Solution to Exercise 3.5.2

By Lemma 3.5.12 and its proof we have

$$G/H_i \times G/H_j \cong \bigoplus_{d\in D_{ij}} G/(H_i^d \cap H_j) \quad \text{as } G\text{-sets, with} \quad G = \dot{\bigcup}_{d\in D_{ij}} H_i d H_j,$$

and the mark of $G/H_i \times G/H_j$ is $m_i \cdot m_j$. So in order to get the orbit structure of G in $G/H_i \times G/H_i$ we simply have to write $m_i \cdot m_i$ as a sum of marks. In GAP this is achieved using the command `IntersectionsTom(t,i,i)`.

```
gap> IntersectionsTom(t,24,24);
[ 4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4 ]
gap> IntersectionsTom(t,25,25);
[ 4, 3, 0, 0, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 2 ]
gap> for j in [24,25] do
>     Print("\n G/H_",j," x G/H_",j," =  ");
>      for i in [1..j] do
>       if IntersectionsTom(t,j,j)[i] <> 0 then
>          Print(IntersectionsTom(t,j,j)[i],".G/H_",i, " + ");
>       fi;
>      od;
>    od;

 G/H_24 x G/H_24 =  4.G/H_1 + 4.G/H_2 + 4.G/H_24
 G/H_25 x G/H_25 =  4.G/H_1 + 3.G/H_2 + 2.G/H_5 + 1.G/H_19 + 2.G/H_25
```

This is almost the TEX-code for the answer:

$$\begin{aligned} G/H_{24} \times G/H_{24} &\cong 4 \cdot G/H_1 \oplus 4 \cdot G/H_2 \oplus 4 \cdot G/H_{24} \\ G/H_{25} \times G/H_{25} &\cong 4 \cdot G/H_1 \oplus 3 \cdot G/H_2 \oplus 2 \cdot G/H_5 \oplus 1 \cdot G/H_{19} \oplus 2 \cdot G/H_{25} \end{aligned}$$

In Table 3.4 on p. 222 one can find the isomorphism types of the H_i. By Lemma 1.2.15 and Lemma 1.2.11 $\mathrm{rk}_K \operatorname{Hom}_{KG}(K(G/H_i), K(G/H_j))$ is equal to the number of double cosets $H_j g H_i$ in G and so can be read of from $m_j \cdot m_i$:

```
gap> IntersectionsTom(t,25,24);
[ 4, 2, 0, 0, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2 ]
gap> Sum(IntersectionsTom(t,25,24));
12
```

Thus $\mathrm{rk}_K \operatorname{Hom}_{KG}(K(G/H_{24}), K(G/H_{25})) = 12$.

From Lemma 3.5.3 (c) and the table of marks of G we see that for $i \in \{24, 25\}$ the group H_i contains exactly one subgroup P_i (with index four) which is conjugate in G to $H_{13} \in \mathrm{Syl}_3(G)$. Likewise we find that H_{24} and H_{25} both have a unique subgroup of index two. Hence $P_i \trianglelefteq H_i$ and $H_i/P_i \cong \mathrm{C}_4$ for $i \in \{24, 25\}$ and thus for $p := 3$

$$H_i \in \mathcal{C}_p(G) := \{H \le G \mid H/\mathbf{O}_p(H) \text{ is a cyclic } p'\text{-group}\}.$$

As mentioned in the exercise it follows from Corollary 4.10.14 that $F(G/H_{24}) \ncong F(G/H_{25})$ for certain fields F of characteristic three. Since $F(G/H_i) \cong F \otimes \mathbb{Z}(G/H_i)$ we conclude that

$$\mathbb{Z}(G/H_{24}) \ncong \mathbb{Z}(G/H_{25}).$$

The other pairs (H_i, H_j) of subgroups with $K(G/H_i)$ having the same permutation character as $K(G/H_j)$ are (H_7, H_8) isomorphic to S_3 and (H_{29}, H_{30}) isomorphic to A_5. Clearly $H_7, H_8 \in \mathcal{C}_3(G)$ and $m_7(H_8) \neq m_8(H_8)$. Also the table of marks shows that $m_{29}(H_7) \neq m_{30}(H_7)$. So the same argument as above proves that

$$\mathbb{Z}(G/H_7) \ncong \mathbb{Z}(G/H_8), \qquad \mathbb{Z}(G/H_{29}) \ncong \mathbb{Z}(G/H_{30}).$$