Solution to Exercise 2.7.6

In the exercise we add the assumption $G = G'$. Otherwise $G = \mathrm{A}_4$ would be the smallest counterexample.

(a) A direct application of Exercise 2.3.3 (with $p = 3$) gives $|\mathbf{C}_G(P_3)| = 3$ for $P_3 \in \mathrm{Syl}_3(G)$. Hence $P_3 = \langle g_3 \rangle$ has order three and $|\mathbf{C}_G(g_3)| = 3$.

If $\eta \in \mathrm{Irr}(G)$ with $\eta(1) = 2$ the congruence relations of Lemma 2.2.2 imply $\eta(g_3) \in \{2, -1\}$ and the orthogonality relations (Theorem 2.1.15, (2.5)) applied to (g_3, g_3) and $(1, g_3)$ show that $\eta(g_3) = -1$ and that there must be a linear character $\mathbf{1}_G \neq \lambda$ with $\lambda(g_3) = 1$, which contradicts $G = G'$. Hence G has no irreducible character η of degree two.

If we were assuming that G is simple, the last paragraph could be replaced by a reference to Exercise 2.2.4 or to Exercise 2.3.3.

(b) Note that $\chi^{[1^2]}(1) = 3$ and $\chi^{[2]}(1) = 6$. From $\chi = \overline{\chi}$ it follows that

$$(\chi^2, \mathbf{1}_G)_G = (\chi, \overline{\chi})_G = 1.$$

Since G has no non-trivial irreducible character of degree ≤ 2 we conclude that

$$\chi^{[1^2]} \in \mathrm{Irr}(G) \qquad \text{and} \qquad \chi^{[2]} = \mathbf{1}_G + \psi$$

with $\psi(1) = 5$ and $\psi \in \mathrm{Irr}(G)$ because ψ cannot have a constituent of degree $\leq 5/2$. If $x \in G \backslash \{1\}$ and $\psi(x) = \psi(1) = 5$ then $6 = \chi^{[2]}(x) = \frac{1}{2}(\chi(x)^2 + \chi(x^2)) < \frac{1}{2}(9+3) = 6$, a contradiction. Hence ψ is faithful. We will prove $\chi^{[1^2]} = \chi$ in (e).

(c) Since $\chi(g_3) = \chi(g_3^2) = 0$ by Theorem 2.3.7, we have $\chi^{[2]}(g_3) = 0$, hence $\psi(g_3) = -1$. (This also follows from the congruence relations.) Since $|\mathbf{C}_G(g_3)| = 3$ there is a unique $\varphi \in \mathrm{Irr}(G)$ with $\varphi(1) = 4$ (and $\varphi(g_3) = 1$), because of the orthogonality relations. These also show that for any $g \in G \setminus g_3^G$ we have $1 + \varphi(g) - \psi(g) = 0$. Furthermore it follows that φ and ψ are the only irreducible characters of degree four and five, respectively

(d), (e) Let $g_5 \in G$ be an element of order five. Since χ is real and faithful $\chi(g_5) \in \{1 + \boldsymbol{\zeta}_5^2 + \boldsymbol{\zeta}_5^3,\ 1 + \boldsymbol{\zeta}_5 + \boldsymbol{\zeta}_5^4\} = \{\frac{1-\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2}\}$. Hence g_5^G is a real (but not rational) conjugacy class and there is $\chi' \in \mathrm{Irr}(G)$, algebraically conjugate to χ. Exercise 2.3.3 (with $p = 5$) shows that $\langle g_5 \rangle = \mathbf{C}_G(g_5) \in \mathrm{Syl}_5(G)$. The congruence relations give $\varphi(g_5) = -\varphi(g_5^2) = -1$. Since

$$5 = |\mathbf{C}_G(g_5)| = 1 + \chi(g_5)^2 + \chi'(g_5)^2 + \varphi(g_5)^2$$

any $\eta \in \mathrm{Irr}(G) \setminus \{\mathbf{1}_G, \chi, \chi', \varphi, \psi\}$ must vanish on all elements of order five; in particular χ, χ' are the only irreducible characters of degree three. Replacing g_5 by g_5^2 if need be we may assume that $\chi(g_5) = \frac{1-\sqrt{5}}{2}$. Then

$$\chi^{[1^2]}(g_5) = \frac{1}{2}(\frac{3-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2}) = \frac{1-\sqrt{5}}{2} = \chi(g_5)$$

and hence $\chi^{[1^2]} = \chi$. If $g \in G$ has order two then $\chi(g) = \pm 1$ and thus $\chi(g) = \chi^{[1^2]}(g) = \frac{\chi(g)^2 - \chi(1)}{2} = -1$. If $p > 5$ is a prime and $g \in G$ has order p then $\chi(g) = 1 + \zeta + \zeta^{-1}$ with a primitive p-th root of unity ζ. Hence there must be $\frac{p-1}{2} > 2$ characters of G algebraically conjugate to χ, which is a contradiction. As a consequence we see that $|G| = 2^k \cdot 3 \cdot 5$ for some $k \geq 2$.

(f) Applying a Galois automorphism to $(\chi\,\chi', \chi)_G = (\chi^2, \chi')_G = 0$ we get $(\chi\,\chi', \chi')_G = 0$. Since also $(\chi\,\chi', \mathbf{1}_G)_G = (\chi\,\chi')_G = 0$, the only possible irreducible constituents of $\chi\,\chi'$ are φ and ψ. Hence

$$\chi\,\chi' = \varphi + \psi.$$

If $g \in G$ has order four then $\chi(g) = \chi'(g) = \pm 1$. Hence $\chi^{[2]}(g) = \frac{1-1}{2} = 1 + \psi(g)$ so $\psi(g) = -1$ and $\varphi(g) = -2$ (by (c)). We get $1 = (\chi\,\chi')(g) \neq \varphi(g) + \psi(g) = -3$, a contradiction.

(g) A simple computation shows:

$$\begin{array}{llll} \chi^2 = \mathbf{1}_G + \chi + \psi & \chi\,\chi' = \varphi + \psi & \chi\,\varphi = \chi' + \varphi + \psi & \chi\,\psi = \chi + \chi' + \varphi + \psi \\ {\chi'}^2 = \mathbf{1}_G + \chi' + \psi & & \chi'\,\varphi = \chi + \varphi + \psi & \chi'\,\psi = \chi + \chi' + \varphi + \psi \end{array}$$

$$\begin{array}{ll} \varphi^2 = 1 + \chi + \chi' + \varphi + \psi & \varphi\,\psi = \chi + \chi' + \psi + 2 \cdot \psi \\ \psi^2 = 1 + \chi + \chi' + 2 \cdot \varphi + 2 \cdot \psi & \end{array}$$

(h) From (g) and Theorem 2.7.3 we conclude that G has exactly five conjugacy classes: $\{1\}$, g_2^G, g_3^G, g_5^G, ${g_5^2}^G$ where g_2 is an involution. Thus the character table of G is

$\lvert \mathbf{C}_G(g) \rvert$	60	4	3	5	5
g	1	g_2	g_3	g_5	g_5^2
$\mathbf{1}_G$	1	1	1	1	1
χ	3	-1	0	ϵ	ϵ'
χ'	3	-1	0	ϵ'	ϵ
φ	4	0	1	-1	-1
ψ	5	1	-1	0	0

with $(\epsilon, \epsilon') := (\frac{1-\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2})$. From this it is apparent that $G \cong \mathrm{A}_5$.