Endomorphism rings of permutation modules

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Abstract

Let k be an algebraically closed field of characteristic p and G a finite group. Then the permutation module k_P^G of G on the cosets of a Sylow p-subgroup P is via Fitting correspondence strongly related to its endomorphism ring $\mathrm{End}_{kG}(k_P^G)$. On the other hand, each Green correspondent in G of a weight module of G occurs as a direct summand of k_P^G. This fact suggests to analyze both structures, the permutation module and the associated endomorphism ring towards hints at a proof for Alperin's weight conjecture. We present a selection of such investigations for different groups and characteristics. In particular we focus on the socle and head constituents of the indecomposable direct summands of k_P^G and of the PIMs of $\mathfrak{E}_{\mathfrak{E}}$.

1 Introduction

This paper is motivated by Alperin's suggestion to investigate his weight conjecture ([Alp87]) through the endomorphism rings of certain permutation modules. Let k be an algebraically closed field of positive characteristic p, G a finite group, and P a Sylow p-subgroup of G. We then write k_P^G for the permutation kG-module on the cosets of P in G, and put $\mathfrak{E} := \mathrm{End}_{kG}(k_P^G)$. In this paper we present a collection of explicit computational results for such endomorphism algebras. In particular, we find their Cartan matrices and study the socles and heads of their PIMs. Alperin showed in [Alp87, La. 1] that the Green correspondents in G of weight modules (which we call weight Green correspondents in the sequel) are isomorphic to indecomposable direct summands of k_P^G. In our computations we identify the weight Green correspondents in a decomposition of k_P^G into indecomposable direct summands.
The special property of $\mathfrak{E}$ being quasi-Frobenius has been examined by Green in [Gre78]. Green's results show, that in the case where $\mathfrak{E}$ is quasi-Frobenius, the number of isomorphism types of simple constituents in the socle of $\mathfrak{E}_{\mathfrak{E}}$ is

equal to the number of indecomposable direct summands of k_P^G and to the number of isomorphism classes of simple kG-modules. We use his results to give the following equivalent formulation of Alperin's weight conjecture in this case. Let k, G, P, and $\mathfrak{E}$ be as above. Assume furthermore that $\mathfrak{E}$ is quasi-Frobenius. Then Alperin's weight conjecture is true for G if and only if each isomorphim type of indecomposable direct summand of k_P^G is a weight Green correspondent.
The question arises if there is a structural relation between the constituents of $\mathrm{soc}(\mathfrak{E}_{\mathfrak{E}})$ and the weight modules not only when $\mathfrak{E}$ is quasi-Frobenius, but in the general case. In this context, we focussed on the analysis of the socle constituents of the PIMs of $\mathfrak{E}_{\mathfrak{E}}$ in correspondence to the indecomposable direct summands of k_P^G. The investigations indicate that it might be worthwhile to concentrate on groups for which the number of simple socle constituents of $\mathfrak{E}_{\mathfrak{E}}$ (up to isomorphism) is equal to the number of simple kG-modules (up to isomorphism). In this case, we might use the simple socle constituents of $\mathfrak{E}_{\mathfrak{E}}$ as an intermediate tool to structurally connect simple kG-modules with weight Green correspondents. There are several million pairs (G, p) for a finite group G and a prime p which we proved to satisfy this assumption. But it should be noted that we have also found one exception, namely M_{11}, $p = 3$ (see Page 20).
The results presented here arose from my dissertation under the supervision of Prof. Gerhard Hiss.

2 Preliminaries

Throughout this section we fix an algebraically closed field k of characteristic $p > 0$ and a finite group G. We assume that all modules are right modules. Homomorphisms are written on the left, and for two homomorphisms φ, ψ, the composition is given as $\varphi \circ \psi(x) = \varphi(\psi(x))$. With this convention, $\mathrm{End}_{kG}(M)$ and $\mathrm{Hom}_{kG}(M, N)$ are right $\mathrm{End}_{kG}(M)$-modules for kG-modules M, N.

Definition 2.1. Let Q be a p-subgroup of G. If S is a simple $kN_G(Q)$-module with vertex Q, we call (Q, S) a **weight of G with respect to Q**. In this case, Q is a **weight subgroup**, S is a **weight module** and the Green correspondent of S in G a **weight Green correspondent**.

If the weight module S belongs to the block $\mathbf{b}$ of $kN_G(Q)$ with defect group $D_{\mathbf{b}}$, then $C_G(D_{\mathbf{b}}) \leq C_G(Q) \leq N_G(Q)$. Hence $\mathbf{B} := \mathbf{b}^G$ is defined. In this case we say that the weight (Q, S) belongs to $\mathbf{B}$. Note that the Green correspondent of S belongs to $\mathbf{B}$ by [Alp93, Cor. 14.4]. Note also that conjugation induces an equivalence relation on the set of weights of G.

Remark 2.2. (a) If $Q \leq G$ is a weight subgroup then Q is p-radical, i. e. Q is equal to the largest normal p-subgroup of $N_G(Q)$ (compare [Lin04, p.

229]).
(b) Let (Q, S) be a weight of G. Then S is a simple projective $kN_G(Q)/Q$-module. Conversely, each projective simple $kN_G(Q)/Q$-module is a weight module after inflation along Q.

We are now ready to state Alperin's weight conjecture, proposed in [Alp87].

Conjecture 2.3 (Alperin's Weight Conjecture)**.** Let $\mathbf{B}$ be a block of G. Then the number of weights (up to conjugation) belonging to $\mathbf{B}$ is equal to the number of simple kG-modules (up to isomorphism) belonging to $\mathbf{B}$.

Note that a weaker form of the conjecture states that the number of weight modules for G (up to conjugation) is equal to the number of simple kG-modules (up to isomorphism). The conjecture has been proved for many different families of groups and blocks, such as for blocks with cyclic defect groups ([Dad66]), for p-solvable groups ([Oku], [IN95]), for symmetric groups ([AF90], [FS82]), for groups of Lie type in defining characteristic ([Cab84]), for general linear groups in non defining characteristic ([AF90]) and for many sporadic groups ([AC95], [An97], [Dad92], [EP99], [Szö98]) etc. Weight modules are strongly related to trivial source modules, as can be seen in the following lemma.

Lemma 2.4 ([Alp87, La. 1])**.** *Let $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. If S is a weight module of G, then its Green correspondent is isomorphic to an indecomposable direct summand of k_P^G.*

In view of a general proof of the weight conjecture, Alperin has suggested to analyze certain alternating sums (the relation of which to the weight conjecture is treated in [KR89]) as well as the endomorphism ring $\mathfrak{E} := \mathrm{End}_{kG}(k_P^G)$ of the permutation module k_P^G of G on the cosets of a Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$. The latter proposal is the motivation for the present work. In [Nae08] we analyzed, for a large number of finite groups in different characteristics, the endomorphism rings of the associated permutation modules k_P^G. We present a selection of this analysis here. The hope is to find patterns in the structural properties of these endomorphsim rings which could give a hint for a proof of the weight conjecture.
As indicated in the previous lemma, we need to identify weight Green correspondents in a decomposition of k_P^G. This is possible by the following observation.

Lemma 2.5 ([Szö98, Prop. 6.1.5])**.** *Let (Q, S) be a weight with $p^\alpha := |G : Q|_p$. Denote the Green correspondent of S by X_S. Then $\dim(X_S)_p = p^\alpha$ and X_S is the only indecomposable direct summand of S^G with dimension not divisible by $p^{\alpha+1}$.*

The above Lemmas give us a strategy to identify weight Green correspondents in a decomposition of k_P^G. We start with a p-radical subgroup Q and check if there are projective simple $kN_G(Q)/Q$-modules. If so, we induce these to G and decompose them into indecomposable modules. By the previous lemma we can easily identify the associated weight Green correspondent.

Remark 2.6. (a) Let X be an indecomposable direct summand of k_P^G. Then X is also a direct summand of S^G, where S is some weight module with vertex P (even if the vertex of X is $Q \subsetneqq P$): We may write $k_P^G = (k_P^{N_G(P)})^G$, and the indecomposable direct summands of $k_P^{N_G(P)}$ are exactly the weight modules with weight subgroup P. This observation helps to find decompositions of induced weight modules in some cases (compare $G = L_2(13)$ in characteristic 3, Page 34).
(b) If $P \trianglelefteq G$ is a normal Sylow p-subgroup, then $\mathfrak{E}$ is isomorphic to (the semisimple algebra) kG/P as k-algebra. In this case P is the only weight subgroup and each simple kG-module is a weight module.
(c) If X is an indecomposable direct summand of k_P^G, then $\mathrm{Hom}_{kG}(k_P^G, X)$ is a PIM of $\mathfrak{E}$ by Fitting correspondence.
(d) Note that each isomorphism type of simple kG-module occurs as head and socle constituent of k_P^G.

Definition 2.7. We denote the covariant and left exact fixed point functor Fix_P by $\mathcal{F}$, i. e. for a finitely generated kG-module M we have

$$\mathcal{F}(M) = \{m \in M : mx = m \text{ for all } x \in P\}$$

and homomorphisms are mapped to the respective restrictions. Note that $\mathcal{F}(M)$ and $\mathrm{Hom}_{kG}(k_P^G, M)$ are isomorphic as k-vector spaces. Via this isomorphism $\mathcal{F}(M)$ may be equipped with a right $\mathfrak{E}$-module structure.

3 Quasi-Frobenius

We keep the assumptions and notation as in the previous section. If the endomorphism ring of the permutation module k_P^G is quasi-Frobenius, we can use the results of Green's work ([Gre78]) to get Alperin's conjecture in terms of the indecomposable direct summands of k_P^G. The decomposition of k_P^G into indecomposable direct summands will be denoted as follows:

$$k_P^G = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n_i} X_{ij}, \tag{1}$$

where $X_{ij} \cong X_{lm}$ if and only if $i = l$. By Fitting correspondence we get an associated decomposition of the regular module $\mathfrak{E}_{\mathfrak{E}}$ into PIMs.

Theorem 3.1 ([Gre78, Thm. 1, Thm. 2]). *Let the notation be as above. Moreover, assume that $\mathfrak{E}$ is quasi-Frobenius, i. e. injective as right $\mathfrak{E}$-module. Then, for any $1 \leq i, l \leq n$:*

(a) $\mathrm{soc}(X_{i,1})$ *and* $\mathrm{hd}(X_{i,1})$ *are simple.*
(b) $\mathcal{F}(\mathrm{soc}(X_{i,1})) = \mathrm{soc}(\mathcal{F}(X_{i,1}))$ *and* $\mathcal{F}(\mathrm{hd}(X_{i,1})) = \mathrm{hd}(\mathcal{F}(X_{i,1}))$*.*
(c) The kG*-modules* $X_{i,1}$ *and* $X_{l,1}$ *are isomorphic if, and only if* $\mathrm{soc}(X_{i,1}) \cong \mathrm{soc}(X_{l,1})$ *or* $\mathrm{hd}(X_{i,1}) \cong \mathrm{hd}(X_{l,1})$*.*
(d) The map $M \mapsto \mathcal{F}(M)$ *induces a bijection between the simple* kG*-modules and the simple* $\mathfrak{E}$*-modules.*

Remark 3.2. (a) For a similar approach as Green's, compare also with [Gec01] and [GH97].
(b) In view of Alperin's weight conjecture, the case where $\mathfrak{E}$ is quasi-Frobenius is of special interest. From Lemma 2.4 we know that the number of weights of kG (up to conjugation) must be less than or equal to the number n of isomorphism types of indecomposable direct summands of k_P^G. On the other hand, by Fitting correspondence, $\mathfrak{E}$ has n isomorphism types of simple modules.
Suppose now that $\mathfrak{E}$ is quasi-Frobenius. Then we have by Theorem 3.1

$$\begin{aligned} |\{\text{weights}\}_{\sim}| &\leq |\{\text{indecomposable direct summands of } k_P^G\}_{\cong}| \\ &= |\{\text{simple } \mathfrak{E}\text{-modules}\}_{\cong}| \\ &= |\{\text{simple } kG\text{-modules}\}_{\cong}|. \end{aligned}$$

This implies that Alperin's conjecture is true if and only if *each* direct summand of k_P^G is a weight Green correspondent. Since each indecomposable direct summand X of k_P^G has trivial source, so has its Green correspondent T. This implies that the vertex Q of T lies in the kernel of T and T has a $kN_G(Q)/Q$-module structure. As Q is the vertex, T is even a projective $kN_G(Q)/Q$-module. Hence Alperin's conjecture is true if and only if the Green correspondent of each indecomposable direct summand of k_P^G is a simple $kN_G(Q)$-module.
Note that the assumption that all indecomposable direct summands of k_P^G are weight Green correspondents does not imply that $\mathfrak{E}$ is quasi-Frobenius, as can be seen in the examples M_{22} in characteristic 2 (compare Page 27) and $U_3(3)$ in characteristic 2 (compare Page 29). Each indecomposable direct summand X of k_P^G is a weight Green correspondent in those examples. But the socle constituents of the corresponding PIMs of $\mathfrak{E}$ show that $\mathfrak{E}$ is not quasi-Frobenius.

Cabanes proved the weight conjecture in [Cab84] for groups of Lie type in defining characteristic. In this case the endomorphism ring $\mathfrak{E}$ is in fact quasi-Frobenius ([Tin80]). Cabanes succeeds in linking the representations of a group of Lie type in defining characteristic and the representations of its parabolic subgroups by means of the Brauer homomorphism. In this way Cabanes shows that the image of each indecomposable direct summand of k_P^G (with vertex Q) under the Brauer homomorphism, i.e. the Green

correspondent of X, is projective simple as $kN_G(Q)/Q$-module and hence a weight module.
As we have seen above, the case when $\mathfrak{E}$ is quasi-Frobenius is of special interest. On the one hand, we then have an equivalent assertion for Alperin's weight conjecture. On the other hand, Cabanes succeeded with the help of this property in proving the weight conjecture for groups of Lie type. It would perhaps be worthwhile to prove Alperin's conjecture just from the assumption that $\mathfrak{E}$ is quasi-Frobenius. This motivates for a characterization of pairs (G, P) such that the endomorphism ring $\mathfrak{E}$ of k_P^G is quasi-Frobenius. Therefore each example below is also analyzed with respect to this property.

4 The socle of $\mathfrak{E}$

Let the notation be as above. With Green's theorem in mind we focus on the socles and heads of the PIMs of $\mathfrak{E}$ and determine them not only in the quasi-Frobenius case, but in all cases. This analysis gives rise to an equivalence relation on the set of PIMs of $\mathfrak{E}$ and on the set of socle constituents of $\mathfrak{E}_{\mathfrak{E}}$, which we describe now.
Let the decomposition of k_P^G be as in (1). We write $\mathfrak{E}_{\mathfrak{E}} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n_i} P_{ij}$, where X_{ij} is associated to P_{ij} by Fitting correspondence.

Remark 4.1. (a) Define the following relation $\sim$ on the set of isomorphism classes of PIMs of $\mathfrak{E}$: $P_{i,1} \sim P_{m,1}$ if and only if $P_{i,1}$ and $P_{m,1}$ have a socle constituent in common. The transitive closure $\simeq$ of $\sim$ then induces an equivalence relation on the set of isomorphism classes of PIMs of $\mathfrak{E}$. The equivalence classes of $\simeq$ are denoted by $\mathcal{P}_l$ for $1 \leq l \leq s$.
(b) The previous equivalence relation obviously induces an equivalence relation on the set of isomorphism classes of indecomposable direct summands of k_P^G by Fitting correspondence.
(c) Let $\mathcal{S} := \{S \leq \mathrm{soc}(\mathfrak{E}_{\mathfrak{E}}) : S \text{ simple}\}_{\cong}$. The equivalence relation in (a) induces an equivalence relation $\approx$ on $\mathcal{S}$ as follows: $S \approx S'$ if and only if there is an equivalence class $\mathcal{P}$ such that S is a socle constituent of $P_{i,1} \in \mathcal{P}$ and S' is a socle constituent of $P_{j,1} \in \mathcal{P}$. The equivalence classes of $\approx$ are denoted by $\mathcal{S}_l$.

Our experiments have led to the idea of investigating groups for which the following hypothesis holds.

Hypothesis 4.2. Let k be an algebraically closed field of characteristic p. Then for G, $P \in \mathrm{Syl}_p(G)$, $\mathfrak{E} := \mathrm{End}_{kG}(k_P^G)$ and $\mathcal{S} := \{S \leq \mathrm{soc}(\mathfrak{E}_{\mathfrak{E}}) : S \text{ simple}\}_{\cong}$ the following holds:
(a) $|\mathcal{S}| = |\{(S, Q) : (S, Q) \text{ is a weight of } G\}_{\sim}|$.
(b) $|\mathcal{S}| = |\{M : M \text{ is a simple } kG\text{-module}\}_{\cong}|$.

Remark 4.3. (a) If one can characterize the groups for which the above hypothesis is satisfied, then Alperin's weight conjecture holds for these group as well. It is therefore a promising question how the groups for which the hypothesis holds can be classified.
(b) We have found one example, namley M_{11} in characteristic 3, for which Hypothesis 4.2 is not satisfied.
(c) GAP provides a data base with all small groups up to order 2000. We have checked and proved the hypothesis on several millions of such groups without finding but the one in (b).
(e) There is a further observation we have made. We fix an equivalence class $\mathcal{P}$ on the PIMs of $\mathfrak{E}$ and let t be the number of simple socle constituents (up to isomorphism) belonging to $\mathcal{P}$ via $\approx$. Assume that there are t simple indecomposable direct summands of k_P^G associated to the PIMs in $\mathcal{P}$ whose p-part of their dimensions is less than the p-part of the dimensions of the remaining summands belonging to $\mathcal{P}$. Then these t summands are weight Green correspondents in all our examples we have analyzed so far. It turns out, that the assumption we have made on t is satisfied in most cases. An exception can be found in $\mathcal{P}_2$ of S_7 in characteristic 2 (see Page 13).

5 Computational methods

In all our computations we used the system GAP4 ([GAP08]) and the MeatAxe ([Mea07]).
We computed the constituents and their multiplicities of the (ordinary) permutation character 1_P^G. For a splitting field K of charactertistic 0 of G, we have $\dim_K(\mathrm{Hom}(K_P^G, M_\chi)) = m_\chi$ for each such constituent χ with corresponding module M_χ and multiplicity m_χ in 1_P^G (see [CR81, Thm. 11.25]). This information helped us to associate direct summands of k_P^G to PIMs of $\mathfrak{E}$. The GAP-function `MeatAxeStringRightCoset(`G`,`P`,`q`)`, which we wrote, first computes generators of the permutation group of G on the right cosets of P in G. Then it returns these generators as permutation matrices over a field of order q in MeatAxe-readible form. These matrices are passed on to the MeatAxe. In [Szö98], Szöke has developed programs which allow to compute the decomposition of modules into indecomposable direct summands as well as the radical and socle series of a module. We used these programs for decomposing the permutation module, the right regular representation for the endomorphism ring, and for determining the socles and heads of the summands. With the help of the Modular Atlas[1], the composition series of each direct summand of k_P^G, the dimensions of the simple modules of $\mathfrak{E}$ in characteristic 0, and the composition series of each PIM of $\mathfrak{E}$, we determined the irreducible consituents of the lift of each indecomposable direct summand of k_P^G. After that we were able to associate an

[1] `http://www.math.rwth-aachen.de/MOC/decomposition/`

indecomposable direct summand of k_P^G to a PIM of $\mathfrak{E}$ (in characteristic p). Finally, we had to determine the weight Green correspondents among the indecomposable direct summands of k_P^G. For this matter, we used the GAP-function `TableOfMarks(`G`)`, to filter possible p-subgroups of P which might be weight subgroups. If Q was such a candidate, we then tested if there were simple projective $kN_G(Q)/Q$-modules. If so, we passed the matrices returned by `MeatAxeRightCoset(`$N_G(Q)$`,`Q`,`q`)` to the MeatAxe, extracted the matrices *mats* for the projective simple modules and induced them in GAP with the function `InducedGModules(`G`,`$N_G(Q)$`,`*mats*`)` to G. This module was then decomposed by the MeatAxe. With Lemma 2.5 we could determine the weight Green correspondent in the induced module.

6 Examples

In this section we present a collection of examples which have been analyzed in [Nae08]. If kG has no projective simple modules, we print two matrices, marked by k_P^G and C_P^G, respectively. The columns of these matrices correspond to the indecomposable direct summands of k_P^G and $\mathfrak{E}_{\mathfrak{E}}$, respectively, and are indexed by their are dimensions. An exponent indicates the multiplicity of the respective summand in the decomposition. The rows of the matrices correspond to the simple kG- and $\mathfrak{E}$-modules, respectively and are indexed by their dimensions. An entry a_{ij} then denotes the multiplicity of the simple module of row i in the summand of column j. Thus C_P^G contains the Cartan invariants of $\mathfrak{E}$. This part of the matrix will be called *the body* of C_P^G. The projective simple kG-modules correspond to projective simple $\mathfrak{E}$-modules and therefore to 1×1-blocks in the Cartan matrix of $\mathfrak{E}$. We omit these summands in the matrix indexed by k_P^G and the associated PIMs in C_P^G. If kG has projective simple modules, we additionally print the decomposition of k_P^G into indecomposable direct summands.
The rows of the matrices indexed by "soc" and "hd" display the isomorphism type of the socle and head of the corresponding direct summand, respectively. If these modules are not simple and if there is no risk of misunderstanding we will omit $\oplus$-signs. The Fitting correspondence is indicated in the first row, labeled by k_P^G, below the body of the matrix C_P^G. The weight Green correspondents are printed boldly.
Moreover, the row labeled by p^x gives the p-part of the direct summand of the corresponding column. Finally, the second to the last row of the matrix is C_P^G indexed by EC and displays the equivalence classes as described in Remark 4.1. The corresponding equivalence classes $\mathcal{S}_i$ are not printed separately but can be read off right from the rows indexed by soc and EC. Instead, we indicate in the last row the number s of constituents belonging to the respective $\mathcal{S}_i$.
We will always denote a weight module of dimension m with vertex Q by

${}^{m}S_{N_G(Q)}$. We write $N := N_G(P)$. As far as possible we have also given the decomposition of the induced weight modules.
The selection of examples cover –apart from the sporadic groups we examined– some particular cases: We chose the pair of groups A_7/S_7 to illustrate how Clifford theory applies. Moreover, M_{11} has in characteristic 5 a cyclic block which is not only a real stem. The group $L_2(17)$ is of special interest, as it has in characteristic 3 a cyclic defect group of order 3^2. Finally we chose some groups of Lie type in defining characteristic to present some of the special properties we have described in the previous sections.

6.1 A_6 mod 2

$\boxed{p = 2,\ G = A_6,\ P \in \mathrm{Syl}_2(G),\ N_G(P) = P,\ |G| = 2^3 \cdot 3^2 \cdot 5}$

The permutation module decomposes into indecomposable direct summands as follows:

$$k_P^G = 1 \oplus 8_1 \oplus 8_2 \oplus 14_1 \oplus 14_2.$$

k_P^G	**1**	$\mathbf{14_1}$	$\mathbf{14_2}$
1	1	2	2
4_1		2	1
4_2		1	2
soc	1	4_1	4_2
hd	1	4_1	4_2

C_P^G	1_1	3_1	3_2
1_1	1		
1_2		2	1
1_3		1	2
k_P^G	**1**	$\mathbf{14_1}$	$\mathbf{14_2}$
2^x	2^0	2	2
soc	1_1	1_2	1_3
hd	1_1	1_2	1_3
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
s	1	1	1

The computations show that all indecomposable direct summands of k_P^G are uniserial.
Note that $A_6 \cong \mathrm{Sp}_4(2)'$, the first derived group of a group of Lie type. We apply Baer's criterion to prove that $\mathfrak{E}$ is quasi-Frobenius. From the Cartan matrix it is easy to find the regular representation of the algebra $\tilde{\mathfrak{E}} := \varepsilon_1\mathfrak{E}\varepsilon_1 \oplus \varepsilon_2\mathfrak{E}\varepsilon_2$, where ε_1 and ε_1 are the idempotents in $\mathfrak{E}$ associated to 3_1 and 3_2, respectively. Considering all possible ideals of $\tilde{\mathfrak{E}}$, and all homomorphisms from those ideals to 3_1 and 3_2, respectively, we see that all such homomorphisms may be extended to $\tilde{\mathfrak{E}}$. Hence $\tilde{\mathfrak{E}}$ and therefore $\mathfrak{E}$ is quasi-Frobenius.
Both indecomposable direct summands 8_1 und 8_2 of k_P^G are simple projective kG-modules. Moreover, the vertices Q_1 and Q_2 of the indecomposable direct summands 14_1 and 14_2 of k_P^G, respectively, have order 4 and are not conjugate in G.

For the induced weight modules (each of dimension 2) we find:

$$\begin{aligned} {}^2S^G_{N(Q_1)} &= \mathbf{14_1} \oplus 8_1 \oplus 8_2, \\ {}^2S^G_{N(Q_2)} &= \mathbf{14_2} \oplus 8_1 \oplus 8_2. \end{aligned}$$

For the trivial weight module we have:

$${}^1S^G_N = \mathbf{1} \oplus 8_1 \oplus 8_2 \oplus 14_1 \oplus 14_2.$$

Finally we notice that each PIM lies in an equivalence class with one element.

6.2 S_6 mod 3

$\boxed{p = 3,\ G = S_6,\ P \in \mathrm{Syl}_3(G),\ |G| = 2^4 \cdot 3^2 \cdot 5}$

The permutation module decomposes into indecomposable direct summands as follows:

$$k_P^G = 1_1 \oplus 1_2 \oplus 9_1 \oplus 9_2 \oplus 10_1 \oplus 10_2 \oplus 20^2.$$

k_P^G	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{10_1}$	$\mathbf{10_2}$	$\mathbf{20}^2$
1_1	1		1	1	
1_2		1	1	1	
4_1			2		1
4_2				2	1
6					2
soc	1_1	1_2	4_1	4_2	6
hd	1_1	1_2	4_1	4_2	6

C_P^G	1_1	1_2	2_1	2_2	4^2
1_1	1				
1_2		1			
1_3			2		
1_4				2	
2					2
k_P^G	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{10_1}$	$\mathbf{10_2}$	$\mathbf{20}$
3^x	3^0	3^0	3^0	3^0	3^0
soc	1_1	1_2	1_3	1_4	2
hd	1_1	1_2	1_5	1_6	2
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$
s	1	1	1	1	1

Remark: The Sylow 3-subgroups are isomorphic to $C_3 \times C_3$.
The indecomposable direct summands 10_1, 10_2 and 20 of k_P^G have simple head and socle but are not uniserial as computation has shown. Moreover, each indecomposable direct summand of k_P^G is a weight Green correspondent. Thus S_6 satisfies Hypothesis 4.2 in characteristic 3 as each PIM of $\mathfrak{E}$ lies in an equivalence class with one element.
The socle of each PIM of $\mathfrak{E}$ is simple. From the Cartan matrix we deduce that $\mathfrak{E}$ is quasi-Frobenius.
All indecomposable direct summands of k_P^G, except for the simple projective ones 9_1 and 9_2, lie in the principal block and have a Sylow 3-subgroup as vertex. The induced weight modules decompose as follows:

$$
\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
{}^{1_1}S_N^G & = & \mathbf{1_1} \oplus 9_1, & {}^{1_2}S_N^G & = & \mathbf{1_2} \oplus 9_2, & {}^{2}S_N^G & = & \mathbf{20}, \\
{}^{1_3}S_N^G & = & \mathbf{10_1}, & {}^{1_4}S_N^G & = & \mathbf{10_2}. & & &
\end{array}
$$

6.3 $A_7 \mod 2$ and $A_7 \mod 3$

$\boxed{p = 2,\ G = A_7,\ P \in \mathrm{Syl}_2(G),\ N_G(P) = P,\ |G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7}$

k_P^G	$\mathbf{1}$	$\mathbf{14_1}$	56^2	64	$\mathbf{70}$	$\mathbf{6}$	20	$\mathbf{14_2}$	$\mathbf{14_3}$
1	1		2	2	2				
14		1	1	3	2				
20			2	1	2				
4_1							1	1	1
4_2							1	1	1
6						1	2	1	1
soc	1	14	20	14	$14 \oplus 20$	6	6	4_1	4_2
hd	1	14	20	14	$14 \oplus 20$	6	6	4_2	4_1

C_P^G	1	3_1	7^2	8	10	2	5	3_2	3_3
1_1	1								
1_2		1		1	1				
2			2	1	2				
1_3		1	1	3	2				
1_4		1	2	2	3				
1_5						1	1		
1_6						1	2	1	1
1_7							1	1	1
1_8							1	1	1
k_P^G	$\mathbf{1}$	$\mathbf{14_1}$	56	64	$\mathbf{70}$	$\mathbf{6}$	20	$\mathbf{14_2}$	$\mathbf{14_3}$
2^x	2^0	2	2^3	2^6	2	2	2^2	2	2
soc	1_1	1_3	2	1_3	$1_3 2$	1_6	1_6	1_8	1_7
hd	1_1	1_2	2	1_3	1_4	1_5	1_6	1_7	1_8
EC	$\mathcal{P}_1$		$\mathcal{P}_2$			$\mathcal{P}_3$		$\mathcal{P}_4$	$\mathcal{P}_5$
s	1		2			1		1	1

Remark: Note that we have omitted the $\oplus$-sign in the socle of the PIM 10 of $\mathfrak{C}$. The weight Green correspondents $6, 14_2, 14_3$ und 70 have a vertex Q_1 of order 4. The induced weight modules decompose as follows:

$$
\begin{array}{lcl@{\qquad}lcl}
{}^{2_1}S_{N(Q_1)}^G & = & \mathbf{70}, & {}^{2_2}S_{N(Q_1)}^G & = & \mathbf{6} \oplus 64, \\
{}^{2_3}S_{N(Q_1)}^G & = & \mathbf{14_2} \oplus 56, & {}^{2_4}S_{N(Q_1)}^G & = & \mathbf{14_3} \oplus 56.
\end{array}
$$

The weight Green correspondent 14_1 has vertex Q_2 of order 4 which is not conjugate in G to Q_1. We have:

$$
{}^{2}S_{N(Q_2)}^G = \mathbf{14_1} \oplus 20 \oplus 64 \oplus 56^2.
$$

From the latter decomposition we see, that the direct summand 20 of k_P^G has a vertex Q_3 of order less than $4 = |Q_2|$, hence $|A_7 : Q_3|_2 \in \{2^2, 2^3\}$. As $|20|_2 = 2^2$ we see that the order of Q_3 must be 2.
As $N_G(P) = P$ we have ${}^1S_{N(Q)}^G = k_P^G$ for the induction of the trivial weight module, and the trivial module is the only weight module with vertex P.

$\boxed{p = 3,\ G = A_7,\ P \in \mathrm{Syl}_3(G),\ |G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7}$

k_P^G	$\mathbf{1}$	$\mathbf{15}^2$	36	$\mathbf{6}^2$	$\mathbf{10_1}$	45_1	$\mathbf{10_2}$	45_2	$\mathbf{28}$	63
1	1					2		2	2	4
15		1	2							
6			1	1						
10_1					1	2		1		1
10_2						1	1	2		1
13						1		1	2	3
soc	1	15	15	6	10_1	10_1	10_2	10_2	13	13
hd	1	15	15	6	10_1	10_1	10_2	10_2	13	13

C_P^G	1	3^2	4_2	2_1^2	2_2	5_1	2_3	5_2	4_2	7
1_1	1									
2_1		1	1							
1_2		1	2							
2_2				1						
1_3					1	1				
1_4					1	2		1		1
1_5							1	1		
1_6						1	1	2		1
1_7									2	2
1_8						1		1	2	3
k_P^G	$\mathbf{1}$	$\mathbf{15}$	36	$\mathbf{6}$	$\mathbf{10_1}$	45_1	$\mathbf{10_2}$	45_2	$\mathbf{28}$	63
3^x	3^0	3	3^2	3	3^0	3^2	3^0	3^2	3^0	3^2
soc	1_1	1_2	1_2	2_2	1_4	1_4	1_6	1_6	1_8	1_8
hd	1_1	2_1	1_2	2_2	1_3	1_4	1_5	1_6	1_7	1_8
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$	$\mathcal{P}_4$		$\mathcal{P}_5$		$\mathcal{P}_6$	
s	1	1		1	1		1		1	

Remark: The Sylow 3-subgroups are isomorphic to $C_3 \times C_3$.
The weight Green correspondents 6 and 15 have vertex Q_1 of order 3. The corresponding induced weight modules decompose as follows:

$$ {}^{3_1}S_{N(Q_1)}^G = \mathbf{6} \oplus 36 \oplus 63 \qquad {}^{3_2}S_{N(Q_1)}^G = \mathbf{15} \oplus 45_1 \oplus 45_2. $$

From this it follows that 36, 45_1, 45_2 and 63 are projective indecomposable direct summands of k_P^G. The remaining weight Green correspondents 1, 28,

10_1, and 10_2 have P as a vertex. The decompositions of the induced weight modules are given as:

$$
\begin{array}{rclrcl}
{}^{1_1}S_N^G & = & \mathbf{1} \oplus 6 \oplus 63, & {}^{1_2}S_N^G & = & \mathbf{28} \oplus 6 \oplus 36, \\
{}^{1_3}S_N^G & = & \mathbf{10_1} \oplus 15 \oplus 45_2, & {}^{1_4}S_N^G & = & \mathbf{10_2} \oplus 15 \oplus 45_1.
\end{array}
$$

6.4 S_7 mod 2 and S_7 mod 3

$\boxed{p = 2,\ G = S_7,\ P \in \mathrm{Syl}_2(G),\ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 7}$

k_P^G	**1**	**6**	20	**28**	**14**	112	64	**70**
1	1					4	2	2
6		1	2	2				
8			1	2				
14					1	2	3	2
20						4	1	2
soc	1	6	6	8	14	20	14	$14 \oplus 20$
hd	1	6	6	8	14	20	14	$14 \oplus 20$

C_P^G	1	2	4	3_1	3_2	7_1	7_2	8
1_1	1							
1_2		1	1					
1_3		1	2	1				
1_4			1	2				
1_5					1		1	1
1_6						4	1	2
1_7					1	1	3	2
1_8					1	2	2	3
k_P^G	**1**	**6**	20	**28**	**14**	112	64	**70**
2^x	2^0	2	2^2	2^2	2	2^4	2^6	2
soc	1_1	1_3	$1_3 1_4$	1_4	1_7	1_6	$1_6 1_7$	$1_6 1_7$
hd	1_1	1_2	1_3	1_4	1_5	1_6	1_7	1_8
EC	$\mathcal{P}_1$		$\mathcal{P}_2$				$\mathcal{P}_3$	
s	1		2				2	

Remark: Our computation has shown, that the indecomposable direct summands 6, 20 and 28 of k_P^G are uniserial and belong to a block whose defect groups have order 2^3. Note that the corresponding PIMs of $\mathfrak{E}$ do not all have simple socles.
The weight Green correspondent 28 has a vertex Q_1 of order 4. Inducing this weight module we get:

$$ {}^{1}S_{N(Q_1)}^G = \mathbf{28} \oplus 112. $$

The weight Green correspondents 70, 14 and 6 have pairwise non conjugate vertices Q_1, Q_2 and Q_3, respectively, of order 8. The decompositions of the induced weight modules are given as:

$$\begin{aligned} {}^2S^G_{N(Q_2)} &= \mathbf{70} \oplus 28 \oplus 112, \\ {}^2S^G_{N(Q_3)} &= \mathbf{14} \oplus 20 \oplus 64 \oplus 112, \\ {}^2S^G_{N(Q_4)} &= \mathbf{6} \oplus 28 \oplus 64 \oplus 112. \end{aligned} \tag{2}$$

The indecomposable direct summand 20 of k_P^G has a vertex of order 2^2 by Equation (2) and the fact that $|20|_2 = 2^2$.
As $N_G(P) = P$, we have ${}^1S_N^G = k_P^G$ for the trivial module.

$p = 3$, $G = S_7$, $P \in \mathrm{Syl}_3(G)$, $|G| = 2^4 \cdot 3^2 \cdot 5 \cdot 7$

k_P^G	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{28_1}$	63_1	$\mathbf{28}_2$	63_2	$\mathbf{20}^2$	90^2	$\mathbf{6}_1^2$	$\mathbf{6}_2^2$	$\mathbf{15}_1^2$	36_1	$\mathbf{15}_2^2$	36_2
1_1	1		1	2	1	2		2						
1_2		1	1	2	1	2		2						
13_1			2	3				1						
13_2					2	3		1						
20				1		1	1	3						
6_1									1			1		
6_2										1				1
15_1											1	2		
15_2													1	2
soc	1_1	1_2	13_1	13_1	13_2	13_2	20	20	6_1	6_2	15_1	15_1	15_2	15_2
hd	1_1	1_2	13_1	13_1	13_2	13_2	20	20	6_1	6_2	15_1	15_1	15_2	15_2

C_P^G	1_1	1_2	4_1	7_1	4_2	7_2	4_3^2	10^2	2_1^2	2_2^2	3_1^2	4_4	3_2^2	4_5
1_1	1													
1_2		1												
1_3			2	2										
1_4			2	3				1						
1_5					2	2								
1_6					2	3		1						
2_1							1	1						
2_2				1		1	1	3						
2_3									1					
2_4										1				
2_5											1	1		
1_7											1	2		
2_6													1	1
1_8													1	2
k_P^G	$\mathbf{1_1}$	$\mathbf{1_2}$	$\mathbf{28_1}$	63_1	$\mathbf{28_2}$	63_2	$\mathbf{20}$	90	$\mathbf{6_1}$	$\mathbf{6_2}$	$\mathbf{15_1}$	36_1	$\mathbf{15_2}$	36_2
3^x	3^0	3^0	3^0	3^2	3^0	3^2	3^0	3^2	3	3	3	3^2	3	3^2
soc	1_1	1_2	1_4	1_4	1_6	1_6	2_2	2_2	2_3	2_4	1_7	1_7	1_8	1_8
hd	1_1	1_2	1_3	1_4	1_5	1_6	2_1	2_2	2_3	2_4	2_5	1_7	2_6	1_8
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$		$\mathcal{P}_3$		$\mathcal{P}_4$		$\mathcal{P}_5$	$\mathcal{P}_6$	$\mathcal{P}_7$		$\mathcal{P}_8$	
s	1	1	1		1		1		1	1	1		1	

Remark: The Sylow 3-subgroups are isomorphic to $C_3 \times C_3$.
The weight Green correspondents $15_1, 15_2, 6_1, 6_2$ have a vertex Q of order 3.
The corresponding induced weight module decompose as follows:

$$\begin{aligned} {}^{3_1}S_{N(Q)}^G &= \mathbf{15_1} \oplus 90, & {}^{3_2}S_{N(Q)}^G &= \mathbf{6_1} \oplus 36_2 \oplus 63_1, \\ {}^{3_3}S_{N(Q)}^G &= \mathbf{15_2} \oplus 90, & {}^{3_4}S_{N(Q)}^G &= \mathbf{6_2} \oplus 36_2 \oplus 63_2. \end{aligned}$$

From the decomposition matrix of kG we see that the indecomposable direct summands 6_1, 6_2, 15_1, 15_2, 36_1 and 36_2 of k_P^G belong to blocks of kG with cyclic defect group of order 3. From this it is clear that the weight Green correspondents among them are uniserial ([Alp93, La. 22.3]). Note that the indecomposable direct summands 36_1, 36_2, 63_1, 63_2 and 90 of k_P^G are projective.
The remaining induced weight modules decompose as follows:

$$\begin{aligned} {}^{1_1}S_N^G &= \mathbf{1_1} \oplus 6_1 \oplus 63_1, & {}^{1_2}S_N^G &= \mathbf{28_1} \oplus 6_1 \oplus 36_1, \\ {}^{1_4}S_N^G &= \mathbf{1_2} \oplus 6_2 \oplus 63_2, & {}^{1_3}S_N^G &= \mathbf{28_2} \oplus 6_2 \oplus 36_2, \\ {}^{2}S_N^G &= \mathbf{20} \oplus 15_1 \oplus 15_2 \oplus 90. \end{aligned}$$

6.5 A_8 mod 3

$\boxed{p = 3,\ G = A_8,\ P \in \mathrm{Syl}_3(G),\ |G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7}$

The permutation module k_P^G decomposes as follows:

$k_P^G = 1 \oplus 45_1^5 \oplus 45_2^5 \oplus 7 \oplus 21^4 \oplus 27^2 \oplus 28 \oplus 34 \oplus 35^2 \oplus 63 \oplus 90 \oplus 99^2 \oplus 162^3 \oplus 225^3.$

k_P^G	$\mathbf{1}$	$\mathbf{21}^4$	63	$\mathbf{7}$	27^2	$\mathbf{34}$	90	$\mathbf{35}^2$	99^2	225^3	162^3	$\mathbf{28}$
1	1					1	2		1	2	1	
21		1	3									
7				1	2	1	2			2	1	
13					1	2	3			1		
28									1	2	3	1
35							1	1	2	4	2	
soc	1	21	21	7	7	13	13	35	35	35	28	28
hd	1	21	21	7	7	13	13	35	35	35	28	28

C_P^G	1	5^4	7_1	3	7_2^2	6	10	7_3^2	15^2	25^3	18^3	4
1_1	1											
4		1	1									
1_2		1	3									
1_3				1	1							
2_1				1	2	1	1					
1_4					1	2	2					
1_5					1	2	3			1		
2_2								1	1	1		
2_3								1	2	2	1	
3_1							1	1	2	4	2	
3_2									1	2	3	1
1_6											1	1
k_P^G	$\mathbf{1}$	$\mathbf{21}$	63	$\mathbf{7}$	27	$\mathbf{34}$	90	$\mathbf{35}$	99	225	162	$\mathbf{28}$
3^x	3^0	3	3^2	3^0	3^3	3^0	3^2	3^0	3^2	3^2	3^4	3^0
soc	1_1	1_2	1_2	2_1	$1_5 2_1$	1_5	1_5	3_1	3_1	3_1	3_2	3_2
hd	1_1	4	1_2	1_3	2_1	1_4	1_5	2_2	2_3	3_1	3_2	1_6
EC	$\mathcal{P}_1$	$\mathcal{P}_1$		$\mathcal{P}_2$				$\mathcal{P}_3$			$\mathcal{P}_4$	
s	1	1		2				1			1	

Remark: The indecomposable direct summands 45_1 and 45_2 of k_P^G are simple projective modules. Moreover the summands 21 and 63 of k_P^G lie in a block with defect 1. The remaining summands belong to the principal block. Let Q_1 be a vertex of order 3 of the summand 21 of k_P^G. The corresponding induced weight module decomposes as follows:

$$^6S_{N(Q_1)}^G = \mathbf{21} \oplus 45_1 \oplus 45_2 \oplus 225.$$

In particular, 225 is a projective indecomposable kG-module. The remaining induced weight modules have vertex P and decompose as follows:

$$
\begin{aligned}
{}^{1_1}S_N^G &= \mathbf{1} \oplus 27 \oplus 90 \oplus 162, \\
{}^{1_2}S_N^G &= \mathbf{7} \oplus 21 \oplus 45_1 \oplus 45_2 \oplus 162, \\
{}^{1_3}S_N^G &= \mathbf{28} \oplus 27 \oplus 63 \oplus 162, \\
{}^{1_4}S_N^G &= \mathbf{34} \oplus 21 \oplus 225, \\
{}^{2}S_N^G &= \mathbf{35} \oplus 21 \oplus 45_1^2 \oplus 45_2^2 \oplus 99 \oplus 225.
\end{aligned}
$$

6.6 $A_9 \mod 2$ and $A_9 \mod 3$

$\boxed{p = 2,\ G = A_9,\ P \in \mathrm{Syl}_2(G),\ N_G(P) = P,\ |G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7}$

k_P^G	$\mathbf{1}$	$\mathbf{8}$	$\mathbf{48}$	384^2	$\mathbf{432}$	$\mathbf{126_1}$	$\mathbf{126_2}$	$\mathbf{120_1}$	$\mathbf{120_2}$	$\mathbf{252}$	$\mathbf{258}$	576
1	1					2	2	2	2	4	4	8
8_1		1		2	2							
48			1	1	2							
160				2	2							
8_2						2	2				2	2
8_3						2	2				2	2
20_1						1	1	1	1	1	1	3
20_2						1	1	1	1	1	1	3
26						2	2			2	4	4
78								1	1	2	1	4
soc	1	8_1	48	160	$48 \oplus 160$	8_2	8_3	20_1	20_2	$26 \oplus 78$	26	78
hd	1	8_1	48	160	$48 \oplus 160$	8_2	8_3	20_2	20_1	$26 \oplus 78$	26	78

C_P^G	1_1	1_2	2	6_1^2	8	6_2	6_3	4_1	4_2	9_1	11	9_2
1_1	1											
1_2		1										
1_3			1	2	3							
1_4			1		1							
2				2	2							
1_5						2	2				2	
1_6						2	2				2	
1_7								1	1	1		1
1_8								1	1	1		1
1_9								1	1	3	2	2
1_{10}						2	2			2	4	1
1_{11}								1	1	2	1	4
k_P^G	**1**	**8**	**48**	384	**432**	$\mathbf{126_1}$	$\mathbf{126_2}$	$\mathbf{120_1}$	$\mathbf{120_2}$	**252**	**258**	576
2^x	2^0	2^3	2^4	2^7	2^4	2	2	2^3	2^3	2^2	2	2^6
soc	1_1	1_2	1_3	2	$1_3 2$	1_5	1_6	$1_{11} 1_8$	$1_{11} 1_7$	$1_{10} 1_{11}$	$1_{10} 1_{11}$	1_{11}
hd	1_1	1_2	1_4	2	1_3	1_5	1_6	1_7	1_8	1_9	1_{10}	1_{11}
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$		$\mathcal{P}_4$	$\mathcal{P}_5$			$\mathcal{P}_6$		
s	1	1		2		1	1			4		

Remark: From the decomposition matrix of kG we know that the indecomposable direct summands 8, 48 and 432 of k_P^G belong to a block with defect group of order 8.
The weight Green correspondent 432 has a vertex Q_1 of order 4. The corresponding induced weight module decomposes as follows:

$$^8S^G_{N(Q_1)} = \mathbf{432} \oplus 576.$$

Moreover, the weight Green correspondents $120_1, 120_2$ and 48 have pairwise non conjugate vertices Q_1, Q_2 and Q_3, respectively, of order 8. We find as decomposition of the corresponding induced weight modules:

$$\begin{aligned} ^8S^G_{N(Q_2)} &= \mathbf{120_1} \oplus 384 \oplus 576, \\ ^8S^G_{N(Q_3)} &= \mathbf{120_2} \oplus 384 \oplus 576, \\ ^8S^G_{N(Q_4)} &= \mathbf{48} \oplus Y, \end{aligned}$$

where Y denotes a sum of indecomposable direct summands whose dimensions have 2-parts greater than 8.
The weight Green correspondent 252 has a vertex Q_5 of order 16. The corresponding induced weight module decomposes as follows:

$$^4S^G_{N(Q_5)} = \mathbf{252} \oplus 432 \oplus 576.$$

Finally, the weight Green correspondents $258, 126_1$ and 126_2 have pairwise non conjugate vertices Q_6, Q_7 and Q_8 of order 32. The induced weight modules decompose as follows:

$$\begin{aligned}
{}^2S^G_{N(Q_6)} &= \mathbf{258} \oplus 48 \oplus 120_1 \oplus 120_2 \oplus 384 \oplus 576,\\
{}^2S^G_{N(Q_7)} &= \mathbf{126_1} \oplus 120_1 \oplus 252 \oplus 384 \oplus 432 \oplus 576,\\
{}^2S^G_{N(Q_8)} &= \mathbf{126_2} \oplus 120_2 \oplus 252 \oplus 384 \oplus 432 \oplus 576.
\end{aligned}$$

As $N_G(P) = P$, the induced trivial weight module decomposes as ${}^1S^G_N = k^G_P$.

$\boxed{p = 3,\ G = A_9,\ P \in \mathrm{Syl}_3(G),\ |G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7}$

k^G_P	**1**	**84**	**118**	**252**	**435**	**27**	**189**	405^2
1	1			3	4			
7		1	2	3	3			
21		2	3		2			
35		1		3	7			
41			1	3	3			
27						1		1
189							1	2
soc	1	21	$7 \oplus 21$	41	35	27	189	189
hd	1	21	$7 \oplus 21$	41	35	27	189	189

C^G_P	1_1	4	6_1	6_2	9	1_2	3	5^2
1_1	1							
1_2		2	1		1			
1_3		1	4	1				
1_4			1	3	2			
1_5		1		2	6			
1_6						1		
1_7							1	1
2							1	2
k^G_P	**1**	**84**	**118**	**252**	**435**	**27**	**189**	405
3^x	3^0	3	3^0	3^2	3	3^3	3^3	3^4
soc	1_1	$1_2 1_3 1_5$	$1_2 1_3^2 1_4$	$1_4 1_5$	$1_4 1_5^2$	1_6	2	2
hd	1_1	1_2	1_3	1_4	1_5	1_6	1_7	2
EC	$\mathcal{P}_1$			$\mathcal{P}_2$		$\mathcal{P}_3$	$\mathcal{P}_4$	
s	1			4		1	1	

Remark: The summand 162 of k_P^G is simple projective.
The weight Green correspondents 27, 189 lie in a block with defect groups of order 3^3. Furthermore, they have a vertex Q_1 of order 3. The decomposition of the induced weight modules is as follows:

$$^{9_1}S^G_{N(Q_1)} = \mathbf{189} \oplus 162 \oplus 405, \qquad ^{9_2}S^G_{N(Q_1)} = \mathbf{27} \oplus 162 \oplus 567.$$

Note that the indecomposable direct summand 405 of k_P^G is projective.
The weight Green correspondents 252, 84 and 435 have vertices of order 9. But only the vertices of 84 and 435 are conjugate in G. The induced weight modules decompose as follows:

$$\begin{aligned} ^{9}S^G_{N(Q_2)} &= \mathbf{252} \oplus 243 \oplus 162^2 \oplus 405^2 \oplus 891, \\ ^{3_1}S^G_{N(Q_3)} &= \mathbf{435} \oplus 405, \\ ^{3_2}S^G_{N(Q_3)} &= \mathbf{84} \oplus 162 \oplus 189 \oplus 405. \end{aligned}$$

Finally, the trivial module and 118 have P as a vertex. Inducing the corresponding weight modules, we get:

$$\begin{aligned} ^{1_1}S^G_{N} &= \mathbf{1} \oplus 27 \oplus 405 \oplus 435 \oplus 252, \\ ^{1_2}S^G_{N} &= \mathbf{118} \oplus 84 \oplus 405 \oplus 162^2 \oplus 189. \end{aligned}$$

6.7 M_{11} mod p for $p = 2, 3, 5, 11$

$\boxed{p = 2,\ G = M_{11},\ P \in \mathrm{Syl}_2(G),\ N_G(P) = P,\ |G| = 2^4 \cdot 3 \cdot 5 \cdot 11}$

The permutation module decomposes as follows:

$$k_P^G = 1 \oplus 16_1 \oplus 16_2 \oplus 10 \oplus 44 \oplus 120 \oplus 144^2.$$

k_P^G	**1**	**10**	**44**	120	144^2
1	1			2	2
10		1		3	1
44			1	2	3
soc	1	10	44	$10 \oplus 44$	44
hd	1	10	44	$10 \oplus 44$	44

C_P^G	1_1	2	4	10	9^2
1_1	1				
1_2		1		1	
1_3			1	1	1
1_4		1	1	4	2
2			1	2	3
k_P^G	**1**	**10**	**44**	120	144
2^x	2^0	2	2^2	2^3	2^4
soc	1_1	1_4	2	$1_4 2$	2
hd	1_1	1_2	1_3	1_4	2
EC	$\mathcal{P}_1$		$\mathcal{P}_2$		
s	1		2		

Remark: The indecomposable direct summands 16_1 and 16_2 of k_P^G are simple projective.
The weight Green correspondent 44 has vertex Q_1 of order 4. The induction of the corresponding weight module gives:

$$^2S^G_{N(Q_1)} = \mathbf{44} \oplus 16_1 \oplus 16_2 \oplus 120 \oplus 144.$$

The weight Green correspondent 10 has vertex Q_2 of order 8. Here we get the following decomposition of the induced weight module:

$$^2S^G_{N(Q_2)} = \mathbf{10} \oplus 16_1 \oplus 16_2 \oplus 144^2.$$

As $N_G(P) = P$ we have $^1S^G_N = k_P^G$ for the trivial weight module.

$\boxed{p = 3,\ G = M_{11},\ P \in \mathrm{Syl}_3(G),\ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11}$

The permutation modules decomposes as:

$$1 \oplus 45^5 \oplus 10 \oplus 11^2 \oplus 54 \oplus 55_1 \oplus 55_2 \oplus 65_1^2 \oplus 65_2^2 \oplus 99^2.$$

k_P^G	**1**	**10**	54	$\mathbf{11}^2$	$\mathbf{55_1}$	$\mathbf{55_2}$	$\mathbf{65_1}^2$	$\mathbf{65_2}^2$	99^2
1	1			1	1	1	1	1	1
5_1			1	1	2	1	1	1	2
5_2			1	1	2	1	1	1	2
10_1		1	2		1				1
10_2						1	2	1	1
10_3						1	1	2	1
24			1		1	1	1	1	2
soc	1	10_1	10_1	5_1	10_2	5_2	$10_2 \oplus 24$	10_3	24
hd	1	10_1	10_1	5_2	10_1	5_1	10_2	$10_3 \oplus 24$	24

C_P^G	1	2	6	3^2	7_1	7_2	9_1^2	9_2^2	11^2
1_1	1								
1_2		1	1						
2_1				1	1				
1_3		1	2		1				1
1_4			1	1	2				1
1_5						1	1	1	1
2_2						1	2	1	1
2_3						1	1	2	1
2_4			1		1	1	1	1	2
k_P^G	**1**	**10**	54	**11**	$\mathbf{55_1}$	$\mathbf{55_2}$	$\mathbf{65_1}$	$\mathbf{65_2}$	99
3^x	3^0	3^0	3^3	3^0	3^0	3^0	3^0	3^0	3^2
soc	1_1	1_3	1_3	1_4	$1_3 1_4 2_1 2_4$	2_2	$2_2 2_4$	$1_4 2_3$	2_4
hd	1_1	1_2	1_3	2_1	1_4	1_5	2_2	2_3	2_4
EC	$\mathcal{P}_1$	$\mathcal{P}_2$				$\mathcal{P}_3$			
s	1	2				6			

Remark: The indecomposable direct summand 45 of k_P^G is simple projective. The remaining weight Green correspondents 1, 55_1, 55_2, 10, 65_1, 65_2 and 11 have the Sylow 3-subgroup P as vertex. We get the following decomposition from the corresponding induced weight module:

$$\begin{array}{rclrcl} {}^{1_1}S_N^G & = & \mathbf{1} \oplus 54, & {}^{1_2}S_N^G & = & \mathbf{55_1}, \\ {}^{1_3}S_N^G & = & \mathbf{10} \oplus 45, & {}^{1_4}S_N^G & = & \mathbf{55_2}, \\ {}^{2_1}S_N^G & = & \mathbf{65_1} \oplus 45, & {}^{2_2}S_N^G & = & \mathbf{65_2} \oplus 45, \\ {}^{2_3}S_N^G & = & \mathbf{11} \oplus 99. & & & \end{array}$$

This is the only example we have found so far, where Hypothesis 4.2 does not hold. To the equivalence class $\mathcal{P}_3$ correspond less weight Green correspondents than socle constituents.

$\boxed{p = 5,\ G = M_{11},\ P \in \mathrm{Syl}_5(G),\ N_G(P) = P,\ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11}$

$$k_P^G = 1 \oplus_{i=1}^{3} 10_i^2 \oplus 45^9 \oplus 55_1^{11} \oplus 11 \oplus 55_2^2 \oplus 16_1 \oplus 16_2 \oplus 60_1^3 \oplus 60_2^3$$

k_P^G	**1**	**11**	55_2^2	$\mathbf{16_1}$	60_1^3	$\mathbf{16_2}$	60_2^3
1	1		1		1		1
11		1	2		1		1
16_1			1	1	2		1
16_2			1		1	1	2
soc	1	11	11	16_1	16_1	16_2	16_2
hd	1	11	11	16_1	16_1	16_2	16_2

C_P^G	1	3	11^2	4_1	12_1^3	4_2	12_2^3
1_1	1						
1_2		1	1				
2_4		1	2		1		1
1_3				1	1		
3_1			1	1	2		1
1_4						1	1
3_2			1		1	1	2
k_P^G	$\mathbf{1}$	$\mathbf{11}$	55_2	$\mathbf{16_1}$	60_1	$\mathbf{16_2}$	60_2
5^x	5^0	5^0	5	5^0	5	5^0	5
soc	1_1	2_4	2_4	3_1	3_1	3_2	3_2
hd	1_1	1_2	2	1_3	3_1	1_4	3_2
EC	$\mathcal{P}_1$	$\mathcal{P}_1$		$\mathcal{P}_2$		$\mathcal{P}_3$	
s	1	1		1		1	

Remark: The indecomposable direct summands 10_1, 10_2, 10_3, 45 and 55 of k_P^G are simple projective modules. Moreover the indecomposable direct summands 55_2, 60_1 and 60_2 are projective kG-modules. The remaining summands of k_P^G are weight Green correspondents. The decomposition of the corresponding weight modules are given as follows:

$$\begin{aligned}
{}^{1_1}S_N^G &= \mathbf{16_1} \oplus 10_1 \oplus 10_2 \oplus 60_2 \oplus 45^3 \oplus 55_1^3,\\
{}^{1_2}S_N^G &= \mathbf{11} \oplus 45 \oplus 55_1^3 \oplus 55_2 \oplus 60_1 \oplus 60_2,\\
{}^{1_3}S_N^G &= \mathbf{1} \oplus 10_3^2 \oplus 45^2 \oplus 55_1^2 \oplus 55_2 \oplus 60_1 \oplus 60_2,\\
{}^{1_4}S_N^G &= \mathbf{16_2} \oplus 10_1 \oplus 10_2 \oplus 60_1 \oplus 45^3 \oplus 55_1^3.
\end{aligned}$$

$\boxed{p = 11,\ G = M_{11},\ P \in \mathrm{Syl}_{11}(G),\ N_G(P) = P,\ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11}$

The permutation module decomposes as follows:

$$k_P^G = 1 \oplus 11 \oplus 44^4 \oplus 55^5 \oplus 45_1 \oplus 45_2 \oplus 45_3 \oplus 45_4 \oplus 77.$$

k_P^G	$\mathbf{1}$	$\mathbf{45_1}$	$\mathbf{45_2}$	$\mathbf{45_3}$	$\mathbf{45_4}$	77
1	1					
9		1	1	1	1	1
10_1		1	1	1	1	1
10_2		1	1	1	1	1
16		1	1	1	1	3
soc	1	9	10_1	10_2	16	16
hd	1	10_2	16	10_1	9	16

C_P^G	1_1	5_1	5_2	5_3	5_4	7
1_1	1					
1_2		1	1	1	1	1
1_3		1	1	1	1	1
1_4		1	1	1	1	1
1_5		1	1	1	1	1
1_6		1	1	1	1	3
k_P^G	$\mathbf{1}$	$\mathbf{45_1}$	$\mathbf{45_2}$	$\mathbf{45_3}$	$\mathbf{45_4}$	77
11^x	11^0	11^0	11^0	11^0	11^0	11
soc	1_1	1_5	1_2	1_3	1_6	1_6
hd	1_1	1_2	1_3	1_4	1_5	1_6
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$	$\mathcal{P}_5$	
s	1	1	1	1	1	

Remark: The indecomposable direct summands 11, 44 and 55 of k_P^G are simple projective. Moreover, the summand 77 of k_P^G is projective. The remaining summands of k_P^G are weight Green correspondents. The decomposition of the respective induced weight module is given as follows:

$$
\begin{aligned}
{}^{1_1}S_N^G &= \mathbf{1} \oplus 11 \oplus 55 \oplus 77, \\
{}^{1_2}S_N^G &= \mathbf{45_1} \oplus 44 \oplus 55, \\
{}^{1_3}S_N^G &= \mathbf{45_2} \oplus 44 \oplus 55, \\
{}^{1_4}S_N^G &= \mathbf{45_3} \oplus 44 \oplus 55, \\
{}^{1_5}S_N^G &= \mathbf{45_4} \oplus 44 \oplus 55.
\end{aligned}
$$

6.8 M_{12} mod 2 **and** M_{12} mod 3

$\boxed{p = 2,\ G = M_{12},\ P \in \mathrm{Syl}_2(G),\ |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11}$

The permutation module decomposes as follows:

$$k_P^G = 1 \oplus 16_1 \oplus 16_1 \oplus 144 \oplus 320^2 \oplus 318 \oplus 350.$$

k_P^G	$\mathbf{1}$	$\mathbf{16_1}$	$\mathbf{16_2}$	$\mathbf{144}$	320^2	$\mathbf{318}$	$\mathbf{350}$
1	1					4	6
16_1		1			1		
16_2			1		1		
144				1	2		
10						5	8
44						6	6
soc	1	16_1	16_2	144	144	44^2	$10 \oplus 44$
hd	1	16_1	16_2	144	144	44^2	$10 \oplus 44$

C_P^G	1_1	1_2	1_3	3	5^2	13	15
1_1	1						
1_2		1					
1_3			1				
1_4				1	1		
2				1	2		
1_5						7	6
1_6						6	9
k_P^G	**1**	$\mathbf{16_1}$	$\mathbf{16_2}$	**144**	320	**318**	**350**
2^x	2^0	2^4	2^4	2^4	2^6	2	2
soc	1_1	1_2	1_3	2	2	$1_5^2 1_6^2$	$1_5 1_6^3$
hd	1_1	1_2	1_3	1_4	2	1_5	1_6
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$		$\mathcal{P}_5$	
s	1	1	1	1		2	

Remark: The indecomposable direct summand 320 of k_P^G is projective. The remaining summands are weight Green correspondents. Among those the summands 16_1, 16_2 and 144 have vertex Q_1 of order 4. For the decomposition of the induced modules we get:

$$\begin{aligned}
{}^{2_1}S_{N(Q_1)}^G &= \mathbf{144} \oplus 320^4 \oplus 384 \oplus 832, \\
{}^{2_2}S_{N(Q_1)}^G &= \mathbf{16_1} \oplus 192 \oplus 320^4 \oplus 384 \oplus 768, \\
{}^{2_3}S_{N(Q_1)}^G &= \mathbf{16_2} \oplus 192 \oplus 320^4 \oplus 384 \oplus 768.
\end{aligned}$$

The indecomposable direct summands 350 and 318 of k_P^G have vertices Q_2 and Q_3, respectively, of order 32 which are not conjugate in G. The decomposition of the induced weight modules is:

$$ {}^2S_{N(Q_2)}^G = \mathbf{350} \oplus 320, \qquad {}^2S_{N(Q_3)}^G = \mathbf{318} \oplus 16_1 \oplus 16_2 \oplus 320. $$

Finally, we have ${}^1S_N^G = k_P^G$, as $N_G(P) = P$.

$\boxed{p = 3,\ G = M_{12},\ P \in \mathrm{Syl}_3(G),\ |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11}$

The permutation modules decomposes as follows:

$$\begin{aligned}
k_P^G \quad = \quad & 1 \oplus 54^2 \oplus 45^2 \oplus 99^2 \oplus 189 \oplus 243^3 \oplus 66_1 \oplus 66_2 \oplus 66_3 \oplus 66_4 \\
& \oplus 175_1 \oplus 175_2 \oplus 297 \oplus 351_1 \oplus 351_2 \oplus 592.
\end{aligned}$$

k_P^G	$\mathbf{1}$	189	$\mathbf{45}^2$	$\mathbf{99}^2$	243^3	$\mathbf{66_1}$	$\mathbf{66_2}$	297	$\mathbf{66_3}$	$\mathbf{66_4}$	351_1	351_2	$\mathbf{175_1}$	$\mathbf{175_2}$	$\mathbf{592}$
1	1					2	2	5	2	2	2	2			4
45_1		2	1		1										
99		1		1	2										
10_1						2	1	2			2	1	1		2
10_2						1	2	2			1	2		1	2
15_1								2	1	1	2	2	1	1	4
15_2								2	1	1	2	2	1	1	4
34						1	1	3	1	1	1	1			2
45_1								1			3	2	2	1	4
45_2								1			2	3	1	2	4
soc	1	45_1	99	45_1	99	10_1	10_2	15_1	15_2	45_1	45_2	34	45_1	45_2	$45_1 45_2 34$
hd	1	45_1	99	45_1	99	10_1	10_2	15_1	15_2	45_2	45_3	34	45_2	45_3	$45_2 45_3 34$

C_P^G	1	7	3^2	5^2	9_1^3	4_1	4_2	11	4_3	4_4	13_1	13_2	9_2	9_3	26
1_1	1														
1_2		2	1		1										
2_1		1	1												
2_2				1	1										
3		1		1	2										
1_3						2	1	1							
1_4						1	2	1							
1_5						1	1	3	1	1	1	1			2
1_6								1	1	1					1
1_7								1	1	1					1
1_8								1			3	2	2	1	4
1_9								1			2	3	1	2	4
1_{10}											2	1	2	1	3
1_{11}											1	2	1	2	3
1_{12}								2	1	1	4	4	3	3	8
k_P^G	$\mathbf{1}$	189	$\mathbf{45}$	$\mathbf{99}$	243	$\mathbf{66_1}$	$\mathbf{66_2}$	297	$\mathbf{66_3}$	$\mathbf{66_4}$	351_1	351_2	$\mathbf{175_1}$	$\mathbf{175_2}$	$\mathbf{592}$
3^x	3^0	3^3	3^2	3^2	3^5	3	3	3^3	3	3	3^3	3^3	3^0	3^0	3^0
soc	1_1	1_2	1_2	3	3	$1_3 1_5$	$1_4 1_5$	1_5	$1_5 1_7$	$1_5 1_6$	1_8	1_9	1_8	1_9	$1_5 1_8 1_9$
hd	1_1	1_2	2_1	2_2	3	1_3	1_4	1_5	1_6	1_7	1_8	1_9	1_{10}	1_{11}	1_{12}
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$							$\mathcal{P}_4$				
s	1	1		1							7				

Remark: The indecomposable direct summand 54 of k_P^G is simple projective. Moreover, the indecomposable direct summands 45, 99, 189 and 243 of k_P^G lie in a block with cyclic defect group of prime order. Among those, 45 and 99 are weight Green correspondents with vertex Q_1 of order 3. Note that the indecomposable direct summands 189 and 243 of k_P^G must be projective modules, as can be seen by the following decomposition of the induced weight modules belonging to this block.

$$
\begin{aligned}
{}^{3_1}S_{N(Q_1)}^G &= \mathbf{99} \oplus 54 \oplus 189 \oplus 243^4 \oplus_{i=1,2} 297_i \oplus 297_3^2 \oplus_{i=1,2} 351_i^2, \\
{}^{3_2}S_{N(Q_1)}^G &= \mathbf{45} \oplus 54^2 \oplus 189^2 \oplus 243^4 \oplus 297_1 \oplus_{i=1,2} (351_i^2 \oplus 378_i).
\end{aligned}
$$

The remaining indecomposable direct summands of k_P^G belong to the principal block. We find 66_1, 66_2 and 66_3, 66_4 to have vertices of order 9, where the first two and the last two modules have conjugate vertices, respectively. We find for the decomposition of the induced modules:

$$
\begin{aligned}
{}^{3_1}S_{N(Q_2)}^G &= \mathbf{66_1} \oplus 54 \oplus 243 \oplus 297,\\
{}^{3_2}S_{N(Q_2)}^G &= \mathbf{66_2} \oplus 243 \oplus 351,\\
{}^{3_1}S_{N(Q_3)}^G &= \mathbf{66_3} \oplus 54 \oplus 243 \oplus 297,\\
{}^{3_2}S_{N(Q_3)}^G &= \mathbf{66_4} \oplus 243 \oplus 351.
\end{aligned}
$$

Finally the weight Green correspondents $1, 175_1, 175_2$ and 592 have a Sylow 3-subgroup as vertex. We get the following decomposition of the corresponding induced of weight modules:

$$
\begin{aligned}
{}^{1_1}S_N^G &= \mathbf{1} \oplus 54^2 \oplus 66_1 \oplus 66_2 \oplus 99 \oplus 243 \oplus 297,\\
{}^{1_1}S_N^G &= \mathbf{175_1} \oplus 45 \oplus 66_3 \oplus 243 \oplus 351_1,\\
{}^{1_1}S_N^G &= \mathbf{175_2} \oplus 45 \oplus 66_4 \oplus 243 \oplus 351_2,\\
{}^{1_1}S_N^G &= \mathbf{592} \oplus 99 \oplus 189.
\end{aligned}
$$

6.9 M_{22} mod 2

$\boxed{p = 2,\ G = M_{22},\ P \in \mathrm{Syl}_2(G),\ |G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11}$

k_P^G	**1**	**848**	**1078**	$\mathbf{616_1}$	$\mathbf{616_2}$	**230**	**76**
1	1	12	14	8	8	4	2
10_1		5	6	3	3	3	2
10_2		5	6	3	3	3	2
34		6	8	5	5	2	1
70_1		1	2	2	2		
70_2		1	2	2	2		
98		4	4	1	1	1	
soc	1	98	$34 \oplus 98$	70_1	70_2	$10_1 \oplus 34$	10_2
hd	1	98	$34 \oplus 98$	70_2	70_1	$10_2 \oplus 34$	10_1

C_P^G	1	11	20	7_1	7_2	9	5
1_1	1						
1_2		4	4	1	1	1	
1_3		4	8	2	2	3	1
1_4		1	2	2	2		
1_5		1	2	2	2		
1_6		1	3			3	2
1_7			1			2	2
k_P^G	**1**	**848**	**1078**	$\mathbf{616_1}$	$\mathbf{616_2}$	**230**	**76**
2^x	2^0	2^4	2	2^3	2^3	2	2^2
soc	1_1	1_2	$1_2^2 1_3 1_4 1_5$	1_5	1_4	$1_2 1_3 1_7$	$1_6 1_7$
hd	1_1	1_2	1_3	1_4	1_5	1_6	1_7
EC	$\mathcal{P}_1$			$\mathcal{P}_2$			
s	1			6			

Remark: All indecomposable direct summands of k_P^G are weight Green correspondents. The weight Green correspondent 848 has vertex Q_1 of order 8. The induced weight module decomposes as follows:

$$^8S_{N(Q_1)}^G = \mathbf{848} \oplus 896_1 \oplus 896_2.$$

The two weight Green correspondents 616_1 and 616_2 have vertex Q_2 of order 16. We get as decomposition of the corresponding induced weight modules:

$$\begin{aligned} ^{8_1}S_{N(Q_2)}^G &= \mathbf{616_1}, \\ ^{8_2}S_{N(Q_2)}^G &= \mathbf{616_2}. \end{aligned}$$

The indecomposable direct summand 76 of k_P^G has vertex Q_3 of order 2^5. Here we get for the induced weight module:

$$^2S_{N(Q_3)}^G = \mathbf{76} \oplus 896_1 \oplus 896_2 \oplus 848 \oplus 1904.$$

The weight Green correspondents 230 and 1078 have non conjugate vertices Q_4 and Q_5, respectively, of order 2^6. We have:

$$\begin{aligned} ^2S_{N(Q_4)}^G &= \mathbf{230} \oplus 616_1 \oplus 616_2 \oplus 848, \\ ^2S_{N(Q_5)}^G &= \mathbf{1078} \oplus 616_1 \oplus 616_2. \end{aligned}$$

As $N_G(P) = P$, the trivial module is the only weight module with vertex P and we have $^1S_N^G = k_P^G$.

6.10 $U_3(3)$ mod 2

$p = 2$, $G = U_3(3)$, $P \in \mathrm{Syl}_2(G)$, $N_G(P) = P$, $\lvert G\rvert = 2^5 \cdot 3^3 \cdot 7$

The permutation module decomposes as follows:

$$k_P^G = 1 \oplus 32_1 \oplus 32_2 \oplus 62_1 \oplus 62_2.$$

k_P^G	$\mathbf{1}$	$\mathbf{62_1}$	$\mathbf{62_2}$
1	1	2	4
6		3	5
14		3	2
soc	1	14	6
hd	1	14	6

C_P^G	1_3	5	6
1_1	1		
1_2		3	2
1_3		2	4
k_P^G	$\mathbf{1}$	$\mathbf{62_1}$	$\mathbf{62_2}$
2^x	2^0	2	2
soc	1_1	1_2	1_3^2
hd	1_1	1_2	1_3
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
s	1	1	1

Remark: All indecomposable direct summands of k_P^G are weight Green correspondents. Note that the endomorphism ring is not quasi-Frobenius. The indecomposable direct summands 32_1 and 32_2 are simple projective modules. The weight Green correspondents 62_1 and 62_2 have non conjugate vertices of order 16. We get the following decomposition for the induced weight modules.

$$^2S^G_{N(Q_1)} = \mathbf{62_2} \oplus 32_1 \oplus 32_2, \qquad ^2S^G_{N(Q_2)} = \mathbf{62_1} \oplus 32_1 \oplus 32_2.$$

As $N_G(P) = P$, we have $^1S^G_N = k^G_P$.

6.11 $G = J_2 \mod 2$

$p = 2,\ G = J_2,\ P \in \mathrm{Syl}_2(G),\ |G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7$

k_P^G	**1**	**154**	**364**	**525_1**	**525_2**	**762_1**	**762_2**	**160**	**288_1**	**288_2**	896
1	1	2	8	9	9	10	10				
6_1		2	6	6	6	7	7				
6_2		2	6	6	6	7	7				
14_1		2	2	3	3	5	5				
14_2		2	2	3	3	5	5				
36		2	4	3	3	3	3				
84			1	3	3	5	5				
64_1									1	1	2
64_2									1	1	2
160								1	1	1	4
soc	1	$14_1 14_2 36$	36	6_1	6_2	84	84	160	64_1	64_2	160
hd	1	$14_1 14_2 36$	36	6_2	6_1	84	84	160	64_1	64_2	160

C_P^G	1	6	8	10_1	10_2	11_1	11_2	2	3_1	3_2	7
1_1	1										
1_2		4	2								
1_3		2	4			1	1				
1_4				3	3	2	2				
1_5				3	3	2	2				
1_6			1	2	2	3	3				
1_7			1	2	2	3	3				
1_8								1	1	1	1
1_9								1	1	1	1
1_{10}									1	1	1
1_{11}											4
k_P^G	**1**	**154**	**364**	**762_1**	**762_2**	**525_1**	**525_2**	**160**	**288_1**	**288_2**	896
2^x	2^0	2	2^2	2	2	2^0	2^0	2^5	2^5	2^5	2^7
soc	1_1	$1_2^2 1_3$	$1_3 1_6 1_7$	$1_4 1_5$	$1_4 1_5$	$1_3 1_4 1_6 1_7$	$1_3 1_5 1_6 1_7$	1_{11}	$1_{10} 1_{11}$	$1_9 1_{11}$	1_{11}
hd	1_1	1_2	1_3	1_4	1_5	1_6	1_7	1_8	1_9	1_{10}	1_{11}
EC	$\mathcal{P}_1$				$\mathcal{P}_2$				$\mathcal{P}_3$		
s	1				6				3		

Remark: The indecomposable direct summands 160, 288_1, 288_2 and 896 of k_P^G belong to the block of G with defect group of order 4. Since the dimensions of the induced weight modules are too large to apply the MeatAxe directly, we use M. Szöke's results in [Szö98] to identify the weight Green correspondents.

6.12 $G = L_2(11) \mod 2$

$p = 2,\ G = L_2(11),\ P \in \mathrm{Syl}_2(G),\ |G| = 2^2 \cdot 3 \cdot 5 \cdot 11$

The permutation module k_P^G decomposes as follows:

$$k_P^G = 1 \oplus 12_1^3 \oplus 12_2^3 \oplus 16_1 \oplus 5_1 \oplus 16_2 \oplus 5_2 \oplus 20 \oplus 10^3.$$

k_P^G	$\mathbf{1}$	16_1	$\mathbf{5_1}$	16_2	$\mathbf{5_2}$	20	$\mathbf{10}^3$
1	1	1		1			
5_1		2	1	1			
5_2		1		2	1		
10						2	1
soc	1	5_1	5_1	5_2	5_2	10	10
hd	1	5_1	5_1	5_2	5_2	10	10

C_P^G	1	4_1	2_1	4_2	2_2	5	4_3^3
1_1	1						
1_2		2	1	1			
1_3		1	1				
1_4		1		2	1		
1_5				1	1		
1_6						2	1
3						1	1
k_P^G	$\mathbf{1}$	16_1	$\mathbf{5_1}$	16_2	$\mathbf{5_2}$	20	$\mathbf{10}$
2^x	2^0	2^4	2^0	2^4	2^0	2^2	2
soc	1	1_2	1_2	1_4	1_4	1_6	1_6
hd	1_1	1_2	1_3	1_4	1_5	1_6	3
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$		$\mathcal{P}_4$	
s	1	1		1		1	

Remark: The indecomposable direct summands 12_1 and 12_2 of k_P^G are simple projective. The summand 10 of k_P^G is a weight Green correspondent with vertex Q_1 of order 2. It lies in a block with cyclic defect group of prime order. We get the following decomposition of the induced weight module:

$${}^2S^G_{N_G(Q_1)} = \mathbf{10} \oplus 12_1^2 \oplus 12_2^2 \oplus 16_1 \oplus 16_2 \oplus 20.$$

From this decomposition we see that the summands 16_1, 16_2 and 20 of K_P^G are projective. Moreover 1, 5_1 and 5_2 are weight Green correspondents with vertex P which occur in the following decomposition of the induced weight module:

$$\begin{aligned}
{}^{1_1}S_N^G &= \mathbf{1} \oplus 10 \oplus 12_1 \oplus 12_2 \oplus 20, \\
{}^{1_2}S_N^G &= \mathbf{5_1} \oplus 10 \oplus 12_1 \oplus 12_2 \oplus 16_1, \\
{}^{1_3}S_N^G &= \mathbf{5_2} \oplus 10 \oplus 12_1 \oplus 12_2 \oplus 16_2.
\end{aligned}$$

6.13 $L_2(13)$ mod 2 and $L_2(13)$ mod 3

$\boxed{p = 2,\ G = L_2(13),\ P \in \mathrm{Syl}_2(G),\ |G| = 2^2 \cdot 3 \cdot 7 \cdot 13}$

$$k_P^G = 1 \oplus 12_1^3 \oplus 12_2^3 \oplus 12_3^3 \oplus 20_1 \oplus 13_1 \oplus 20_2 \oplus 13_2 \oplus 28^2 \oplus 14^3.$$

k_P^G	**1**	20_1	$\mathbf{13_1}$	20_2	$\mathbf{13_2}$	28^2	$\mathbf{14}^3$
1	1	2	1	2	1		
6_1		2	1	1	1		
6_2		1	1	2	1		
14						2	1
soc	1	6_1	6_1	6_2	6_2	14	14
hd	1	6_1	6_2	6_2	6_1	14	14

C_P^G	1	5_1	4_1	5_2	4_2	7^2	5_3^3
1_1	1						
1_2		2	1	1	1		
1_3		1	1	1	1		
1_4		1	1	2	1		
1_5		1	1	1	1		
2						2	1
3						1	1
k_P^G	**1**	20_1	$\mathbf{13_1}$	20_2	$\mathbf{13_2}$	28	**14**
2^x	2^0	2^2	2^0	2^2	2^0	2^2	2
soc	1_1	1_2	1_2	1_4	1_4	2	2
hd	1_1	1_2	1_3	1_4	1_5	2	3
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$		$\mathcal{P}_4$	
s	1	1		1		1	

Remark: The indecomposable direct summands 12_i of k_P^G for $1 \le i \le 3$ are simple projective. The weight module 14 has vertex Q_1 of order 2. The decomposition of the associated induced weight module is:

$$^2S^G_{N(Q_1)} = \mathbf{14} \oplus 20_1 \oplus 20_2 \oplus 28^2 \oplus 36^2.$$

Note that the summands $20_1, 20_2$ and 28 are projective.
The weight Green correspondents $1, 13_1, 13_2$ have a Sylow 2-subgroup as vertex. We get the following decomposition of the associated induced weight modules.

$$\begin{aligned} {}^{1_1}S_N^G &= \mathbf{1} \oplus 14 \oplus 20_1 \oplus 20_2 \oplus_{1\le i\le 3} 12_i, \\ {}^{1_2}S_N^G &= \mathbf{13_1} \oplus 14 \oplus 28 \oplus_{1\le i\le 3} 12_i, \\ {}^{1_3}S_N^G &= \mathbf{13_2} \oplus 14 \oplus 28 \oplus_{1\le i\le 3} 12_i. \end{aligned}$$

We remark that all indecomposable direct summands of k_P^G are uniserial as direct computation has shown.

$\boxed{p = 3,\ G = L_2(13),\ P \in \mathrm{Syl}_2(G),\ |G| = 2^2 \cdot 3 \cdot 7 \cdot 13}$

The permutation module k_P^G decomposes as follows:

$$k_P^G = 1 \oplus 12_1^4 \oplus 12_2^4 \oplus 12_3^4 \oplus 7_1 \oplus 7_2 \oplus 13 \oplus 21_1^2 \oplus 21_2^2 \oplus 27^4.$$

k_P^G	1	$\mathbf{13}$	27^4	$\mathbf{7_1}$	21_1^2	$\mathbf{7_2}$	21_2^2
1	1		1				
13		1	2				
7_1				1	2		1
7_2					1	1	2
soc	1	13	13	7_1	7_1	7_2	7_2
hd	1	13	13	7_1	7_1	7_2	7_2

C_P^G	1_1	5	9^4	3_1	7_1^2	3_2	7_2^2
1_1	1						
1_2		1	1				
4		1	2				
1_3				1	1		
2_1				1	2		1
1_4						1	1
2_2					1	1	2
k_P^G	$\mathbf{1}$	$\mathbf{13}$	27	$\mathbf{7_1}$	21_1	$\mathbf{7_2}$	21_2
3^x	3^0	3^0	3^3	3^0	3	3^0	3
soc	1_1	4	4	2_1	2_1	2_2	2_2
hd	1_1	1_2	4	1_3	2_1	1_4	2_2
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$		$\mathcal{P}_4$	
s	1	1		1		1	

Remark: The indecomposable direct summands 12_1, 12_2 and 12_3 of k_P^G are simple projective. The remaining weight Green correspondents have the Sylow-3-subgroup P of order 3 as vertex. For the decomposition of the induced weight modules, we have:

$$\begin{aligned}
{}^{1_1}S_N^G &= \mathbf{1} \oplus 27^2 \oplus_{1 \le i \le 3} 12_i,\\
{}^{1_2}S_N^G &= \mathbf{13} \oplus 21_1 \oplus 21_2 \oplus_{1 \le i \le 3} 12_i,\\
{}^{1_3}S_N^G &= \mathbf{7_1} \oplus 21_1 \oplus 27 \oplus_{1 \le i \le 3} 12_i,\\
{}^{1_4}S_N^G &= \mathbf{7_2} \oplus 21_2 \oplus 27 \oplus_{1 \le i \le 3} 12_i.
\end{aligned}$$

The last decomposition could not be computed directly. An application of Remark 2.6 (a) shows that this decomposition must be as above when considering the first three decompositions and that of k_P^G.

6.14 $G = L_2(17)$ mod 2 **and** $L_2(17)$ mod 3

$p = 2$, $G = L_2(17)$, $P \in \mathrm{Syl}_2(G)$, $|G| = 2^4 \cdot 3^2 \cdot 17$

The permutation module k_P^G decomposes into indecomposable direct summands as follows:

$$k_P^G = \oplus_{1\leq i\leq 4} 16_i \oplus 1 \oplus 44_1 \oplus 44_2.$$

k_P^G	$\mathbf{1}$	$\mathbf{44_1}$	$\mathbf{44_2}$
1	1	4	4
8_1		3	2
8_2		2	3
soc	1	8_1	8_2
hd	1	8_1	8_2

C_P^G	1_5	5_1	5_2
1_1	1		
1_2		3	2
1_3		2	3
k_P^G	$\mathbf{1}$	$\mathbf{44_1}$	$\mathbf{44_2}$
2^x	2^0	2^2	2^2
soc	1_1	1_2	1_3
hd	1_1	1_2	1_3
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$
s	1	1	1

Remark: The indecomposable direct summands 16_i of k_P^G for $1 \leq i \leq 4$ are simple projective. The weight Green correspondents 44_1 and 44_2 have non conjugate vertices Q_1 and Q_2, respectively, of order 4. The associated induced weight modules decompose as follows:

$$\begin{aligned} {}^1S^G_{N(Q_1)} &= \mathbf{44_1} \oplus 16_1 \oplus 16_2 \oplus 16_3 \oplus 16_4^2 \oplus 80, \\ {}^1S^G_{N(Q_2)} &= \mathbf{44_2} \oplus 16_1 \oplus 16_2 \oplus 16_3 \oplus 16_4^2 \oplus 80. \end{aligned}$$

As $N_G(P) = P$ we have $k_P^G = {}^1S_N^G$.

$p = 3$, $G = L_2(17)$, $P \in \mathrm{Syl}_3(G)$, $|G| = 2^4 \cdot 3^2 \cdot 17$

The permutation module k_P^G decomposes into indecomposable direct summands as follows:

$$k_P^G = 1 \oplus 9_1 \oplus 9_2 \oplus_{1\leq i\leq 3} 18_i^2 \oplus 81 \oplus 64.$$

k_P^G	1	81	**64**
1	1	1	
16		5	4
soc	1	16	16
hd	1	16	16

C_P^G	1_1	9	8
1_1	1		
1_2		5	4
1_3		4	4
k_P^G	**1**	81	**64**
3^x	3^0	3^4	3^0
soc	1_1	1_2	1_2
hd	1_1	1_2	1_3
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	
s	1	1	

Note: The indecomposable direct summands 9_1, 9_2, 18_1, 18_2, and 18_3 of k_P^G are simple projective while the weight Green correspondent 64 has vertex P. For the associated induced weight module and the induced trivial weight module we get the following decompositions:

$${}^{1_1}S_N^G \;=\; \mathbf{1} \oplus 9_1 \oplus 9_2 \oplus 18_1^2 \oplus 81, \qquad {}^{1_2}S_N^G \;=\; \mathbf{64} \oplus 18_2^2 \oplus 18_3^2.$$

6.15 $G = L_2(19) \mod 2$ and $G = L_2(19) \mod 3$

$\boxed{p = 2,\ G = L_2(19),\ P \in \mathrm{Syl}_2(G),\ |G| = 2^2 \cdot 3^2 \cdot 5 \cdot 19}$

$$k_P^G = 1 \oplus_{1\le i\le 4} 20_i^5 \oplus_{i=1,2} (28_i^2 \oplus 18_i^3 \oplus 36_i^3) \oplus 9_1 \oplus 9_2.$$

k_P^G	1	$\mathbf{9_1}$	28_1^2	$\mathbf{9_2}$	28_2^2	36_1^3	$\mathbf{18_1}^3$	36_2^3	$\mathbf{18_2}^3$
1	1		1		1				
9_1		1	2		1				
9_2			1	1	2				
18_1						2	1		
18_2								2	1
soc	1	9_1	9_1	9_2	9_2	18_1	18_1	18_2	18_2
hd	1	9_1	9_1	9_2	9_2	18_1	18_1	18_2	18_2

C_P^G	1_1	3_1	7_2^2	3_2	7_1^2	9_1^3	6_1^3	9_2^3	6_2^3
1_1	1								
1_2		1	1						
2_1		1	2		1				
1_3				1	1				
2_2			1	1	2				
3_1						2	1		
3_2						1	1		
3_3								2	1
3_4								1	1
k_P^G	$\mathbf{1}$	$\mathbf{9_1}$	28_1	$\mathbf{9_2}$	28_2	36_1	$\mathbf{18_1}$	36_2	$\mathbf{18_2}$
2^x	2^0	2^0	2^2	2^2	2^0	2^2	2	2^2	2
soc	1_1	2_1	2_1	2_2	2_2	3_1	3_1	3_3	3_3
hd	1_1	1_2	2_1	1_3	2_2	3_1	3_2	3_3	3_4
EC	$\mathcal{P}_1$	$\mathcal{P}_2$		$\mathcal{P}_3$		$\mathcal{P}_4$		$\mathcal{P}_5$	
s	1	1		1		1		1	

Note: The weight Green correspondents $18_1, 18_2$ have vertex Q_1 of order 2. We get the following decompositions for the corresponding induced weight modules:

$$\begin{aligned} {}^{2_1}S^G_{N(Q_1)} &= \mathbf{18}_1 \oplus_{i=1,2} 36_1 \oplus 36_2^2 \oplus_{1\le i\le 4} 20_i^2, \\ {}^{2_2}S^G_{N(Q_1)} &= \mathbf{18}_2 \oplus_{i=1,2} 36_2 \oplus 36_1^2 \oplus_{1\le i\le 4} 20_i^2. \end{aligned}$$

The induced weight modules with a Sylow 2-subgroup as vertex decompose as follows:

$$\begin{aligned} {}^{1_1}S^G_N &= \mathbf{1} \oplus_{i=1,2} (18_i \oplus 36_i \oplus 28_i) \oplus 20_1^3 \oplus_{2\le i\le 4} 20_i, \\ {}^{1_2}S^G_N &= \mathbf{9_1} \oplus_{i=1,2} (18_i \oplus 36_i) \oplus 28_1 \oplus 20_1 \oplus_{i=2,3,4} 20_i^2, \\ {}^{1_3}S^G_N &= \mathbf{9_2} \oplus_{i=1,2} (18_i \oplus 36_i) \oplus 28_2 \oplus 20_1 \oplus_{i=2,3,4} 20_i^2. \end{aligned}$$

$\boxed{p = 3,\ G = L_2(19),\ P \in \mathrm{Syl}_3(G),\ |G| = 2^2\cdot 3^2\cdot 5\cdot 19}$

The permutation module k_P^G decomposes into indecomposable direct summands as follows:

$$k_P^G = 1 \oplus_{i=1,2} 9_i \oplus_{1\le i\le 4} 18_i^2 \oplus 19 \oplus 99^2.$$

k_P^G	1	**19**	99^2
1	1		4
19		1	5
soc	1	19	19
hd	1	19	19

C_P^G	1_1	3	11^2
1_1	1		
1_2		1	1
2		1	5
k_P^G	**1**	**19**	99
3^x	3^0	3^0	3^2
soc	1_1	2	2
hd	1_1	1_2	2
EC	$\mathcal{P}_1$	$\mathcal{P}_2$	
s	1	1	

Remark: The Sylow 3-subgroup P is a cyclic group of order 3^2.
The indecomposable direct summand 99 of k_P^G is not a weight Green correspondent while the remaining are. Note that 9_i and 18_i for $i = 1, 2$ are simple projective. We get the following decompositions for the induced weight modules with P as a defect group:

$$\begin{aligned} {}^{1_1}S_N^G &= \mathbf{1} \oplus 99 \oplus 9_1 \oplus 9_2 \oplus 18_1^2 \oplus 18_2^2, \\ {}^{1_2}S_N^G &= \mathbf{19} \oplus 99 \oplus 18_3^2 \oplus 18_4^2. \end{aligned}$$

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