Solution to Exercise 1.2.4

We put $\Omega := \{1, \ldots, n\}$ and assume $G \leq \mathrm{S}_n$ acts transitively on Ω.

(1) Let $U \leq_{KG} V$ and $U \not\leq \mathrm{Inv}_G(V)$. Choose $0 \neq v \in U$ with $l(v)$ minimal. We may assume that $v \notin \mathrm{Inv}_G(V)$. We choose the notation so that $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $v = \sum_{i=1}^r \alpha_i \omega_i$, hence $\alpha_i \neq 0$ for $i = 1, \ldots, r$. If $r = 1$ then $\omega_1 \in U$ and $U = V$, because G acts transitively on Ω. So let $r > 1$. Replacing v by $\alpha_1^{-1} v$ if need be, we may also assume that $\alpha_1 = 1$.

(a) If $G = \mathrm{S}_n$ we put $g_j := (\omega_1\ \omega_j) \in G$ and get for $j = 2, \ldots, r$

$$v_j := v - \alpha_j^{-1} g_j\, v = (\alpha_j - \alpha_j^{-1})\, \omega_j + \sum_{\substack{i=2 \\ i \neq j}}^{r} \alpha_i (1 - \alpha_j^{-1})\, \omega_i \in U.$$

Then $l(v_j) < r$ and from the minimality of $r = l(v)$ we conclude that $v_j = 0$ for $j = 2, \ldots, r$. It follows that $\alpha_j = 1$ for $j = 1, \ldots, r$ or $r = 2$ and $v = \omega_1 - \omega_2$.

Assume that $\alpha_j = 1$ for $j = 1, \ldots, r$. Since $v \notin \mathrm{Inv}_G(V)$ we have $r < n$. Then $v'' := v - (\omega_1,\ \omega_n)\, v = \omega_1 - \omega_n \in U$ and $l(v'') = 2$.

Thus we have $r = 2$ and $\omega_1 - \omega \in U$ for all $\omega \in \Omega$, since G acts doubly transitively on Ω and so $U \geq \mathrm{Inv}^G(V)$.

(b) Now let $G := \mathrm{A}_n$. If $r \leq n - 2$ we may argue exactly as in (a) replacing the transposition $(\omega_1\ \omega_j)$ by $(\omega_1\ \omega_j)(\omega_{n-1}\ \omega_n)$ and $(\omega_1\ \omega_n)$ by $(\omega_1\ \omega_n\ \omega_{n-1})$ to show that $U \geq \mathrm{Inv}^G(V)$.

So we may assume that $r \geq n - 1$, hence $r \geq 3$ (and $r \geq 4$ if $n \geq 5$). For $3 \leq j \leq r$ we put $g'_j := (\omega_1\ \omega_2\ \omega_j) \in \mathrm{A}_n$ and

$$v'_j := v - \alpha_j^{-1} g'_j\, v = (\alpha_2 - \alpha_j^{-1})\, \omega_2 + (\alpha_j - \alpha_2 \alpha_j^{-1})\, \omega_j + \sum_{\substack{i=3 \\ i \neq j}}^{r} \alpha_i (1 - \alpha_j^{-1})\, \omega_i \in U.$$

Then $v'_j = 0$ because $l(v'_j) < r$. If $r \geq 4$ the last sum is not empty and we conclude that $\alpha_j = 1$ for $j = 2, \ldots, r$. Replacing $(\omega\ \omega_1)$ by $(\omega\ \omega_1)(\omega_3\ \omega_4)$ in the argument of (a) we obtain $U \geq \mathrm{Inv}^G(V)$. In particular (i) is proved.

So it remains to consider the case $r = 3$ and, consequently $n = 4$. Here $v'_3 = 0$ means

$$\alpha_2 = \alpha_3^{-1}, \qquad \alpha_3 = \alpha_2 \alpha_3^{-1} \qquad \text{hence} \qquad \alpha_3^3 = 1, \qquad \alpha_2 = \alpha_3^2.$$

If $X^2 + X + 1 \in K[X]$ is irreducible then $\alpha_2 = \alpha_3 = 1$ and $v' - (\omega_1\ \omega_4)(\omega_2\ \omega_3)\, v' = \omega_1 - \omega_4 \in U$, a contradiction. In particular (iii) is proved.

Now let $\alpha := \alpha_2$ be a primitve third root of unity in K and $\alpha_3 = \alpha^2$. Put

$$\begin{aligned} v' &:= (\omega_1\ \omega_4)(\omega_2\ \omega_3)\, v = \alpha^2\, \omega_2 + \alpha\, \omega_3 + \omega_4 \\ v'' &:= (\omega_1\ \omega_4\ \omega_2)\, v = \alpha\, \omega_1 + \alpha^2\, \omega_3 + \omega_4 \end{aligned}$$

Then

$$w := v'' - \alpha\, v + v' = -2\,\omega_3 + 2\,\omega_4 \in U$$

and $l(w) = 2$ if $\operatorname{char} K \neq 2$, a contradiction, which proves (ii).

Note: If $\operatorname{char} K = 2$ then $W := \langle v,\ v' \rangle_K \leq_{KG} V$ and affords (with respect to the basis $(v,\ v')$) the matrix representation

$$(\omega_1\ \omega_2\ \omega_3) \mapsto \begin{bmatrix} \alpha & \alpha \\ 0 & 1 \end{bmatrix}, \quad (\omega_1\ \omega_4)(\omega_2\ \omega_3) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

With respect to the basis $(\sum_{i=1}^{4} \omega_i, v)$ the module W affords the matrix representation

$$(\omega_1\ \omega_2\ \omega_3) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \quad (\omega_1\ \omega_4)(\omega_2\ \omega_3) \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$