kP-modules. Since εkM is a direct summand of a permutation kP-module, it is itself a permutation kP-module. Thus $\mathrm{Res}_P(V)$ is an endopermutation module. □

If ε is not P-invariant, we consider the stabilizer Q of ε in P and apply the lemma to QM. Then induction yields a Morita equivalence between the block εkQM and the kG-block covering εkM (see [2, Theorem 6.4.1]).

Recall that we are interested in $\mathrm{Ind}_P^G(k)$. The structure of this permutation module is revealed in the following proposition.

Proposition 3.2. *We have $\mathrm{Ind}_P^G(k) \cong kM$ with P acting by conjugation and M by right multiplication.*

Choose a set of representatives $\varepsilon_1, \ldots, \varepsilon_s$ for the P-orbits on the centrally primitive idempotents of kM. Write Q_i for the stabilizer of ε_i in P, and let V_i be a simple kQ_iM-submodule of $\varepsilon_i kM$, $1 \leq i \leq s$. Then

$$\mathrm{Ind}_P^G(k) \cong \bigoplus_{i=1}^{s} \mathrm{Ind}_{Q_iM}^G(\varepsilon_i kM)$$

and

$$\mathrm{End}_{kG}(\mathrm{Ind}_P^G(k)) \cong \bigoplus_{i=1}^{s} \mathrm{End}_{kG}(\mathrm{Ind}_{Q_iM}^G(\varepsilon_i kM)).$$

Moreover,

$$\mathrm{End}_{kG}(\mathrm{Ind}_{Q_iM}^{G}(\varepsilon_i kM)) \cong \mathrm{End}_{kQ_i}(\mathrm{Res}_{Q_i}^{Q_iM}(V_i))$$

for $1 \leq i \leq s$.

Proof. The first assertion is clear. Fix i, $1 \leq i \leq s$, and let ε denote the sum of the centrally primitive idempotents in the P-orbit of ε_i. Then $\varepsilon kM \cong \mathrm{Ind}_{Q_iM}^{G}(\varepsilon_i kM)$ as kG-modules. Also,

$$\mathrm{End}_{kG}(\mathrm{Ind}_{Q_iM}^{G}(\varepsilon_i kM)) \cong \mathrm{End}_{kQ_iM}(\varepsilon_i kM) \cong \mathrm{End}_{kQ_i}(\mathrm{Res}_{Q_i}^{Q_iM}(V_i)),$$

the first isomorphism arising from [2, Theorem 6.4.1], and the second from Lemma 3.1(b). Finally, the assertion about $\mathrm{End}_{kG}(\mathrm{Ind}_{P}^{G}(k))$ follows from the fact that the direct summands $\mathrm{Ind}_{Q_iM}^{G}(\varepsilon_i kM)$ of $\mathrm{Ind}_{P}^{G}(k)$ lie in distinct blocks of kG. □

Thus in order to study

$$\mathrm{End}_{kG}(\mathrm{Ind}_{P}^{G}(k))$$

in the p-nilpotent group G, we have to investigate the endomorphism rings $\mathrm{End}_{kQ_i}(\mathrm{Res}_{Q_i}^{Q_iM}(V_i))$.

3.3. **Groups with a cyclic Sylow p-subgroup.** We now assume that a Sylow subgroup P of G is cyclic of order $p^n > 1$. For $0 \leq i \leq n$ we write D_i for the unique subgroup of P of order p^i, and put $N := N_G(D_1)$ and $C := C_G(D_1)$.

3.3.1. *The structure of C and N.* The following lemma restricts the structure of C and of N.

Lemma 3.3. *Assume that $O_p(G) \neq 1$. Then C is p-nilpotent, $G = N$ and G/C is cyclic of order dividing $p-1$. In particular, G is p-solvable.*

Proof. Notice that D_1 is the unique subgroup of order p in any p-subgroup of G containing D_1. Since $1 \neq O_p(G) \leq P$, it follows that D_1 is contained in $O_p(G)$ and hence is normal in G. In particular, D_1 is the unique subgroup of G of order p and $G = N$. Since $C \trianglelefteq G$ and $|G/C| \mid p-1$, it suffices to prove the first assertion.

Now p does not divide $|C' \cap Z(G)|$ by a well known transfer argument (see e.g., [7, Theorem (5.6)]). Hence D_1 is not contained in C' and thus C' is a p'-group. The result follows. □

3.3.2. *The p-nilpotent case.* We first aim at investigating the situation in C. Thus assume in addition that G is p-nilpotent, i.e., $G = PM$ with $M = O_{p'}(G)$ being a normal p-complement.

Every p-block of kG has a unique simple module, and since P is cyclic, every indecomposable module in such a block is uniserial and uniquely determined by its composition factor V and its composition length ℓ. We write $J(\ell, V)$ for such an indecomposable kG-module, using a similar convention for subgroups of G.

Lemma 3.4. *Let V be a simple G-invariant kM-module and let ε denote the centrally primitive idempotent of kM corresponding to V. Then ε is G-invariant and we may use the notation of Section 3.2.*

Suppose that

$$\mathrm{Res}^G_P(V) = J(\ell_1, k) \oplus J(\ell_2, k) \oplus \ldots \oplus J(\ell_r, k)$$

with positive integers ℓ_i, $1 \leq i \leq r$. Then

$$\varepsilon kM \cong J(\ell_1, V) \oplus J(\ell_2, V) \oplus \ldots \oplus J(\ell_r, V)$$

as a kG-module.

Proof. By Lemma 3.1(b), we have $\mathrm{End}_{kG}(\varepsilon kM) \cong \mathrm{End}_{kP}(V)$. The isomorphism induces a bijection $\pi \mapsto \pi'$ between the centrally primitive idempotents of $\mathrm{End}_{kG}(\varepsilon kM)$ and those of $\mathrm{End}_{kP}(V)$, such that $\pi\mathrm{End}_{kG}(\varepsilon kM)\pi \cong \pi'\mathrm{End}_{kP}(V)\pi'$. The dimension of $\pi\mathrm{End}_{kG}(\varepsilon kM)\pi$ equals the composition length of the direct summand $\pi\varepsilon M$ of εM. This implies the result. $\square$

By Lemma 3.1(d), the module $\mathrm{Res}^G_P(V)$ is an endopermutation kP-module. Thus every indecomposable direct summand of $\mathrm{Res}^G_P(V)$ is an endopermutation module as well. The indecomposable endopermutation modules of a cyclic p-group are classified (see [12, Exercise (28.3)]).

Since the dimension of V is prime to p, the vertex of V equals P. Let S denote a source of V. Thus $S \mid \mathrm{Res}^G_P(V)$ and $V \mid \mathrm{Ind}^G_P(S)$. In particular, S is an indecomposable endopermutation kP-module with vertex P. Moreover,

$$\mathrm{Res}^G_P(V) \mid \mathrm{Res}^G_P(\mathrm{Ind}^G_P(S)).$$

By Mackey's theorem, the indecomposable summands of $\mathrm{Res}^G_P(V)$ are of the form $\mathrm{Ind}^P_{P^g \cap P}(T)$, where $g \in G$ and T is an indecomposable summand of $\mathrm{Res}^P_{P^g \cap P}(S)$.

3.3.3. *The situation in N.* Now assume that G has a normal subgroup C such that G/C is cyclic of order dividing $p-1$ and that $C = PM$ is p-nilpotent.

Lemma 3.5. *Let V be simple kC-module, and let $W = J(\ell, V)$ be a uniserial kC-module of composition length $\ell \leq p^n$ (for the notation see Subsection 3.3.2). Write H for the inertia group of V in G, and put $e := |H : C|$.*

Since $|G/C|$ is prime to p, $\mathrm{Ind}_C^G(V)$ is semisimple,

$$\mathrm{Ind}_C^G(V) = V_1 \oplus \ldots \oplus V_e$$

with pairwise non-isomorphic simple kG-modules V_i. Moreover,

$$\mathrm{Ind}_C^G(W) \cong V_{1,\ell} \oplus \ldots \oplus V_{e,\ell},$$

where $V_{i,\ell}$ denotes the indecomposable kG-module with head isomorphic to V_i and composition length ℓ.

Proof. This is just an application of Clifford theory. □

Notice that the indecomposable direct summands of $\mathrm{Ind}_C^G(W)$ form an orbit under the square of the Heller operator Ω^2 of kG.

3.3.4. *The general case.* From the preceding considerations we obtain information about the indecomposable direct summands of $\mathrm{Ind}_P^G(k)$ in a block of kG. Let $\mathbf{B}$ be block of kG with defect group $1 \neq Q \leq P$, and let $\mathbf{b}$ denote the Brauer correspondent of $\mathbf{B}$ in N. We also choose a block $\mathbf{c}$ of kC covered by $\mathbf{b}$. An indecomposable direct summand of $\mathrm{Ind}_P^G(k)$ lying in $\mathbf{B}$ will be called a $\mathbf{B}$-component of $\mathrm{Ind}_P^G(k)$; an analogous notation is used for the blocks $\mathbf{b}$ and $\mathbf{c}$. The indecomposable $\mathbf{b}$-modules and $\mathbf{c}$-modules are uniserial, and we write $\ell(U)$ for the composition length of an indecomposable $\mathbf{b}$-module. The Green correspondence between the indecomposable modules of $\mathbf{B}$ and those of $\mathbf{b}$ is denoted by f.

Proposition 3.6. *Let V be the simple $\mathbf{c}$-module and let S be the source of V. Then S is an indecomposable endopermutation kQ-module with vertex Q and trivial D_1-action, and the following statements hold for any non-projective $\mathbf{B}$-component U of $\mathrm{Ind}_P^G(k)$.*

(a) *If U has vertex $R \leq Q$, then $\ell(f(U)) = |Q : R|\ell(T)$, where T is an indecomposable direct summand of $\mathrm{Res}_R^Q(S)$.*

(b) *$\Omega^2(U)$ is a $\mathbf{B}$-component of $\mathrm{Ind}_P^G(k)$.*

(c) *If $\mathbf{B}$ is the principal block, then $\ell(f(U)) = 1$.*

Proof. Clearly, V and hence S have vertex Q, since Q is a defect group of $\mathbf{c}$. Also, D_1 acts trivially on V, hence also on S.

We have $\mathrm{Ind}_P^G(k) \cong \mathrm{Ind}_N^G\left(\mathrm{Ind}_P^N(k)\right)$, and f sets up a vertex preserving one-to-one correspondence between the non-projective $\mathbf{B}$-components of $\mathrm{Ind}_P^G(k)$ and the non-projective $\mathbf{b}$-components of $\mathrm{Ind}_P^N(k)$. Hence $f(U)$ is a $\mathbf{b}$-component of $\mathrm{Ind}_P^N(k) \cong \mathrm{Ind}_C^N\left(\mathrm{Ind}_P^C(k)\right)$. It follows that there is a $\mathbf{c}$-component W of $\mathrm{Ind}_P^C(k)$ such that $f(U) \mid \mathrm{Ind}_C^N(W)$.

By Lemma 3.5 and the subsequent remark, $\ell(f(U)) = \ell(W)$ and $\Omega^2(f(U)) \mid \mathrm{Ind}_C^N(W)$. Since the Green correspondence commutes with the Heller operator, the latter proves (b).

Write $M := O_{p'}(C)$. By Proposition 3.2 and Lemma 3.4, applied to QM, the number $\ell(W)$ equals the length of an indecomposable direct summand of $\mathrm{Res}_Q^{QM}(V')$, where V' is a simple submodule of $\mathrm{Res}_{QM}^C(V)$. Clearly S is a source of V', and hence (a) follows from the considerations after Lemma 3.4.

Finally, if $\mathbf{B}$ is the principal block, then $Q = P$ and V is the trivial module. This implies $\ell(f(U)) = 1$. □

3.3.5. *The proof of Theorem 1.4.* Let $\mathbf{A}$ be as in Theorem 1.4 and let $\mathbf{B}$ be a block of $\mathbf{A}$. If G is p-solvable, the Brauer tree of $\mathbf{B}$ is a star with its exceptional node at the center (see [4, Lemma X.4.1]). This implies that any $\mathbf{B}$-module is uniserial, and we are done.

Suppose then that $|P| = p$ or that $\mathbf{B}$ is the principal block. If U is a non-projective indecomposable summand of Y contained in $\mathbf{B}$, we have $\ell(f(U)) = 1$ by Proposition 3.6. By [1, Lemma 22.3], this implies that U is uniserial, thus proving the first claim of Theorem 1.4.

Let us now turn to the proof of the remaining assertions of the theorem. By replacing $\mathbf{E}$ by its basic algebra, we may assume that Y satisfies Hypothesis 1.1(a). The second part of this hypothesis is clearly satisfied by $\mathrm{Ind}_P^G(k)$ and hence by Y. In Cases (b) and (c), the indecomposable direct summands of Y are either projective or of maximal vertex with a simple Green correspondent (see Proposition 3.6). In particular, the hypothesis of Corollary 1.3 is satisfied. Fix j, $1 \leq j \leq n$. If Y_j is projective, then $\mathbf{E}_j$ has a simple socle by Corollary 2.10(b). Suppose that Y_j is not projective. Then Y_j is uniserial, and the composition factors of Y_j, from top to bottom, arise from a cyclic walk around a vertex of the Brauer tree of the block containing Y_j (see [1, Theorem 22.1, Lemma 22.3]).

Let $\varphi \in \varepsilon_j \mathbf{E} \varepsilon_i$ for some $1 \leq i \leq n$ be such that $[\varphi] \in \mathcal{K}_{S,j}$ is maximal, where $S := \varphi(Y) \leq Y_j$. We claim that S is simple. If not, let S_0 and S_1 denote the socle and the head of S, respectively. Suppose first that Y_i is projective. Let Y_l denote the indecomposable direct summand of Y which is in the Ω^2-orbit of Y_j and which has socle S_1 (such a direct summand exists by Proposition 3.6(c)). Then there is an embedding

$\psi : Y_l \to Y_i$ such that $\psi(Y_l) \not\leq \ker(\varphi)$. Indeed, the multiplicity of S_0 as a composition factor of Y_i is equal to the multiplicity of S_0 as a composition factor of Y_l. Thus $\varphi\psi \neq 0$, contradicting the maximality of $[\varphi]$. Finally, suppose that Y_i is not projective. Then there is an $1 \leq l \leq n$ and a homomorphism $\psi : Y_l \to Y_i$ with $\psi(Y_l) = \mathrm{rad}(Y_i)$. Again, $\varphi\psi \neq 0$, a contradiction.

We have thus proved that S is simple, which implies $S = \mathrm{soc}(Y_j)$. It now follows from Lemma 2.16 and the fact that there is at most one non-projective direct summand of Y with a given head, that $\mathcal{K}_{S,j}$ has a unique maximal element, and thus $\mathrm{soc}(\mathbf{E}_j)$ is simple. The last assertion follows from the fact that there are at most two non-isomorphic indecomposable direct summands of Y with isomorphic socles.

3.3.6. *Some examples.* In order to describe the modular Hecke algebra $\mathbf{E} = \mathrm{End}_{kG}(\mathrm{Ind}_P^G(k))$ in case P is cyclic, we have to determine the $\mathbf{B}$-components of $\mathrm{Ind}_P^G(k)$ for the blocks $\mathbf{B}$ of kG. By Lemma 3.5 and Proposition 3.6, this can be done locally, i.e. inside the p-nilpotent group $C = C_G(P)$. Mazza has shown in [8], that all indecomposable endopermutation kP-modules occur as sources of simple modules in p-nilpotent groups with Sylow subgroup P. But by Lemma 3.4, apart from the sources, we have to take into account all indecomposable summands of the restrictions of the simple kC-modules to P.

Let us use Mazza's construction to consider two specific examples. Let P denote the cyclic group of order 7^2. Then P acts on a group M of order $13^3 \cdot 97^3$, the direct product of two extraspecial groups. By [8, 4.1, Theorem 5.3], the semidirect product PM has a simple module V of dimension $13 \cdot 97$ such that $Res_P^{PM}(V) = (J(6,k) \oplus J(7,k)) \otimes (J(48,k) \oplus J(49,k)) \cong J(42,k) \oplus J(43,k) \oplus J(49,k)^{24}$. (The indecomposable direct summands of these tensor products can be computed with [11].)

Next, let P denote the cyclic group of order 17^2, and let $q = 577 = 2 \cdot 17^2 - 1$. By Mazza's construction, we get an action of P on the extraspecial q-group M_1 of order q^3 and exponent q in such a way that the semidirect product PM_1 has a representation V_1 of dimension 577 and $\mathrm{Res}_P^{PM_1}(V_1) = J(17^2 - 1, k) \oplus J(17^2, k)$. Now let M_2 denote the extraspecial group 2_-^{8+1} of minus type and order 512. Its automorphism group is an extension of an elementary abelian group of order 256 by the orthogonal group $O^-(8,2)$ (see [13, Theorem 1]). Let Q denote a Sylow 17-subgroup of $\mathrm{Aut}(M_2)$. A computation with GAP (see [5]) shows that Q has exactly two fixed points in its action on M_2 by conjugation. There is a simple kM_2-module V_2 of dimension 16, unique up to isomorphism. Letting P act on M_2 via the projection $P \to Q$, we find that V_2 extends to a kPM_2-module, also denoted by V_2. Since P has exactly one fixed point on $V_2^* \otimes_k V_2$, we have $\mathrm{Res}_P^{PM_2}(V_2) = J(16, k)$. Combining, we obtain an action of P on $M := M_1 \times M_2$, and a simple kPM-module $V = V_1 \otimes_k V_2$ such that $\mathrm{Res}_P^{PM}(V) = J(273, k) \oplus J(17^2, k)^{31}$. Thus the corresponding 17-block of PM contains a unique non-projective direct summand of $\mathrm{Ind}_P^{PM}(k)$, namely $J(273, V)$.

It is at least feasible, though not very likely, that there is a non-17-solvable group G such that $PM = C_G(D_1)/D_1$, where D_1 is the subgroup of order 17 of a cyclic defect group $\hat{P}$ of order 17^3 of a block $\mathbf{B}$ of G, whose Brauer tree is a straight line with 4 edges, say. Then, by the results of this section, the non-projective $\mathbf{B}$-component Y of $\mathrm{Ind}_{\hat{P}}^G(k)$ would consist of a direct sum of four indecomposable modules whose Brauer correspondents all have length 273. In this case, the heads of these modules would not be simple. Moreover, by the results of our first section, the Hom-functor F corresponding to $\mathbf{E} = \mathrm{End}_{kG}(Y)$ would not have the property that $F(S)$ has a simple socle for all simple $\mathbf{A}$-modules S. Such a hypothetical configuration could presumably only be ruled with the help of the classification of the finite simple groups.

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