Solution to Exercise 2.2.6

Let $\mathrm{Irr}(G) = \{\mathbf{1}_G = \chi_1, \chi_2, \ldots, \chi_6\}$ with $\chi_i(1) \leq \chi_j(1)$ for $i \leq j$. From Remark 2.2.9 we conclude that $\chi_i(g) \in \mathbb{Z}$ for $i \in \{5, 6\}$ and all $g \in G$. Also, by Lemma 2.2.6 $|\chi_i(g)| = 1$ for $1 \leq i \leq 4$ and $g \in G$. The Orthogonality Relations (Theorem 2.1.15) imply that

$$(\chi_5(g), \chi_6(g)) \in \{(0,0),\ (2,-1),\ (-2,0)\} \qquad \text{for all} \qquad g \in G \setminus \{1\}$$

and, consequently $|\,\mathbf{C}_G(g)| \in \{4,\ 9,\ 8\}$.

By Corollary 1.5.12 we know that $|G| = 72$ and $|G'| = 18$. Then G' contains an element g_2 of order three and also an involution, say g_3. Since $|\,\mathbf{C}_G(g_2)| \notin \{4,\ 8\}$ we know $(\chi_5(g_2), \chi_6(g_2)) = (2, -1)$. Similarly we get $(\chi_5(g_3), \chi_6(g_3)) = (-2, 0)$. Since $(\chi_i, \chi_i)_G = 1$ for all $1 \leq i \leq 6$, we see that $\chi_i(g) = 0$ for $i = 5, 6$ and $1 \neq g \notin g_2^G \cup g_3^G$. This means that we have found the first four columns and the last two rows of the character table of G. In particular we see that $\ker \chi_5$ is an elementary abelian group P_3 of order 9, so that G/P_3 is a non-abelian group of order 8 and hence isomorphic to Q_8 or D_8. Using Example 2.1.19 we complete the character table of G:

$\lvert\,\mathbf{C}_G(g_i)\rvert$	72	9	8	4	4	4
g_i	g_1	g_2	g_3	g_4	g_5	g_6
χ_1	1	1	1	1	1	1
χ_2	1	1	1	-1	1	1
χ_3	1	1	1	1	-1	1
χ_4	1	1	1	1	1	-1
χ_5	2	2	-2	0	0	0
χ_6	8	-1	0	0	0	0

where $g_4,\ g_5,\ g_6$ are representatives of the conjugacy classes of G, which are not contained in G'. It is clear that $G \cong P_3 \rtimes P_2$ with $P_2 \in \mathrm{Syl}_2(G)$, which is isomorphic to Q_8 or D_8.