Solution to Exercise 3.4.2

(a) Let $G = \langle x \rangle$. By Example 1.1.24 every irreducible matrix representation of G is equivalent to

$$\boldsymbol{\delta}_f \colon x^j \mapsto (M_f)^j \qquad 1 \leq j \leq n$$

for some monic irreducible polynomial $f \in \mathbb{Q}[X]$ dividing $X^n - 1$, in other words, some cyclotomic polynomial $\Phi_d := \prod_{(d,i)=1}(X - \boldsymbol{\zeta}_d^i) \in \mathbb{Q}[X]$ with $d \mid n$. Here M_f is the companion matrix of f (see Example 1.1.24). Let χ_d be the character of $\boldsymbol{\delta}_{\Phi_d}$. Then

$$\chi_d(x^j) = \sum_{\substack{1 \leq i \leq d \\ (d,i)=1}} \boldsymbol{\zeta}_d^{ij} \quad (1 \leq j \leq n) \qquad \text{and} \qquad \operatorname{Irr}(G) = \{ \chi_d \mid \ d \mid n \}.$$

(b) For every $d \mid n$ there is a unique subgroup $H_d := \langle x^{n/d} \rangle \leq G$ of index d. Then $\theta_d := (\mathbf{1}_{H_d})^G$ is the unique transitive permutation character of degree d.

(c) Let $\delta_d^0 \colon H_d \to \mathbb{Q}^\times$ be the trivial representation. and let $B_d := (1, x, \ldots, x^{d-1})$. Then $[\delta_d^{0\,G}(x)]_{B_d} = \boldsymbol{\delta}_{X^d - 1}(x)$ and, since $X^d - 1 = \prod_{m \mid d} \Phi_d$,

$$\theta_d = \sum_{\substack{1 \leq m \leq d \\ m \mid d}} \chi_m = \sum_{m=1}^{d} \epsilon_{m,d}\, \chi_m \qquad \text{with} \qquad \epsilon_{m,d} := \begin{cases} 0 & \text{if } m \nmid d \\ 1 & \text{if } m \mid d \end{cases}$$

Let $\mu_{\mathbb{N}}$ be the Möbius function of the poset $\mathbb{N}$ ordered by divisibilty (see Definition 2.5.3); thus $[\mu_{\mathbb{N}}(i,j)]_{1 \leq i,j \leq d} = [\epsilon_{i,j}]_{1 \leq i,j \leq d}^{-1}$. Hence

$$\chi_d = \sum_{m=1}^{d} \mu_{\mathbb{N}}(m,d)\, \theta_m = \sum_{\substack{1 \leq m \leq d \\ m \mid d}} \mu(\frac{d}{m})\, \theta_m,$$

see Example 2.5.4.

(d) It follows from Lemma 3.2.7 (b) that

$$\theta_d(g) = \begin{cases} d & \text{if } \ g \in H_d, \\ 0 & \text{if } \ g \notin H_d \end{cases}$$

for any $d \mid n$. From this we get $\theta_d(g) = \frac{d}{n}\theta_n(g^d)$ for $g \in G$ since $g \in H_d$ if and only if $g^d \in H_n = \{1\}$.