Endomorphism rings of permutation modules

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Abstract

Let \( k \) be an algebraically closed field of characteristic \( p \) and \( G \) a finite group. Then the permutation module \( kG_P \) of \( G \) on the cosets of a Sylow \( p \)-subgroup \( P \) is via Fitting correspondence strongly related to its endomorphism ring \( \text{End}_{kG}(kG_P) \). On the other hand, each Green correspondent in \( G \) of a weight module of \( G \) occurs as a direct summand of \( kG_P \). This fact suggests to analyze both structures, the permutation module and the associated endomorphism ring towards hints at a proof for Alperin’s weight conjecture. We present a selection of such investigations for different groups and characteristics. In particular we focus on the socle and head constituents of the indecomposable direct summands of \( kG_P \) and of the PIMs of \( \mathcal{E}_\mathcal{S} \).

1 Introduction

This paper is motivated by Alperin’s suggestion to investigate his weight conjecture ([Alp87]) through the endomorphism rings of certain permutation modules. Let \( k \) be an algebraically closed field of positive characteristic \( p \), \( G \) a finite group, and \( P \) a Sylow \( p \)-subgroup of \( G \). We then write \( kG_P \) for the permutation \( kG \)-module on the cosets of \( P \) in \( G \), and put \( \mathcal{E} := \text{End}_{kG}(kG_P) \). In this paper we present a collection of explicit computational results for such endomorphism algebras. In particular, we find their Cartan matrices and study the socles and heads of their PIMs. Alperin showed in [Alp87, La. 1] that the Green correspondents in \( G \) of weight modules (which we call weight Green correspondents in the sequel) are isomorphic to indecomposable direct summands of \( kG_P \). In our computations we identify the weight Green correspondents in a decomposition of \( kG_P \) into indecomposable direct summands.

The special property of \( \mathcal{E} \) being quasi-Frobenius has been examined by Green in [Gre78]. Green’s results show, that in the case where \( \mathcal{E} \) is quasi-Frobenius, the number of isomorphism types of simple constituents in the socle of \( \mathcal{E}_\mathcal{S} \) is
equal to the number of indecomposable direct summands of $k_G^P$ and to the number of isomorphism classes of simple $kG$-modules. We use his results to give the following equivalent formulation of Alperin’s weight conjecture in this case. Let $k$, $G$, $P$, and $\mathfrak{E}$ be as above. Assume furthermore that $\mathfrak{E}$ is quasi-Frobenius. Then Alperin’s weight conjecture is true for $G$ if and only if each isomorphism type of indecomposable direct summand of $k_G^P$ is a weight Green correspondent.

The question arises if there is a structural relation between the constituents of $\text{soc}(\mathfrak{E}_G)$ and the weight modules not only when $\mathfrak{E}$ is quasi-Frobenius, but in the general case. In this context, we focussed on the analysis of the socle constituents of the PIMs of $\mathfrak{E}_G$ in correspondence to the indecomposable direct summands of $k_G^P$. The investigations indicate that it might be worthwhile to concentrate on groups for which the number of simple socle constituents of $\mathfrak{E}_G$ (up to isomorphism) is equal to the number of simple $kG$-modules (up to isomorphism). In this case, we might use the simple socle constituents of $\mathfrak{E}_G$ as an intermediate tool to structurally connect simple $kG$-modules with weight Green correspondents. There are several million pairs $(G, p)$ for a finite group $G$ and a prime $p$ which we proved to satisfy this assumption. But it should be noted that we have also found one exception, namely $M_{11}$, $p = 3$ (see Page 20).

The results presented here arose from my dissertation under the supervision of Prof. Gerhard Hiss.

2 Preliminaries

Throughout this section we fix an algebraically closed field $k$ of characteristic $p > 0$ and a finite group $G$. We assume that all modules are right modules. Homomorphisms are written on the left, and for two homomorphisms $\varphi, \psi$, the composition is given as $\varphi \circ \psi(x) = \varphi(\psi(x))$. With this convention, $\text{End}_{kG}(M)$ and $\text{Hom}_{kG}(M, N)$ are right $\text{End}_{kG}(M)$-modules for $kG$-modules $M, N$.

Definition 2.1. Let $Q$ be a $p$-subgroup of $G$. If $S$ is a simple $kN_G(Q)$-module with vertex $Q$, we call $(Q, S)$ a weight of $G$ with respect to $Q$. In this case, $Q$ is a weight subgroup, $S$ is a weight module and the Green correspondent of $S$ in $G$ a weight Green correspondent.

If the weight module $S$ belongs to the block $b$ of $kN_G(Q)$ with defect group $D_b$, then $C_G(D_b) \leq C_G(Q) \leq N_G(Q)$. Hence $B := b_G^Q$ is defined. In this case we say that the weight $(Q, S)$ belongs to $B$. Note that the Green correspondent of $S$ belongs to $B$ by [Alp93, Cor. 14.4]. Note also that conjugation induces an equivalence relation on the set of weights of $G$.

Remark 2.2. (a) If $Q \leq G$ is a weight subgroup then $Q$ is $p$-radical, i.e. $Q$ is equal to the largest normal $p$-subgroup of $N_G(Q)$ (compare [Lin04, p.
(b) Let \((Q, S)\) be a weight of \(G\). Then \(S\) is a simple projective \(kN_G(Q)/Q\)-module. Conversely, each projective simple \(kN_G(Q)/Q\)-module is a weight module after inflation along \(Q\).

We are now ready to state Alperin’s weight conjecture, proposed in [Alp87].

**Conjecture 2.3** (Alperin’s Weight Conjecture). Let \(B\) be a block of \(G\). Then the number of weights (up to conjugation) belonging to \(B\) is equal to the number of simple \(kG\)-modules (up to isomorphism) belonging to \(B\).

Note that a weaker form of the conjecture states that the number of weight modules for \(G\) (up to conjugation) is equal to the number of simple \(kG\)-modules (up to isomorphism). The conjecture has been proved for many different families of groups and blocks, such as for blocks with cyclic defect groups ([Dad66]), for \(p\)-solvable groups ([Oku], [IN95]), for symmetric groups ([AF90], [FS82]), for groups of Lie type in defining characteristic ([Cab84]), for general linear groups in non defining characteristic ([AF90]) and for many sporadic groups ([AC95], [An97], [Dad92], [EP99], [Szö98]) etc. Weight modules are strongly related to trivial source modules, as can be seen in the following lemma.

**Lemma 2.4** ([Alp87, La. 1]). Let \(P \in \text{Syl}_p(G)\) be a Sylow \(p\)-subgroup of \(G\). If \(S\) is a weight module of \(G\), then its Green correspondent is isomorphic to an indecomposable direct summand of \(kG_P\).

In view of a general proof of the weight conjecture, Alperin has suggested to analyze certain alternating sums (the relation of which to the weight conjecture is treated in [KR89]) as well as the endomorphism ring \(\mathcal{E} := \text{End}_{kG}(kG_P)\) of the permutation module \(kG_P\) of \(G\) on the cosets of a Sylow \(p\)-subgroup \(P \in \text{Syl}_p(G)\). The latter proposal is the motivation for the present work. In [Nae08] we analyzed, for a large number of finite groups in different characteristics, the endomorphism rings of the associated permutation modules \(kG_P\). We present a selection of this analysis here. The hope is to find patterns in the structural properties of these endomorphism rings which could give a hint for a proof of the weight conjecture.

As indicated in the previous lemma, we need to identify weight Green correspondents in a decomposition of \(kG_P\). This is possible by the following observation.

**Lemma 2.5** ([Szö98, Prop. 6.1.5]). Let \((Q, S)\) be a weight with \(p^a := |G : Q|_p\). Denote the Green correspondent of \(S\) by \(X_S\). Then \(\dim(X_S)_p = p^a\) and \(X_S\) is the only indecomposable direct summand of \(S^G\) with dimension not divisible by \(p^{a+1}\).

The above Lemmas give us a strategy to identify weight Green correspondents in a decomposition of \(kG_P\). We start with a \(p\)-radical subgroup \(Q\) and
check if there are projective simple $kN_G(Q)/Q$-modules. If so, we induce these to $G$ and decompose them into indecomposable modules. By the previous lemma we can easily identify the associated weight Green correspondent.

**Remark 2.6.** (a) Let $X$ be an indecomposable direct summand of $k_G^P$. Then $X$ is also a direct summand of $S^G$, where $S$ is some weight module with vertex $P$ (even if the vertex of $X$ is $Q \subseteq P$): We may write $k_G^P = (k_N^G(P))^G$, and the indecomposable direct summands of $k_N^G(P)$ are exactly the weight modules with weight subgroup $P$. This observation helps to find decompositions of induced weight modules in some cases (compare $G = L_2(13)$ in characteristic 3, Page 34).

(b) If $P \unlhd G$ is a normal Sylow $p$-subgroup, then $E$ is isomorphic to (the semisimple algebra) $kG/P$ as $k$-algebra. In this case $P$ is the only weight subgroup and each simple $kG$-module is a weight module.

(c) If $X$ is an indecomposable direct summand of $k_G^P$, then $\text{Hom}_{kG}(k_G^P, X)$ is a PIM of $E$ by Fitting correspondence.

(d) Note that each isomorphism type of simple $kG$-module occurs as head and socle constituent of $k_G^P$.

**Definition 2.7.** We denote the covariant and left exact fixed point functor $\text{Fix}_P$ by $\mathcal{F}$, i.e. for a finitely generated $kG$-module $M$ we have

$$\mathcal{F}(M) = \{m \in M : mx = m \text{ for all } x \in P\}$$

and homomorphisms are mapped to the respective restrictions. Note that $\mathcal{F}(M)$ and $\text{Hom}_{kG}(k_G^P, M)$ are isomorphic as $k$-vector spaces. Via this isomorphism $\mathcal{F}(M)$ may be equipped with a right $E$-module structure.

### 3 Quasi-Frobenius

We keep the assumptions and notation as in the previous section. If the endomorphism ring of the permutation module $k_G^P$ is quasi-Frobenius, we can use the results of Green’s work ([Gre78]) to get Alperin’s conjecture in terms of the indecomposable direct summands of $k_G^P$. The decomposition of $k_G^P$ into indecomposable direct summands will be denoted as follows:

$$k_G^P = \bigoplus_{i=1}^n \bigoplus_{j=1}^{n_i} X_{ij},$$

where $X_{ij} \cong X_{lm}$ if and only if $i = l$. By Fitting correspondence we get an associated decomposition of the regular module $E_E$ into PIMs.

**Theorem 3.1** ([Gre78, Thm. 1, Thm. 2]). Let the notation be as above. Moreover, assume that $E$ is quasi-Frobenius, i.e. injective as right $E$-module. Then, for any $1 \leq i, l \leq n$:
Quasi-Frobenius

(a) \(\text{soc}(X_{i,1})\) and \(\text{hd}(X_{i,1})\) are simple.
(b) \(\mathcal{F}(\text{soc}(X_{i,1})) = \text{soc}(\mathcal{F}(X_{i,1}))\) and \(\mathcal{F}(\text{hd}(X_{i,1})) = \text{hd}(\mathcal{F}(X_{i,1}))\).
(c) The \(kG\)-modules \(X_{i,1}\) and \(X_{l,1}\) are isomorphic if, and only if \(\text{soc}(X_{i,1}) \cong \text{soc}(X_{l,1})\) or \(\text{hd}(X_{i,1}) \cong \text{hd}(X_{l,1})\).
(d) The map \(M \mapsto \mathcal{F}(M)\) induces a bijection between the simple \(kG\)-modules and the simple \(\mathcal{E}\)-modules.

Remark 3.2. (a) For a similar approach as Green’s, compare also with [Gec01] and [GH97].
(b) In view of Alperin’s weight conjecture, the case where \(\mathcal{E}\) is quasi-Frobenius is of special interest. From Lemma 2.4 we know that the number of weights of \(kG\) (up to conjugation) must be less than or equal to the number \(n\) of isomorphism types of indecomposable direct summands of \(kG_P\). On the other hand, by Fitting correspondence, \(\mathcal{E}\) has \(n\) isomorphism types of simple modules.

Suppose now that \(\mathcal{E}\) is quasi-Frobenius. Then we have by Theorem 3.1

\[
|\{\text{weights}\}| \leq |\{\text{indecomposable direct summands of } kG_P\}| \leq |\{\text{simple } \mathcal{E}\text{-modules}\}| \leq |\{\text{simple } kG\text{-modules}\}|.
\]

This implies that Alperin’s conjecture is true if and only if \textit{each} direct summand of \(kG_P\) is a weight Green correspondent. Since each indecomposable direct summand \(X\) of \(kG_P\) has trivial source, so has its Green correspondent \(T\). This implies that the vertex \(Q\) of \(T\) lies in the kernel of \(T\) and \(T\) has a \(kN_G(Q)/Q\)-module structure. As \(Q\) is the vertex, \(T\) is even a projective \(kN_G(Q)/Q\)-module. Hence Alperin’s conjecture is true if and only if the Green correspondent of each indecomposable direct summand of \(kG_P\) is a simple \(kN_G(Q)\)-module.

Note that the assumption that all indecomposable direct summands of \(kG_P\) are weight Green correspondents does not imply that \(\mathcal{E}\) is quasi-Frobenius, as can be seen in the examples \(M_{22}\) in characteristic 2 (compare Page 27) and \(U_3(3)\) in characteristic 2 (compare Page 29). Each indecomposable direct summand \(X\) of \(kG_P\) is a weight Green correspondent in those examples. But the socle constituents of the corresponding PIMs of \(\mathcal{E}\) show that \(\mathcal{E}\) is not quasi-Frobenius.

Cabanes proved the weight conjecture in [Cab84] for groups of Lie type in defining characteristic. In this case the endomorphism ring \(\mathcal{E}\) is in fact quasi-Frobenius ([Tin80]). Cabanes succeeds in linking the representations of a group of Lie type in defining characteristic and the representations of its parabolic subgroups by means of the Brauer homomorphism. In this way Cabanes shows that the image of each indecomposable direct summand of \(kG_P\) (with vertex \(Q\)) under the Brauer homomorphism, i.e. the Green
4 The socle of $\mathfrak{E}$

Let the notation be as above. With Green’s theorem in mind we focus on the socles and heads of the PIMs of $\mathfrak{E}$ and determine them not only in the quasi-Frobenius case, but in all cases. This analysis gives rise to an equivalence relation on the set of PIMs of $\mathfrak{E}$ and on the set of socle constituents of $\mathfrak{E}$, which we describe now.

Let the decomposition of $kG_P$ be as in (1). We write $\mathfrak{E} = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n_i} P_{ij}$, where $X_{ij}$ is associated to $P_{ij}$ by Fitting correspondence.

Remark 4.1. (a) Define the following relation $\sim$ on the set of isomorphism classes of PIMs of $\mathfrak{E}$: $P_{i,1} \sim P_{m,1}$ if and only if $P_{i,1}$ and $P_{m,1}$ have a socle constituent in common. The transitive closure $\simeq$ of $\sim$ then induces an equivalence relation on the set of isomorphism classes of PIMs of $\mathfrak{E}$. The equivalence classes of $\simeq$ are denoted by $P_l$ for $1 \leq l \leq s$.

(b) The previous equivalence relation obviously induces an equivalence relation on the set of isomorphism classes of indecomposable direct summands of $kG_P$ by Fitting correspondence.

(c) Let $S := \{S \leq \text{soc}(\mathfrak{E}) : S \text{ simple}\}$. The equivalence relation in (a) induces an equivalence relation $\approx$ on $S$ as follows: $S \approx S'$ if and only if there is an equivalence class $\mathcal{P}$ such that $S$ is a socle constituent of $P_{i,1} \in \mathcal{P}$ and $S'$ is a socle constituent of $P_{j,1} \in \mathcal{P}$. The equivalence classes of $\approx$ are denoted by $S_l$.

Our experiments have led to the idea of investigating groups for which the following hypothesis holds.

Hypothesis 4.2. Let $k$ be an algebraically closed field of characteristic $p$. Then for $G, P \in \text{Syl}_p(G)$, $\mathfrak{E} := \text{End}_{kG}(kG_P)$ and $S := \{S \leq \text{soc}(\mathfrak{E}) : S \text{ simple}\} \simeq$ the following holds:

(a) $|S| = |\{(S, Q) : (S, Q) \text{ is a weight of } G\}|$.

(b) $|S| = |\{M : M \text{ is a simple } kG\text{-module}\}|$. 
Remark 4.3. (a) If one can characterize the groups for which the above hypothesis is satisfied, then Alperin’s weight conjecture holds for these group as well. It is therefore a promising question how the groups for which the hypothesis holds can be classified.
(b) We have found one example, namely $M_{11}$ in characteristic 3, for which Hypothesis 4.2 is not satisfied.
(c) GAP provides a data base with all small groups up to order 2000. We have checked and proved the hypothesis on several millions of such groups without finding but the one in (b).
(e) There is a further observation we have made. We fix an equivalence class $\mathcal{P}$ on the PIMs of $\mathfrak{E}$ and let $t$ be the number of simple socle constituents (up to isomorphism) belonging to $\mathcal{P}$ via $\approx$. Assume that there are $t$ simple indecomposable direct summands of $k^G_P$ associated to the PIMs in $\mathcal{P}$ whose $p$-part of their dimensions is less than the $p$-part of the dimensions of the remaining summands belonging to $\mathcal{P}$. Then these $t$ summands are weight Green correspondents in all our examples we have analyzed so far. It turns out, that the assumption we have made on $t$ is satisfied in most cases. An exception can be found in $\mathcal{P}_2$ of $S_7$ in characteristic 2 (see Page 13).

5 Computational methods

In all our computations we used the system GAP4 ([GAP08]) and the MeatAxe ([Mea07]).

We computed the constituents and their multiplicities of the (ordinary) permutation character $1^G_P$. For a splitting field $K$ of characteristic 0 of $G$, we have $\dim_K(\text{Hom}(K^G_P, M_\chi)) = m_\chi$ for each such constituent $\chi$ with corresponding module $M_\chi$ and multiplicity $m_\chi$ in $1^G_P$ (see [CR81, Thm. 11.25]). This information helped us to associate direct summands of $k^G_P$ to PIMs of $\mathfrak{E}$. The GAP-function $\text{MeatAxeStringRightCoset}(G, P, q)$, which we wrote, first computes generators of the permutation group of $G$ on the right cosets of $P$ in $G$. Then it returns these generators as permutation matrices over a field of order $q$ in MeatAxe-readable form. These matrices are passed on to the MeatAxe. In [Szö98], Szöke has developed programs which allow to compute the decomposition of modules into indecomposable direct summands as well as the radical and socle series of a module. We used these programs for decomposing the permutation module, the right regular representation for the endomorphism ring, and for determining the socles and heads of the summands. With the help of the Modular Atlas\footnote{http://www.math.rwth-aachen.de/MOC/decomposition/}, the composition series of each direct summand of $k^G_P$, the dimensions of the simple modules of $\mathfrak{E}$ in characteristic 0, and the composition series of each PIM of $\mathfrak{E}$, we determined the irreducible constituents of the lift of each indecomposable direct summand of $k^G_P$. After that we were able to associate an
indecomposable direct summand of $k^G_P$ to a PIM of $\mathfrak{E}$ (in characteristic $p$). Finally, we had to determine the weight Green correspondents among the indecomposable direct summands of $k^G_P$. For this matter, we used the GAP-function TableOfMarks$(G)$, to filter possible $p$-subgroups of $P$ which might be weight subgroups. If $Q$ was such a candidate, we then tested if there were simple projective $kN_G(Q)/Q$-modules. If so, we passed the matrices returned by MeatAxeRightCoset$(N_G(Q),Q,q)$ to the MeatAxe, extracted the matrices $\text{mats}$ for the projective simple modules and induced them in GAP with the function InducedGModules$(G,N_G(Q),\text{mats})$ to $G$. This module was then decomposed by the MeatAxe. With Lemma 2.5 we could determine the weight Green correspondent in the induced module.

6 Examples

In this section we present a collection of examples which have been analyzed in [Nae08]. If $kG$ has no projective simple modules, we print two matrices, marked by $k^G_P$ and $C^G_P$, respectively. The columns of these matrices correspond to the indecomposable direct summands of $k^G_P$ and $\mathfrak{E}_G$, respectively, and are indexed by their are dimensions. An exponent indicates the multiplicity of the respective summand in the decomposition. The rows of the matrices correspond to the simple $kG$- and $\mathfrak{E}$-modules, respectively and are indexed by their dimensions. An entry $a_{ij}$ then denotes the multiplicity of the simple module of row $i$ in the summand of column $j$. Thus $C^G_P$ contains the Cartan invariants of $\mathfrak{E}$. This part of the matrix will be called the body of $C^G_P$. The projective simple $kG$-modules correspond to projective simple $\mathfrak{E}$-modules and therefore to $1 \times 1$-blocks in the Cartan matrix of $\mathfrak{E}$. We omit these summands in the matrix indexed by $k^G_P$ and the associated PIMs in $C^G_P$. If $kG$ has projective simple modules, we additionally print the decomposition of $k^G_P$ into indecomposable direct summands.

The rows of the matrices indexed by "soc" and "hd" display the isomorphism type of the socle and head of the corresponding direct summand, respectively. If these modules are not simple and if there is no risk of misunderstanding we will omit $\oplus$-signs. The Fitting correspondence is indicated in the first row, labeled by $k^G_P$, below the body of the matrix $C^G_P$. The weight Green correspondents are printed boldly. Moreover, the row labeled by $p^x$ gives the $p$-part of the direct summand of the corresponding column. Finally, the second to the last row of the matrix is $C^G_P$ indexed by EC and displays the equivalence classes as described in Remark 4.1. The corresponding equivalence classes $\mathcal{S}_i$ are not printed separately but can be read off right from the rows indexed by soc and EC. Instead, we indicate in the last row the number $s$ of constituents belonging to the respective $\mathcal{S}_i$.

We will always denote a weight module of dimension $m$ with vertex $Q$ by
We write $N := N_G(P)$. As far as possible we have also given the decomposition of the induced weight modules. The selection of examples cover –apart from the sporadic groups we examined– some particular cases: We chose the pair of groups $A_7/S_7$ to illustrate how Clifford theory applies. Moreover, $M_{11}$ has in characteristic 5 a cyclic block which is not only a real stem. The group $L_2(17)$ is of special interest, as it has in characteristic 3 a cyclic defect group of order $3^2$. Finally we chose some groups of Lie type in defining characteristic to present some of the special properties we have described in the previous sections.

### 6.1 $A_6 \mod 2$

$p = 2$, $G = A_6$, $P \in \text{Syl}_2(G)$, $N_G(P) = P$, $|G| = 2^3 \cdot 3^2 \cdot 5$

The permutation module decomposes into indecomposable direct summands as follows:

$$k^G_P = 1 \oplus 8_1 \oplus 8_2 \oplus 14_1 \oplus 14_2.$$  

<table>
<thead>
<tr>
<th>$k^G_P$</th>
<th>$1$</th>
<th>$14_1$</th>
<th>$14_2$</th>
<th>$C^G_P$</th>
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<th>$3_1$</th>
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<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
<td>$3_1$</td>
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<tr>
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<td>$2$</td>
<td>$1$</td>
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<td></td>
</tr>
<tr>
<td>$k^G_P$</td>
<td>$4_2$</td>
<td>$1$</td>
<td>$2$</td>
<td>$1$</td>
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<td></td>
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</tr>
<tr>
<td>soc</td>
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<td>$4_1$</td>
<td>$4_2$</td>
<td>$2^e$</td>
<td>$2^0$</td>
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<tr>
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<td>$1_2$</td>
<td>$1_3$</td>
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The computations show that all indecomposable direct summands of $k^G_P$ are uniserial.

Note that $A_6 \cong \text{Sp}_4(2)'$, the first derived group of a group of Lie type. We apply Baer’s criterion to prove that $E$ is quasi-Frobenius. From the Cartan matrix it is easy to find the regular representation of the algebra $\mathcal{E} := \varepsilon_1 \mathcal{E}_{\varepsilon_1} \oplus \varepsilon_2 \mathcal{E}_{\varepsilon_2}$, where $\varepsilon_1$ and $\varepsilon_1$ are the idempotents in $\mathcal{E}$ associated to $3_1$ and $3_2$, respectively. Considering all possible ideals of $\mathcal{E}$, and all homomorphisms from those ideals to $3_1$ and $3_2$, respectively, we see that all such homomorphisms may be extended to $\mathcal{E}$. Hence $\mathcal{E}$ and therefore $E$ is quasi-Frobenius.

Both indecomposable direct summands $8_1$ und $8_2$ of $k^G_P$ are simple projective $kG$-modules. Moreover, the vertices $Q_1$ and $Q_2$ of the indecomposable direct summands $14_1$ and $14_2$ of $k^G_P$, respectively, have order 4 and are not conjugate in $G$. 


For the induced weight modules (each of dimension 2) we find:

\[ 2S^G_{N(Q_1)} = 14_1 \oplus 8_1 \oplus 8_2, \]
\[ 2S^G_{N(Q_2)} = 14_2 \oplus 8_1 \oplus 8_2. \]

For the trivial weight module we have:

\[ 1S^G_N = 1 \oplus 8_1 \oplus 8_2 \oplus 14_1 \oplus 14_2. \]

Finally we notice that each PIM lies in an equivalence class with one element.

### 6.2 $S_6 \mod 3$

| $p = 3$, $G = S_6$, $P \in \text{Syl}_3(G)$, $|G| = 2^4 \cdot 3^2 \cdot 5$ |

The permutation module decomposes into indecomposable direct summands as follows:

\[ k^G_P = 1_1 \oplus 1_2 \oplus 9_1 \oplus 9_2 \oplus 10_1 \oplus 10_2 \oplus 20^2. \]

<table>
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<th>$10_2$</th>
<th>$20^2$</th>
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**Remark:** The Sylow 3-subgroups are isomorphic to $C_3 \times C_3$.
The indecomposable direct summands $10_1$, $10_2$ and $20$ of $k^G_P$ have simple head and socle but are not uniserial as computation has shown. Moreover, each indecomposable direct summand of $k^G_P$ is a weight Green correspondent. Thus $S_6$ satisfies Hypothesis 4.2 in characteristic 3 as each PIM of $E$ lies in an equivalence class with one element.
The socle of each PIM of $E$ is simple. From the Cartan matrix we deduce that $E$ is quasi-Frobenius.
All indecomposable direct summands of $k^G_P$, except for the simple projective ones $9_1$ and $9_2$, lie in the principal block and have a Sylow 3-subgroup as vertex. The induced weight modules decompose as follows:
\[ S_G^1 = 1_1 \oplus 9_1, \quad S_G^{12} = 1_2 \oplus 9_2, \quad S_G^{2} = 20, \]
\[ S_G^{13} = 10_1, \quad S_G^{14} = 10_2. \]

### 6.3 $A_7 \mod 2$ and $A_7 \mod 3$

\[ p = 2, \quad G = A_7, \quad P \in \text{Syl}_2(G), \quad N_G(P) = P, \quad |G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \]

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**Remark:** Note that we have omitted the $\oplus$-sign in the socle of the PIM 10 of $\mathfrak{C}$. The weight Green correspondents $6, 14_2, 14_3$ and $70$ have a vertex $Q_1$ of order 4. The induced weight modules decompose as follows:

\[ S_G^{21}(Q_1) = 70, \quad S_G^{22}(Q_1) = 6 \oplus 64, \]
\[ S_G^{23}(Q_1) = 14_2 \oplus 56, \quad S_G^{24}(Q_1) = 14_3 \oplus 56. \]

The weight Green correspondent $14_1$ has vertex $Q_2$ of order 4 which is not conjugate in $G$ to $Q_1$. We have:

\[ S_G^{25}(Q_2) = 14_1 \oplus 20 \oplus 64 \oplus 56^2. \]
From the latter decomposition we see, that the direct summand 20 of $k^G_P$ has a vertex $Q_3$ of order less than $4 = |Q_2|$, hence $|A_7 : Q_3|_2 \in \{2^2, 2^3\}$. As $|20|_2 = 2^2$ we see that the order of $Q_3$ must be 2.

As $N_G(P) = P$ we have $1S^G_{N(Q)} = k^G_P$ for the induction of the trivial weight module, and the trivial module is the only weight module with vertex $P$.

\[
p = 3, \ G = A_7, \ P \in \text{Syl}_3(G), \ |G| = 2^3 \cdot 3^2 \cdot 5 \cdot 7
\]

\[
\begin{array}{cccccccc}
k^G_P & 1 & 15^2 & 36 & 6^2 & 10_1 & 10_2 & 45_1 & 45_2 & 28 & 63 \\
\hline
1 & 1 \\
15 & 1 & 2 \\
6 & 1 & 1 \\
10_1 & 1 & 2 & 1 & 1 \\
10_2 & 1 & 1 & 2 & 1 \\
13 & 1 & 1 & 2 & 3 \\
\hline
soc & 1 & 15 & 15 & 6 & 10_1 & 10_1 & 10_2 & 10_2 & 13 & 13 \\
hd & 1 & 15 & 15 & 6 & 10_1 & 10_1 & 10_2 & 10_2 & 13 & 13 \\
\hline
C^G_P & 1 & 3^2 & 4_2 & 2^1 \cdot 2_2 & 5_1 & 2_3 & 5_2 & 4_2 & 7 \\
\hline
1_1 & 1 \\
2_1 & 1 & 1 \\
2_2 & 1 & 2 \\
1_3 & 1 & 1 \\
1_4 & 1 & 2 & 1 & 1 \\
1_5 & 1 & 1 \\
1_6 & 1 & 1 & 2 & 1 \\
1_7 & 2 & 2 \\
1_8 & 1 & 1 & 2 & 3 \\
\hline
k^G_P & 1 & 15 & 36 & 6 & 10_1 & 10_2 & 45_1 & 45_2 & 28 & 63 \\
3^2 & 3^0 & 3^0 & 3^2 & 3^0 & 3^0 & 3^2 & 3^0 & 3^2 \\
\hline
soc & 1_1 & 1_2 & 1_2 & 2_2 & 1_4 & 1_4 & 1_6 & 1_6 & 1_8 & 1_8 \\
hd & 1_1 & 2_1 & 1_2 & 2_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & 1_8 \\
EC & P_1 & P_2 & P_3 & P_4 & P_5 & P_6 \\
s & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Remark: The Sylow 3-subgroups are isomorphic to $C_3 \times C_3$.

The weight Green correspondents 6 and 15 have vertex $Q_1$ of order 3. The corresponding induced weight modules decompose as follows:

\[
3^1 S^G_{N(Q_1)} = 6 \oplus 36 \oplus 63 \quad 3^2 S^G_{N(Q_1)} = 15 \oplus 45_1 \oplus 45_2.
\]

From this it follows that 36, 45_1, 45_2 and 63 are projective indecomposable direct summands of $k^G_P$. The remaining weight Green correspondents 1, 28,
10_1 and 10_2 have P as a vertex. The decompositions of the induced weight modules are given as:

\[ 1^1 S^G_N = 1 \oplus 6 \oplus 63, \quad 1^2 S^G_N = 28 \oplus 6 \oplus 36, \]
\[ 1^3 S^G_N = 10_1 \oplus 15 \oplus 45_2, \quad 1^4 S^G_N = 10_2 \oplus 15 \oplus 45_1. \]

6.4 \( S_7 \mod 2 \) and \( S_7 \mod 3 \)

\[ p = 2, \ G = S_7, \ P \in \text{Syl}_2(G), \ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \]

\[
\begin{array}{cccccccc}
\hline
k^G_P & 1 & 6 & 20 & 28 & 14 & 112 & 64 & 70 \\
\hline
1 & 1 & 1 & 4 & 2 & 2 & 6 & 1 & 2 \\
6 & 1 & 2 & 2 & 8 & 1 & 2 & 14 & 1 & 2 \\
8 & 1 & 2 & 3 & 2 & 14 & 1 & 2 & 3 & 2 \\
14 & 1 & 2 & 3 & 2 & 14 & 1 & 2 & 3 & 2 \\
20 & 4 & 1 & 2 & 14 & 1 & 2 & 3 & 2 & 3 \\
\hline
\text{soc} & 1 & 6 & 6 & 8 & 14 & 20 & 14 & 14 \oplus 20 & 14 \oplus 20 \\
\text{hd} & 1 & 6 & 6 & 8 & 14 & 20 & 14 & 14 \oplus 20 & 14 \oplus 20 \\
\hline
C^G_P & 1 & 2 & 4 & 3_1 & 3_2 & 7_1 & 7_2 & 8 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
3 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
4 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
5 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
6 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
7 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
8 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
\hline
k^G_P & 1 & 6 & 20 & 28 & 14 & 112 & 64 & 70 \\
\hline
2^x & 2^0 & 2^1 & 2^2 & 2^1 & 2^2 & 2^3 & 2^4 & 2^5 & 2^6 \\
\text{soc} & 1 & 13 & 1_1 & 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & 1_8 \\
\text{hd} & 1 & 13 & 1_1 & 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 & 1_8 \\
\text{EC} & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 \\
\hline
s & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
\hline
\end{array}
\]

Remark: Our computation has shown, that the indecomposable direct summands 6, 20 and 28 of \( k^G_P \) are uniserial and belong to a block whose defect groups have order \( 2^3 \). Note that the corresponding PIMs of \( \mathcal{E} \) do not all have simple socles.

The weight Green correspondent 28 has a vertex \( Q_1 \) of order 4. Inducing this weight module we get:

\[ 1^1 S^G_{N(Q_1)} = 28 \oplus 112. \]
The weight Green correspondents 70, 14 and 6 have pairwise non conjugate vertices $Q_1$, $Q_2$ and $Q_3$, respectively, of order 8. The decompositions of the induced weight modules are given as:

\[
\begin{align*}
2SG_N(Q_2) &= 70 \oplus 28 \oplus 112, \\
2SG_N(Q_3) &= 14 \oplus 20 \oplus 64 \oplus 112, \\
2SG_N(Q_4) &= 6 \oplus 28 \oplus 64 \oplus 112.
\end{align*}
\]

(2)

The indecomposable direct summand 20 of $k^G_P$ has a vertex of order $2^2$ by Equation (2) and the fact that $|20|_2 = 2^2$.

As $N_G(P) = P$, we have $1S^G_N = k^G_P$ for the trivial module.

\[
p = 3, \ G = S_7, \ P \in \text{Syl}_3(G), \ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 7
\]
Remark: The Sylow 3-subgroups are isomorphic to $C_3 \times C_3$.
The corresponding induced weight module decompose as follows:

\[
\begin{align*}
3_1S_{N(Q)}^G &= 15_1 \oplus 90, \\
3_2S_{N(Q)}^G &= 6_1 \oplus 36_2 \oplus 63_1, \\
3_3S_{N(Q)}^G &= 15_2 \oplus 90, \\
3_4S_{N(Q)}^G &= 6_2 \oplus 36_2 \oplus 63_2.
\end{align*}
\]

From the decomposition matrix of $kG$ we see that the indecomposable direct summands $6_1, 6_2, 15_1, 15_2, 36_1,$ and $36_2$ of $k_P^G$ belong to blocks of $kG$ with cyclic defect group of order 3. From this it is clear that the weight Green correspondents among them are uniserial ([Alp93, La. 22.3]). Note that the indecomposable direct summands $36_1, 36_2, 63_1, 63_2$ and $90$ of $k_P^G$ are projective.

The remaining induced weight modules decompose as follows:

\[
\begin{align*}
1_1S_N^G &= 1_1 \oplus 6_1 \oplus 63_1, \\
1_2S_N^G &= 1_2 \oplus 6_2 \oplus 63_2, \\
2_1S_N^G &= 20 \oplus 15_1 \oplus 15_2 \oplus 90.
\end{align*}
\]

6.5 $A_8 \mod 3$

$p = 3, G = A_8, P \in \text{Syl}_3(G), |G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$
The permutation module $k^G_P$ decomposes as follows:

$$k^G_P = 1 \oplus 45^5 \oplus 45^5 \oplus 7 \oplus 21^4 \oplus 27^2 \oplus 28 \oplus 34 \oplus 35^2 \oplus 63 \oplus 90 \oplus 99^2 \oplus 162^3 \oplus 225^3.$$

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| soc     | $1$ | $21$  | $21$ | $7$ | $7$    | $13$ | $13$   | $35$ | $35$   | $28$ | $28$ |
| hd      | $1$ | $21$  | $21$ | $7$ | $7$    | $13$ | $13$   | $35$ | $35$   | $28$ | $28$ |

| $C^G_P$ | $1$ | $5^4$ | $7_1$ | $3$ | $7_2^2$ | $6$ | $10$   | $7_3$ | $15^2$ | $25^3$ | $18^3$ | $4$ |
|---------|-----|--------|------|-----|--------|------|--------|------|--------|------|------|
| $1$     |     |        |      |     |        |      |        |      |        |      |      |
| $4$     |     | $1$    | $1$  |     |        |      |        |      |        |      |      |
| $12$    |     | $1$    | $3$  |     |        |      |        |      |        |      |      |
| $13$    |     | $1$    | $1$  |     |        |      |        |      |        |      |      |
| $21$    |     | $1$    | $2$  | $1$ | $1$    |      |        |      |        |      |      |
| $24$    |     | $1$    | $2$  | $2$ | $1$    |      |        |      |        |      |      |
| $25$    |     | $1$    | $2$  | $3$ | $1$    |      |        |      |        |      |      |
| $31$    |     | $1$    | $1$  | $2$ | $2$    | $4$  | $2$    |      |        |      |      |
| $32$    |     | $1$    | $2$  | $3$ | $1$    |      |        |      |        |      |      |
| $15$    |     | $1$    | $1$  |     | $1$    |      |        |      |        |      |      |

| $k^P_P$ | $1$ | $21$  | $63$ | $7$ | $27$   | $34$ | $90$   | $35$ | $99$   | $225$ | $162$ | $28$ |
|---------|-----|--------|------|-----|--------|------|--------|------|--------|------|------|
| $1$     |     | $1$    |      |     |        |      |        |      |        |      |      |
| $3^x$   |     | $3^x$  | $3^2$ | $3^9$ | $3^3$ | $3^3$ | $3^{2^3}$ | $3^9$ | $3^2$ | $3^2$ | $3^3$ | $3^3$ |
| soc     | $1$ | $12$  | $12$ | $21$ | $15_2$ | $15$ | $15$   | $31$ | $31$   | $31$ | $32$ | $32$ |
| hd      | $1$ | $4$   | $12$ | $13$ | $21$   | $14$ | $15$   | $22$ | $23$   | $31$ | $32$ | $16$ |

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**Remark:** The indecomposable direct summands $45_1$ and $45_2$ of $k^G_P$ are simple projective modules. Moreover the summands $21$ and $63$ of $k^G_P$ lie in a block with defect $1$. The remaining summands belong to the principal block.

Let $Q_1$ be a vertex of order $3$ of the summand $21$ of $k^G_P$. The corresponding induced weight module decomposes as follows:

$$^6G^G_{N(Q_1)} = 21 \oplus 45_1 \oplus 45_2 \oplus 225.$$  

In particular, $225$ is a projective indecomposable $kG$-module. The remaining induced weight modules have vertex $P$ and decompose as follows:
6.6 $A_9 \mod 2$ and $A_9 \mod 3$

$p = 2, G = A_9, P \in \text{Syl}_2(G), N_G(P) = P, |G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$

\[
1^{11} S^G_N = 1 \oplus 27 \oplus 90 \oplus 162,
\]
\[
1^{12} S^G_N = 7 \oplus 21 \oplus 45_1 \oplus 45_2 \oplus 162,
\]
\[
1^{13} S^G_N = 28 \oplus 27 \oplus 63 \oplus 162,
\]
\[
1^{14} S^G_N = 34 \oplus 21 \oplus 225,
\]
\[
2^{12} S^G_N = 35 \oplus 21 \oplus 45_1 \oplus 45_2 \oplus 99 \oplus 225.
\]

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Remark: From the decomposition matrix of $kG$ we know that the indecomposable direct summands $8$, $48$ and $432$ of $kP$ belong to a block with defect group of order $8$.

The weight Green correspondent $432$ has a vertex $Q_1$ of order $4$. The corresponding induced weight module decomposes as follows:

$$8_S^{G}_{N(Q_1)} = 432 \oplus 576.$$ 

Moreover, the weight Green correspondents $120_1$, $120_2$ and $48$ have pairwise non conjugate vertices $Q_1$, $Q_2$ and $Q_3$, respectively, of order $8$. We find as decomposition of the corresponding induced weight modules:

$$8_S^{G}_{N(Q_2)} = 120_1 \oplus 384 \oplus 576,$n
$$8_S^{G}_{N(Q_3)} = 120_2 \oplus 384 \oplus 576,$n
$$8_S^{G}_{N(Q_4)} = 48 \oplus Y,$n

where $Y$ denotes a sum of indecomposable direct summands whose dimensions have 2-parts greater than $8$.

The weight Green correspondent $252$ has a vertex $Q_5$ of order $16$. The corresponding induced weight module decomposes as follows:

$$4_S^{G}_{N(Q_5)} = 252 \oplus 432 \oplus 576.$$
Finally, the weight Green correspondents 258, 1261 and 1262 have pairwise non-conjugate vertices $Q_6$, $Q_7$ and $Q_8$ of order 32. The induced weight modules decompose as follows:

\[
2S^G_{N(Q_6)} = 258 \oplus 48 \oplus 120_1 \oplus 120_2 \oplus 384 \oplus 576, \\
2S^G_{N(Q_7)} = 126_1 \oplus 120_1 \oplus 252 \oplus 384 \oplus 432 \oplus 576, \\
2S^G_{N(Q_8)} = 126_2 \oplus 120_2 \oplus 252 \oplus 384 \oplus 432 \oplus 576.
\]

As $N_G(P) = P$, the induced trivial weight module decomposes as $^1S^G_N = k^G_P$.

$p = 3$, $G = A_9$, $P \in \text{Syl}_3(G)$, $|G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7$

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soc 1 2 7 21 41 35 27 189 189
hd 1 2 7 21 41 35 27 189 189

EC $\mathcal{P}_1$ | $\mathcal{P}_2$ | $\mathcal{P}_3$ | $\mathcal{P}_4$

s 1 | 4 | 1 | 1
Remark: The summand 162 of \( k_G^P \) is simple projective. The weight Green correspondents 27, 189 lie in a block with defect groups of order \( 3^3 \). Furthermore, they have a vertex \( Q_1 \) of order 3. The decomposition of the induced weight modules is as follows:

\[
9_1 S_{N(Q_1)}^G = 189 \oplus 162 \oplus 405, \quad 9_2 S_{N(Q_1)}^G = 27 \oplus 162 \oplus 567.
\]

Note that the indecomposable direct summand 405 of \( k_G^P \) is projective. The weight Green correspondents 252, 84 and 435 have vertices of order 9. But only the vertices of 84 and 435 are conjugate in \( G \). The induced weight modules decompose as follows:

\[
9 S_{N(Q_2)}^G = 252 \oplus 243 \oplus 162^2 \oplus 405^2 \oplus 891, \\
3_1 S_{N(Q_3)}^G = 435 \oplus 405, \\
3_2 S_{N(Q_3)}^G = 84 \oplus 162 \oplus 189 \oplus 405.
\]

Finally, the trivial module and 118 have \( P \) as a vertex. Inducing the corresponding weight modules, we get:

\[
11 S_N^G = 1 \oplus 27 \oplus 405 \oplus 435 \oplus 252, \\
12 S_N^G = 118 \oplus 84 \oplus 405 \oplus 162^2 \oplus 189.
\]

6.7 \( M_{11} \mod p \) for \( p = 2, 3, 5, 11 \)

\[ p = 2, G = M_{11}, P \in \text{Syl}_2(G), N_G(P) = P, |G| = 2^4 \cdot 3 \cdot 5 \cdot 11 \]

The permutation module decomposes as follows:

\[
k_G^P = 1 \oplus 16_1 \oplus 16_2 \oplus 10 \oplus 44 \oplus 120 \oplus 144^2.
\]

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Remark: The indecomposable direct summands $16_1$ and $16_2$ of $k_P^G$ are simple projective.

The weight Green correspondent $44$ has vertex $Q_1$ of order $4$. The induction of the corresponding weight module gives:

$$2^G S^G_{N(Q_1)} = 44 \oplus 16_1 \oplus 16_2 \oplus 120 \oplus 144.$$ 

The weight Green correspondent $10$ has vertex $Q_2$ of order $8$. Here we get the following decomposition of the induced weight module:

$$2^G S^G_{N(Q_2)} = 10 \oplus 16_1 \oplus 16_2 \oplus 144^2.$$ 

As $N_G(P) = P$ we have $1^G S^G_N = k_P^G$ for the trivial weight module.

\[ p = 3, \ G = M_{11}, \ P \in \text{Syl}_3(G), \ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11 \]

The permutation modules decomposes as:

$$1 \oplus 45^5 \oplus 10 \oplus 11^2 \oplus 54 \oplus 55_1 \oplus 55_2 \oplus 65_1^2 \oplus 65_2^2 \oplus 99^2.$$
Remark: The indecomposable direct summand 45 of $k_G^P$ is simple projective. The remaining weight Green correspondents 1, 55, 10, 65, and 11 have the Sylow 3-subgroup $P$ as vertex. We get the following decomposition from the corresponding induced weight module:

$$
\begin{align*}
1^{11} S_N^G &= 1 \oplus 54, & 1^{12} S_N^G &= 55_1, \\
1^{13} S_N^G &= 10 \oplus 45, & 1^{14} S_N^G &= 55_2, \\
2^{21} S_N^G &= 65_1 \oplus 45, & 2^{22} S_N^G &= 65_2 \oplus 45, \\
2^{23} S_N^G &= 11 \oplus 99.
\end{align*}
$$

This is the only example we have found so far, where Hypothesis 4.2 does not hold. To the equivalence class $P_3$ correspond less weight Green correspondents than socle constituents.

\[ p = 5, \ G = M_{11}, \ P \in \text{Syl}_5(G), \ N_G(P) = P, \ |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11 \]

$$
\begin{array}{c|cccccccc}
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n\hline
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1_2 & 1 & 1 \\
2_1 & 1 & 1 & 1 \\
1_3 & 1 & 2 & 1 & 1 \\
1_4 & 1 & 1 & 1 & 1 \\
1_5 & 1 & 1 & 1 & 1 \\
2_2 & 1 & 2 & 1 & 1 \\
2_3 & 1 & 1 & 2 & 1 \\
2_4 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
k^G_P & 1 & 10 & 54 & 11 & 55_1 & 55_2 & 65_1 & 65_2 & 99 \\
3^2 & 3^0 & 3^0 & 3^3 & 3^6 & 3^0 & 3^0 & 3^0 & 3^2 & 3^2 \\
soc & 1_1 & 1_3 & 1_3 & 1_4 & 1_3_1 & 1_2 & 1_4 & 1_5 & 2_1 & 2_4 & 1_4 & 2_3 \ 2_4 \\
hd & 1_1 & 1_2 & 1_3 & 1_2 & 1_4 & 1_5 & 1_5 & 1_5 & 1_5 & 1_5 \\
EC & \mathcal{P}_1 & \mathcal{P}_2 & \mathcal{P}_3 \\
s & 1 & 2 & 6
\end{array}
$$
Remark: The indecomposable direct summands 10₁, 10₂, 10₃, 45 and 55 of $k^G_P$ are simple projective modules. Moreover the indecomposable direct summands 55₁, 60₁ and 60₂ are projective $kG$-modules. The remaining summands of $k^G_P$ are weight Green correspondents. The decomposition of the corresponding weight modules are given as follows:

$$11^1 S^G_N = 16₁ \oplus 10₁ \oplus 10₂ \oplus 60₂ \oplus 45^3 \oplus 55^3,$$

$$12^1 S^G_N = 11 \oplus 45 \oplus 55^3 \oplus 55₂ \oplus 60₁ \oplus 60₂,$$

$$13^1 S^G_N = 1 \oplus 10^2_2 \oplus 45^2 \oplus 55^2 \oplus 55₂ \oplus 60₁ \oplus 60₂,$$

$$14^1 S^G_N = 16₂ \oplus 10₁ \oplus 10₂ \oplus 60₁ \oplus 45^3 \oplus 55^3.$$

$p = 11, G = M_{11}, P \in \text{Syl}_{11}(G), N_G(P) = P, |G| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$

The permutation module decomposes as follows:

$$k^G_P = 1 \oplus 11 \oplus 44^4 \oplus 55^5 \oplus 45₁ \oplus 45₂ \oplus 45₃ \oplus 45₄ \oplus 77.
Remark: The indecomposable direct summands 11, 44 and 55 of $k^G_P$ are simple projective. Moreover, the summand 77 of $k^G_P$ is projective. The remaining summands of $k^G_P$ are weight Green correspondents. The decomposition of the respective induced weight module is given as follows:

\[
\begin{align*}
{_{11}S}^G_N &= 1 \oplus 11 \oplus 55 \oplus 77, \\
{_{12}S}^G_N &= 45_1 \oplus 44 \oplus 55, \\
{_{13}S}^G_N &= 45_2 \oplus 44 \oplus 55, \\
{_{14}S}^G_N &= 45_3 \oplus 44 \oplus 55, \\
{_{15}S}^G_N &= 45_4 \oplus 44 \oplus 55.
\end{align*}
\]

6.8 $M_{12}$ mod 2 and $M_{12}$ mod 3

$p = 2, G = M_{12}, P \in \text{Syl}_2(G), |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$

The permutation module decomposes as follows:

\[
k^G_P = 1 \oplus 16_1 \oplus 16_1 \oplus 144 \oplus 320^2 \oplus 318 \oplus 350.
\]
Remark: The indecomposable direct summand $320$ of $k^G_P$ is projective. The remaining summands are weight Green correspondents. Among those the summands $16_1$, $16_2$ and $144$ have vertex $Q_1$ of order 4. For the decomposition of the induced modules we get:

$$\begin{align*}
21S^G_{N(Q_1)} &= 144 \oplus 320 \oplus 384 \oplus 832, \\
22S^G_{N(Q_1)} &= 16_1 \oplus 192 \oplus 320 \oplus 384 \oplus 768, \\
23S^G_{N(Q_1)} &= 16_2 \oplus 192 \oplus 320 \oplus 384 \oplus 768.
\end{align*}$$

The indecomposable direct summands $350$ and $318$ of $k^G_P$ have vertices $Q_2$ and $Q_3$, respectively, of order 32 which are not conjugate in $G$. The decomposition of the induced weight modules is:

$$\begin{align*}
2S^G_{N(Q_2)} &= 350 \oplus 320, \\
2S^G_{N(Q_3)} &= 318 \oplus 16_1 \oplus 16_2 \oplus 320.
\end{align*}$$

Finally, we have $1S^G_N = k^G_P$, as $N_G(P) = P$.

$p = 3, G = M_{12}, P \in Syl_3(G), |G| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$

The permutation modules decomposes as follows:

$$k^G_P = 1 \oplus 54^2 \oplus 45^2 \oplus 99^2 \oplus 189 \oplus 243^3 \oplus 66_1 \oplus 66_2 \oplus 66_3 \oplus 66_4 \oplus 175_1 \oplus 175_2 \oplus 297 \oplus 351_1 \oplus 351_2 \oplus 592.$$
Remark: The indecomposable direct summand 54 of $kG_P$ is simple projective. Moreover, the indecomposable direct summands 45, 99, 189 and 243 of $kG_P$ lie in a block with cyclic defect group of prime order. Among those, 45 and 99 are weight Green correspondents with vertex $Q_1$ of order 3. Note that the indecomposable direct summands 189 and 243 of $kG_P$ must be projective modules, as can be seen by the following decomposition of the induced weight modules belonging to this block.

\[ 3_{1}^{5}S^2_{N(Q_1)} = 99 \oplus 54 \oplus 189 \oplus 243^{4} \oplus _{i=1,2} 297_{i} \oplus 297_{3} \oplus _{i=1,2} 351_{2}^{3} \]

\[ 3_{2}^{5}S^2_{N(Q_1)} = 45 \oplus 54^{2} \oplus 189^{2} \oplus 243^{4} \oplus 297_{1} \oplus _{i=1,2} (35^{1}_{2} \oplus 378_{i}) \]
The remaining indecomposable direct summands of $k^G_P$ belong to the principal block. We find $66_1$, $66_2$ and $66_3$, $66_4$ to have vertices of order 9, where the first two and the last two modules have conjugate vertices, respectively. We find for the decomposition of the induced modules:

\[ 3_1 S^G_N(Q_2) = 66_1 \oplus 54 \oplus 243 \oplus 297, \]
\[ 3_2 S^G_N(Q_2) = 66_2 \oplus 243 \oplus 351, \]
\[ 3_1 S^G_N(Q_3) = 66_3 \oplus 54 \oplus 243 \oplus 297, \]
\[ 3_2 S^G_N(Q_3) = 66_4 \oplus 243 \oplus 351. \]

Finally the weight Green correspondents $1, 175_1, 175_2$ and $592$ have a Sylow 3-subgroup as vertex. We get the following decomposition of the corresponding induced of weight modules:

\[ 1_1 S^G_N = 1 \oplus 54^2 \oplus 66_1 \oplus 66_2 \oplus 99 \oplus 243 \oplus 297, \]
\[ 1_1 S^G_N = 175_1 \oplus 45 \oplus 66_3 \oplus 243 \oplus 351_1, \]
\[ 1_1 S^G_N = 175_2 \oplus 45 \oplus 66_4 \oplus 243 \oplus 351_2, \]
\[ 1_1 S^G_N = 592 \oplus 99 \oplus 189. \]

### 6.9 $M_{22}$ mod 2

$p = 2, G = M_{22}, P \in \text{Syl}_2(G), |G| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

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<th>$1078$</th>
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\[ \begin{array}{c|ccccccc} 
\text{C}^G_p \backslash \text{k}^G_p & 1 & 11 & 20 & 7_1 & 7_2 & 9 & 5 \\
\hline
1_1 & 1 \\
1_2 & 4 & 4 & 1 & 1 & 1 \\
1_3 & 4 & 8 & 2 & 2 & 3 & 1 \\
1_4 & 1 & 2 & 2 & 2 \\
1_5 & 1 & 2 & 2 & 2 \\
1_6 & 1 & 3 & 3 & 2 \\
1_7 & 1 & 6 \\
\hline
\text{k}^G_p \backslash \text{C}^G_p & 1 & 848 & 1078 & 616_1 & 616_2 & 230 & 76 \\
\hline
2^x & 2^0 & 2^4 & 2^3 & 2^3 & 2 & 2^2 \\
soc & 1_1 & 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 \\
hd & 1_1 & 1_2 & 1_3 & 1_4 & 1_5 & 1_6 & 1_7 \\
\text{EC} & P_1 & P_2 \\
s & 1 & 6 \\
\end{array} \]

Remark: All indecomposable direct summands of \( k^G_P \) are weight Green correspondents. The weight Green correspondent 848 has vertex \( Q_1 \) of order 8. The induced weight module decomposes as follows:

\[ 8^1 S^G_N(Q_1) = 848 \oplus 896_1 \oplus 896_2. \]

The two weight Green correspondents 616_1 and 616_2 have vertex \( Q_2 \) of order 16. We get as decomposition of the corresponding induced weight modules:

\[ 8^1 S^G_N(Q_2) = 616_1, \]

\[ 8^2 S^G_N(Q_2) = 616_2. \]

The indecomposable direct summand 76 of \( k^G_P \) has vertex \( Q_3 \) of order \( 2^5 \). Here we get for the induced weight module:

\[ 2^2 S^G_N(Q_3) = 76 \oplus 896_1 \oplus 896_2 \oplus 848 \oplus 1904. \]

The weight Green correspondents 230 and 1078 have non conjugate vertices \( Q_4 \) and \( Q_5 \), respectively, of order \( 2^6 \). We have:

\[ 2^2 S^G_N(Q_4) = 230 \oplus 616_1 \oplus 616_2 \oplus 848, \]

\[ 2^2 S^G_N(Q_5) = 1078 \oplus 616_1 \oplus 616_2. \]

As \( N_G(P) = P \), the trivial module is the only weight module with vertex \( P \) and we have \( 1^G_N = k^G_P \).

6.10 \( U_3(3) \) mod 2

\[ p = 2, G = U_3(3), P \in \text{Syl}_2(G), N_G(P) = P, |G| = 2^5 \cdot 3^3 \cdot 7 \]
The permutation module decomposes as follows:

\[ k_G^P = 1 \oplus 32_1 \oplus 32_2 \oplus 62_1 \oplus 62_2. \]

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| \([C_P^G]^{kG}\) | 13   | 1    | 5    | 6    |
|------------------|------|------|------|
| 1                | 1    |      |
| 1_2              |    3 | 2    |
| 1_3              |    2 | 4    |
| soc              | 1    | 14   | 6    |
| hd               | 1    | 14   | 6    |

### Remark:
All indecomposable direct summands of \(k_G^P\) are weight Green correspondents. Note that the endomorphism ring is not quasi-Frobenius. The indecomposable direct summands 32_1 and 32_2 are simple projective modules. The weight Green correspondents 62_1 and 62_2 have non conjugate vertices of order 16. We get the following decomposition for the induced weight modules.

\[ ^2S_{N(Q_1)}^G = 62_2 \oplus 32_1 \oplus 32_2, \quad ^2S_{N(Q_2)}^G = 62_1 \oplus 32_1 \oplus 32_2. \]

As \(N_G(P) = P\), we have \(^1S_N^G = k_P^G\).
6.11 \( G = J_2 \mod 2 \)

\[ p = 2, \; G = J_2, \; P \in \text{Syl}_2(G), \; |G| = 2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \]

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**Remark:** The indecomposable direct summands 160, 288_1, 288_2 and 896 of \( k^G_p \) belong to the block of \( G \) with defect group of order 4. Since the dimensions of the induced weight modules are too large to apply the MeatAxe directly, we use M. Szöke’s results in [Szö98] to identify the weight Green correspondents.

6.12 \( G = L_2(11) \mod 2 \)

\[ p = 2, \; G = L_2(11), \; P \in \text{Syl}_2(G), \; |G| = 2^2 \cdot 3 \cdot 5 \cdot 11 \]
The permutation module $k_G^P$ decomposes as follows:

$$k_G^P = 1 \oplus 12^3 \oplus 12^3_2 \oplus 16_1 \oplus 5_1 \oplus 16_2 \oplus 5_2 \oplus 20 \oplus 10^3.$$ 

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**Remark:** The indecomposable direct summands $12_1$ and $12_2$ of $k_G^P$ are simple projective. The summand $10$ of $k_G^P$ is a weight Green correspondent with vertex $Q_1$ of order 2. It lies in a block with cyclic defect group of prime order. We get the following decomposition of the induced weight module:

$$2S_{NG(Q_1)}^G = 10 \oplus 12^3 \oplus 12^3_2 \oplus 16_1 \oplus 16_2 \oplus 20.$$ 

From this decomposition we see that the summands $16_1$, $16_2$ and $20$ of $K_G^N$ are projective. Moreover $1$, $5_1$ and $5_2$ are weight Green correspondents with vertex $P$ which occur in the following decomposition of the induced weight module:

$$1S_N^G = 1 \oplus 10 \oplus 12_1 \oplus 12_2 \oplus 20,$$

$$1_2S_N^G = 5_1 \oplus 10 \oplus 12_1 \oplus 12_2 \oplus 16_1,$$

$$1_3S_N^G = 5_2 \oplus 10 \oplus 12_1 \oplus 12_2 \oplus 16_2.$$
6.13 \( L_2(13) \mod 2 \) and \( L_2(13) \mod 3 \)

\[ p = 2, \ G = L_2(13), \ P \in \text{Syl}_2(G), \ |G| = 2^3 \cdot 3 \cdot 7 \cdot 13 \]

\[ k_p^G = 1 \oplus 12_1^3 \oplus 12_2^3 \oplus 12_3^3 \oplus 20_1 \oplus 13_1 \oplus 20_2 \oplus 13_2 \oplus 28^2 \oplus 14^3. \]

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**Remark:** The indecomposable direct summands 12_i of \( k_p^G \) for 1 \( \leq \) i \( \leq \) 3 are simple projective. The weight module 14 has vertex \( Q_1 \) of order 2. The decomposition of the associated induced weight module is:

\[ {}^2 S_{N(Q_1)}^G = 14 \oplus 20_1 \oplus 20_2 \oplus 28^2 \oplus 36^2. \]

Note that the summands 20_1, 20_2 and 28 are projective.

The weight Green correspondents 1, 13_1, 13_2 have a Sylow 2-subgroup as vertex. We get the following decomposition of the associated induced weight modules.

\[ 1_1 S_N^G = 1 \oplus 14 \oplus 20_1 \oplus 20_2 \oplus 1 \leq i \leq 3 \ 12_i, \]
\[ 1_2 S_N^G = 13_1 \oplus 14 \oplus 28 \oplus 1 \leq i \leq 3 \ 12_i, \]
\[ 1_3 S_N^G = 13_2 \oplus 14 \oplus 28 \oplus 1 \leq i \leq 3 \ 12_i. \]
We remark that all indecomposable direct summands of \( k^G_P \) are uniserial as direct computation has shown.

\[ p = 3, \ G = L_2(13), \ P \in \text{Syl}_2(G), \ |G| = 2^2 \cdot 3 \cdot 7 \cdot 13 \]

The permutation module \( k^G_P \) decomposes as follows:

\[ k^G_P = 1 \oplus 12_1^4 \oplus 12_2^4 \oplus 12_3^4 \oplus 7_1 \oplus 7_2 \oplus 13 \oplus 21_1^2 \oplus 21_2^2 \oplus 27^4. \]

\[
\begin{array}{c|cccccc}
 & 1 & 13 & 27^4 & 7_1 & 21_1^2 & 7_2 & 21_2^2 \\
\hline
1 & 1 & 1 & & & & \\
13 & & & 1 & 2 & & \\
7_1 & & & 1 & 2 & 1 & \\
7_2 & & & & 1 & 1 & 2 \\
\text{soc} & 1 & 13 & 13 & 7_1 & \bar{7}_1 & \bar{7}_2 & \bar{7}_2 \\
\text{hd} & 1 & 13 & 13 & 7_1 & 7_1 & \bar{7}_2 & \bar{7}_2 \\
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1_3 \\
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1_4 \\
2_2 \\
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\text{soc} \\
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\]

Remark: The indecomposable direct summands \( 12_1, 12_2 \) and \( 12_3 \) of \( k^G_P \) are simple projective. The remaining weight Green correspondents have the Sylow-3-subgroup \( P \) of order 3 as vertex. For the decomposition of the induced weight modules, we have:

\[
\begin{align*}
1_1 S_N^G &= 1 \oplus 27^2 \oplus 1 \leq i \leq 3 \ 12_i, \\
1_2 S_N^G &= 13 \oplus 21_1 \oplus 21_2 \oplus 1 \leq i \leq 3 \ 12_i, \\
1_3 S_N^G &= 7_1 \oplus 21_1 \oplus 27 \oplus 1 \leq i \leq 3 \ 12_i, \\
1_4 S_N^G &= 7_2 \oplus 21_2 \oplus 27 \oplus 1 \leq i \leq 3 \ 12_i.
\end{align*}
\]
The last decomposition could not be computed directly. An application of Remark 2.6 (a) shows that this decomposition must be as above when considering the first three decompositions and that of $k^G_P$.

6.14 $G = L_2(17) \mod 2$ and $L_2(17) \mod 3$

$p = 2, G = L_2(17), P \in \text{Syl}_2(G), |G| = 2^4 \cdot 3^2 \cdot 17$

The permutation module $k^G_P$ decomposes into indecomposable direct summands as follows:

\[ k^G_P = \oplus_{1 \leq i \leq 4} 16_i \oplus 1 \oplus 44_1 \oplus 44_2. \]

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**Remark:** The indecomposable direct summands $16_i$ of $k^G_P$ for $1 \leq i \leq 4$ are simple projective. The weight Green correspondents $44_1$ and $44_2$ have non conjugate vertices $Q_1$ and $Q_2$, respectively, of order 4. The associated induced weight modules decompose as follows:

\[ ^1S^G_{N(Q_1)} = 44_1 \oplus 16_1 \oplus 16_2 \oplus 16_3 \oplus 16_4^2 \oplus 80, \]
\[ ^1S^G_{N(Q_2)} = 44_2 \oplus 16_1 \oplus 16_2 \oplus 16_3 \oplus 16_4^2 \oplus 80. \]

As $N_G(P) = P$ we have $k^G_P = ^1S^G_N$.

$p = 3, G = L_2(17), P \in \text{Syl}_3(G), |G| = 2^4 \cdot 3^2 \cdot 17$

The permutation module $k^G_P$ decomposes into indecomposable direct summands as follows:

\[ k^G_P = 1 \oplus 9_1 \oplus 9_2 \oplus 1 \oplus_{1 \leq i \leq 3} 18_i^2 \oplus 81 \oplus 64. \]
Note: The indecomposable direct summands $9_1$, $9_2$, $18_1$, $18_2$, and $18_3$ of $k_P^G$ are simple projective while the weight Green correspondent $64$ has vertex $P$. For the associated induced weight module and the induced trivial weight module we get the following decompositions:

$$^{11}S^G_N = 1 \oplus 9_1 \oplus 9_2 \oplus 18_1^2 \oplus 81, \quad ^{12}S^G_N = 64 \oplus 18_2^2 \oplus 18_3^2.$$ 

6.15 $G = L_2(19) \pmod 2$ and $G = L_2(19) \pmod 3$

$p = 2, G = L_2(19), \ P \in \text{Syl}_2(G), |G| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$

$$k_P^G = 1 \oplus 1 \oplus 20_1^5 \oplus_1 2 \left(28_2^2 \oplus 18_1^3 \oplus 36_1^3\right) \oplus 9_1 \oplus 9_2.$$ 

$$k_P^G \begin{array}{c|cccccc}
1 & 1 & 81 & 64 & 11 & 9 & 8 \\
16 & 1 & 1 &  & & & \\
soc & 1 & 16 & 16 & 1 & 5 & 4 \\
hd & 1 & 16 & 16 & 1 & 4 & 4 \\
\end{array}$$ 

$$C_P^G \begin{array}{c|cccccc}
1 & 1 &  & & & & \\
12 & 1 & 5 & 4 & & & \\
13 & 4 & 4 & & & & \\
\end{array}$$

Note: The indecomposable direct summands $9_1$, $9_2$, $18_1$, $18_2$, and $18_3$ of $k_P^G$ are simple projective while the weight Green correspondent $64$ has vertex $P$. For the associated induced weight module and the induced trivial weight module we get the following decompositions:

$$^{11}S^G_N = 1 \oplus 9_1 \oplus 9_2 \oplus 18_1^2 \oplus 81, \quad ^{12}S^G_N = 64 \oplus 18_2^2 \oplus 18_3^2.$$ 

6.15 $G = L_2(19) \pmod 2$ and $G = L_2(19) \pmod 3$

$p = 2, G = L_2(19), \ P \in \text{Syl}_2(G), |G| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$

$$k_P^G = 1 \oplus 1 \oplus 20_1^5 \oplus_1 2 \left(28_2^2 \oplus 18_1^3 \oplus 36_1^3\right) \oplus 9_1 \oplus 9_2.$$ 

$$k_P^G \begin{array}{c|cccccccc}
1 & 1 & 9_1 & 28_1^2 & 9_2 & 28_2^2 & 36_1^3 & 18_1^3 & 36_2^3 & 18_2^3 \\
9_1 & 1 & 1 & 1 & & & & & & \\
9_2 & & 2 & 1 & & & & & & \\
18_1 & 1 & 1 & 2 & & & & & & \\
18_2 & & & & & & & & & & \\
\end{array}$$ 

Note: The indecomposable direct summands $9_1$, $9_2$, $18_1$, $18_2$, and $18_3$ of $k_P^G$ are simple projective while the weight Green correspondent $64$ has vertex $P$. For the associated induced weight module and the induced trivial weight module we get the following decompositions:

$$^{11}S^G_N = 1 \oplus 9_1 \oplus 9_2 \oplus 18_1^2 \oplus 81, \quad ^{12}S^G_N = 64 \oplus 18_2^2 \oplus 18_3^2.$$ 

6.15 $G = L_2(19) \pmod 2$ and $G = L_2(19) \pmod 3$

$p = 2, G = L_2(19), \ P \in \text{Syl}_2(G), |G| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$

$$k_P^G = 1 \oplus 1 \oplus 20_1^5 \oplus_1 2 \left(28_2^2 \oplus 18_1^3 \oplus 36_1^3\right) \oplus 9_1 \oplus 9_2.$$ 

$$k_P^G \begin{array}{c|cccccccc}
1 & 1 & 9_1 & 28_1^2 & 9_2 & 28_2^2 & 36_1^3 & 18_1^3 & 36_2^3 & 18_2^3 \\
9_1 & 1 & 1 & 1 & & & & & & \\
9_2 & & 2 & 1 & & & & & & \\
18_1 & 1 & 1 & 2 & & & & & & \\
18_2 & & & & & & & & & & \\
\end{array}$$ 

Note: The indecomposable direct summands $9_1$, $9_2$, $18_1$, $18_2$, and $18_3$ of $k_P^G$ are simple projective while the weight Green correspondent $64$ has vertex $P$. For the associated induced weight module and the induced trivial weight module we get the following decompositions:

$$^{11}S^G_N = 1 \oplus 9_1 \oplus 9_2 \oplus 18_1^2 \oplus 81, \quad ^{12}S^G_N = 64 \oplus 18_2^2 \oplus 18_3^2.$$ 

6.15 $G = L_2(19) \pmod 2$ and $G = L_2(19) \pmod 3$

$p = 2, G = L_2(19), \ P \in \text{Syl}_2(G), |G| = 2^2 \cdot 3^2 \cdot 5 \cdot 19$

$$k_P^G = 1 \oplus 1 \oplus 20_1^5 \oplus_1 2 \left(28_2^2 \oplus 18_1^3 \oplus 36_1^3\right) \oplus 9_1 \oplus 9_2.$$ 

$$k_P^G \begin{array}{c|cccccccc}
1 & 1 & 9_1 & 28_1^2 & 9_2 & 28_2^2 & 36_1^3 & 18_1^3 & 36_2^3 & 18_2^3 \\
9_1 & 1 & 1 & 1 & & & & & & \\
9_2 & & 2 & 1 & & & & & & \\
18_1 & 1 & 1 & 2 & & & & & & \\
18_2 & & & & & & & & & & \\
\end{array}$$
6 Examples

\[
\begin{array}{|c|cccccccc|}
\hline
C^G_P & 1 & 3 & 7^2 & 3^2 & 7_1 & 9^2 & 6_1 & 9_2 & 6_2 \\
\hline
1_1 & 1 & & & & & & & & \\
1_2 & 1 & 1 & & & & & & & \\
2_1 & 1 & 2 & 1 & & & & & & \\
1_3 & & 1 & 1 & & & & & & \\
2_2 & & 1 & 1 & 2 & & & & & \\
3_1 & & & 2 & 1 & & & & & \\
3_2 & & & 1 & 1 & & & & & \\
3_3 & & & & 2 & 1 & & & & \\
3_4 & & & & 1 & 1 & & & & \\
\hline
k^G_P & 1 & 9_1 & 28_1 & 9_2 & 28_2 & 36_1 & 18_1 & 36_2 & 18_2 \\
\hline
& 2^1 & 2^0 & 2^2 & 2^0 & 2^2 & 2 & 2^2 & 2 & \\
soc & 1_1 & 2_1 & 2_1 & 2_2 & 2_2 & 3_1 & 3_1 & 3_3 & 3_3 \\
hd & 1_1 & 1_2 & 2_1 & 1_3 & 2_2 & 3_1 & 3_2 & 3_3 & 3_4 \\
EC & P_1 & P_2 & P_3 & P_4 & P_5 \\
s & 1 & 1 & 1 & 1 & 1 & \\
\hline
\end{array}
\]

Note: The weight Green correspondents 18_1, 18_2 have vertex Q_1 of order 2. We get the following decompositions for the corresponding induced weight modules:

\[
\begin{align*}
2_1 S^G_N(Q_1) &= 18_1 \oplus_{i=1,2} 36_1 \oplus 36_2 \oplus_{1 \leq i \leq 4} 20_i^2, \\
2_2 S^G_N(Q_1) &= 18_2 \oplus_{i=1,2} 36_2 \oplus 36_1 \oplus_{1 \leq i \leq 4} 20_i^2.
\end{align*}
\]

The induced weight modules with a Sylow 2-subgroup as vertex decompose as follows:

\[
\begin{align*}
1_1 S^G_N &= 1 \oplus_{i=1,2} (18_i \oplus 36_i \oplus 28_i) \oplus 20_1^3 \oplus_{2 \leq i \leq 4} 20_i, \\
1_2 S^G_N &= 9_1 \oplus_{i=1,2} (18_i \oplus 36_i) \oplus 28_i \oplus 20_i \oplus_{i=2,3,4} 20_i^2, \\
1_3 S^G_N &= 9_2 \oplus_{i=1,2} (18_i \oplus 36_i) \oplus 28_2 \oplus 20_1 \oplus_{i=2,3,4} 20_i^2.
\end{align*}
\]

\(p = 3, G = L_2(19), P \in \text{Syl}_3(G), |G| = 2^2 \cdot 3^2 \cdot 5 \cdot 19\)

The permutation module \(k^G_P\) decomposes into indecomposable direct summands as follows:

\[
k^G_P = 1 \oplus_{i=1,2} 9_i \oplus_{1 \leq i \leq 4} 18_i^2 \oplus 19 \oplus 99^2.
\]
Remark: The Sylow 3-subgroup $P$ is a cyclic group of order $3^2$. The indecomposable direct summand $99$ of $k^G_P$ is not a weight Green correspondent while the remaining are. Note that $9_i$ and $18_i$ for $i = 1, 2$ are simple projective. We get the following decompositions for the induced weight modules with $P$ as a defect group:

\[
\begin{align*}
1^1 S_N^G &= 1 \oplus 99 \oplus 9_1 \oplus 9_2 \oplus 18_2^2 \oplus 18_3^2, \\
1^2 S_N^G &= 19 \oplus 99 \oplus 18_3^2 \oplus 18_4^2.
\end{align*}
\]

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References


