Solution to Exercise 3.3.3

By Theorem 3.3.8 and (3.11) in Lemma 3.3.9

$$\lambda = \lambda' \quad \text{if and only if} \quad \chi_\lambda(\sigma) = 0 \quad \text{for all} \quad \sigma \in \mathrm{S}_n \setminus \mathrm{A}_n\,.$$

Observe that (abbreviating $\chi := \chi_\lambda,\ \ \chi' := \chi_{\lambda'} = \epsilon_{\mathrm{S}_n} \cdot \chi$)

$$\begin{aligned} \delta_{\chi,\chi'} = (\chi, \chi')_{\mathrm{S}_n} &= \frac{1}{|\,\mathrm{S}_n\,|}\Big(\sum_{\sigma\in\mathrm{A}_n} \chi(\sigma)\chi(\sigma^{-1}) - \sum_{\sigma\in\mathrm{S}_n\setminus\mathrm{A}_n} \chi(\sigma)\chi(\sigma^{-1})\Big) \\ &= \frac{1}{2}(\chi_{\mathrm{A}_n}, \chi_{\mathrm{A}_n})_{\mathrm{A}_n} - \frac{1}{|\,\mathrm{S}_n\,|} \sum_{\sigma\in\mathrm{S}_n\setminus\mathrm{A}_n} \chi(\sigma)\chi(\sigma^{-1}). \end{aligned}$$

Thus $(\chi_{\mathrm{A}_n}, \chi_{\mathrm{A}_n})_{\mathrm{A}_n} = 2$ if $\chi = \chi'$. On the other hand, if $\chi \neq \chi'$ we conclude

$$\sum_{\sigma\in\mathrm{A}_n} \chi(\sigma)\chi(\sigma^{-1}) = \sum_{\sigma\in\mathrm{S}_n\setminus\mathrm{A}_n} \chi(\sigma)\chi(\sigma^{-1})$$

and hence $1 = (\chi,\chi)_{\mathrm{S}_n} = \frac{1}{2}(\chi_{\mathrm{A}_n}, \chi_{\mathrm{A}_n})_{\mathrm{A}_n} + \frac{1}{2}(\chi_{\mathrm{A}_n}, \chi_{\mathrm{A}_n})_{\mathrm{A}_n}$.

Note that the argument can be generalized and strenghtened to give:

Lemma: *Let λ be a linear character of a finite group G with* $\ker\lambda = N$ *and* $\chi \in \mathrm{Irr}(G)$. *Then $\chi_N \in \mathrm{Irr}(N)$ if and only if $\lambda^i \cdot \chi \neq \lambda^j \cdot \chi$ for* $0 \leq i \neq j < n :=$ $[G : H]$. *If* $\chi = \lambda \cdot \chi$ then $(\chi_N, \chi_N)_N = n$.

Proof: (a) Since $N \trianglelefteq G$ Definition 3.2.6 shows that

$$(\chi_N)^G(g) = \begin{cases} 0 & \text{for } g \in G \setminus N \\ n \cdot \chi(g) & \text{for } g \in N. \end{cases}$$

Writing ρ for the inflation of the regular character $\rho_{G/N}$ to G we thus get

$$(\chi_N)^G = \rho \cdot \chi = \sum_{i=0}^{n-1} \lambda^i \cdot \chi.$$

By Frobenius Reciprocity we have $((\chi_N)^G, \lambda^i \cdot \chi)_G = (\chi_N, \chi_N)_N$. Thus χ_N is irreducible if and only if $\lambda^i \cdot \chi \neq \lambda^j \cdot \chi$ for $0 \leq i \neq j < n$.

(b) As above we have $\chi = \lambda \cdot \chi$ if and only if $\chi(g) = 0$ for all $g \in G \setminus N$. Hence $(\chi_N, \chi_N) = n$ in that case.