Solution to Exercise 2.3.3

By assumption and Theorem 2.3.4 $P \neq \{1\}$ and consequently $\mathbf{Z}(P) \neq \{1\}$. Let $g \in \mathbf{Z}(P)$ be of order p and let δ be a representation affording χ. Then $p \nmid |g^G|$ because $P \subseteq \mathbf{C}_G(g)$. Since $\mathbf{Z}(G) = \{1\}$ and χ is faithful, Theorem 2.3.7 implies that $\chi(g) = 0$.

Assume that the eigenvalues of $\delta(g)$ are $\boldsymbol{\zeta}_p^{i_1}, \ldots, \boldsymbol{\zeta}_p^{i_p}$ with $0 \leq i_1, \ldots, i_p < p$ and let $f := \sum_{j=1}^{p} X^{i_j} \in \mathbb{Q}[X]$. Then $f(\boldsymbol{\zeta}_p) = \chi(g) = 0$. Hence f is a multiple of the minimal polynomial $\sum_{j=0}^{p-1} X^j$ of $\boldsymbol{\zeta}_p$ over $\mathbb{Q}$. Since $\deg f < p$ we conclude $f = \sum_{j=0}^{p-1} X^j$ and so the eigenvalues of $\delta(g)$ are $\boldsymbol{\zeta}_p^0, \boldsymbol{\zeta}_p, \ldots, \boldsymbol{\zeta}_p^{p-1}$. If $A \in \mathbb{C}^{p \times p}$ satisfies

$$A \cdot \operatorname{diag}(\boldsymbol{\zeta}_p^0, \ldots, \boldsymbol{\zeta}_p^{p-1}) = \operatorname{diag}(\boldsymbol{\zeta}_p^0, \ldots, \boldsymbol{\zeta}_p^{p-1}) \cdot A$$

then A must be a diagonal matrix. Since δ is faithful we see that $\mathbf{C}_G(g)$ is abelian and hence so also its subgroup P.

Now let $x \in \mathbf{C}_G(P) \setminus \{1\}$. Then $P \subseteq \mathbf{C}_G(x)$ and hence $p \nmid |x^G|$. Using Theorem 2.3.7 again we see that $\chi(x) = 0$. Thus

$$(\, \chi|_{\mathbf{C}_G(P)} \,, \mathbf{1}_{\mathbf{C}_G(P)} \,)_{\mathbf{C}_G(P)} = \frac{1}{|\,\mathbf{C}_G(P)|} \cdot p.$$

We conclude that $|\,\mathbf{C}_G(P)| \mid p$ and hence $P = \mathbf{C}_G(P) \cong \mathrm{C}_p$.