Solution to Exercise 1.6.3

By definition

$$v = \sum_{\omega \in \Omega} \alpha_\omega \, \omega \in \mathrm{Inv}_H V \qquad \text{with} \qquad \alpha_\omega \in K$$

if and only if $\alpha_\omega = \alpha_{h\,\omega}$ for all $\omega \in \Omega$ and $h \in H$, that is, if and only if α_ω is constant on the orbits of H on Ω. Thus $v \in \mathrm{Inv}_H V$ if and only if v can be written as $v = \sum_{1=1}^r \alpha_i \, \mathcal{O}_i^+$ with $\alpha_i := \alpha_{\omega_i}$ if $\omega_i \in \mathcal{O}_i$. Since $\Omega = \mathcal{O}_1 \dot\cup \ldots \dot\cup \, \mathcal{O}_r$ it follows that $(\mathcal{O}_1^+, \ldots, \mathcal{O}_r^+)$ is a K-basis of $\mathrm{Inv}_H V$.

Since $h\,H^+ = H^+$ for all $h \in H$ it is clear that $H^+ K\Omega \subseteq \mathrm{Inv}_H(K\Omega)$. From $h\,\mathcal{O}_i^+ = \mathcal{O}_i^+$ for all $h \in H$ and $1 \le i \le r$ we conclude that $H^+\,\mathcal{O}_i^+ = |H|\,\mathcal{O}_i^+$. If $|H|$ is invertible in K we see that $\mathcal{O}_i^+ \subseteq H^+\,K\Omega$ for all i and thus

$$H^+ K\Omega = \mathrm{Inv}_H(K\Omega).$$

We fix $g \in G$ and put

$$\mathcal{O}_{j,i} := \{\, \omega \in \mathcal{O}_j \mid g\,\omega \in \mathcal{O}_i \,\} \qquad (1 \le i, j \le r).$$

Then we get for any $\omega \in \mathcal{O}_{j,i}$

$$e_H\,g\,\omega = \frac{1}{|H|} \sum_{h \in H} h\,g\,\omega = \frac{1}{|H|} |\,\mathrm{Stab}_H(g\,\omega)|\,\mathcal{O}_i^+ = \frac{1}{|\mathcal{O}_i|}\mathcal{O}_i^+.$$

Hence

$$\begin{aligned} e_H g e_H\,\mathcal{O}_j^+ = e_H g\,\mathcal{O}_j^+ \quad &= \quad \sum_{\omega \in \mathcal{O}_j} e_H\,g\,\omega = \sum_{i=1}^r \sum_{\omega \in \mathcal{O}_{j,i}} \frac{1}{|\mathcal{O}_i|}\mathcal{O}_i^+ \\ &= \quad \sum_{i=1}^r |\mathcal{O}_{j,i}|\,\frac{1}{|\mathcal{O}_i|}\mathcal{O}_i^+. \end{aligned}$$

Recall that empty sums are zero, thus $\sum_{\omega \in \mathcal{O}_{j,i}} \frac{1}{|\mathcal{O}_i|}\mathcal{O}_i^+ = 0$ if $\mathcal{O}_{j,i} = \emptyset$.