Solution to Exercise 3.6.11

G acts on $\operatorname{End}_K V$ (see Definition 1.1.37) and we first have to see that E is G-invariant in $\operatorname{End}_K V$. So let $g \in G$, $h \in N$ $v \in V$ and $\alpha \in E$. Then

$$g \cdot \alpha(h\,v) = g\,\alpha(g^{-1}h\,v) = g\,h^g\,\alpha(g^{-1}\,v) = h\,g\,\alpha(g^{-1}\,v) = h\,((g\cdot\alpha)(v)).$$

Thus $g \cdot \alpha \in E$. Now assume also that $\alpha' \in \operatorname{End}_{KN} V$. Then

$$(g \cdot (\alpha \circ \alpha'))(v) = g\,(\alpha \circ \alpha'(g^{-1}v)) = g\,\alpha(g^{-1}\,g\,\alpha'(g^{-1}v)) = (g\cdot\alpha\ \circ g\cdot\alpha')(v).$$

So $\varphi_g \colon E \to E,\ \alpha \mapsto g \cdot \alpha$ is a ring automorphism and

$$\varphi\colon G \to \operatorname{Aut} E,\ g \mapsto \varphi_g \qquad \text{is a group homomorphism.}$$

We now assume that V_N is simple. Then E is a division ring by Schur's Lemma. If K is finite E is also finite, hence a field by Wedderburn's Theorem. Since V is necessarily simple as well, E is a finite field extension of $L := \operatorname{End}_{KG} V$. Clearly $\varphi_g(\alpha) = \alpha$ for $\alpha \in L$, so $\operatorname{im}\varphi \leq \operatorname{Gal}(E/L)$. Furthermore, if $\alpha \in E$ and $g \in N$ then $g \cdot \alpha = \alpha$, hence $N \leq \ker\varphi$. If $g \in \mathbf{C}_G(N)$ observe that $g^\bullet \colon V \to V,\ v \mapsto g\,v$ is in E and $g\cdot\alpha = g^\bullet \circ \alpha \circ (g^{-1})^\bullet = \alpha$, because E is commutative. Hence $N\,\mathbf{C}_G(N) \leq \ker\varphi$.

Embedding K into L and identifying V with $LV = L \otimes_K V$, we van view V as an absolutely simple LG-module. Let $E = \psi_1 \otimes L \oplus \cdots \oplus \psi_t \otimes L$ with $\mathrm{id}_V = \psi_1, \ldots, \psi_t \in E$, so $t = [E : L]$. Then

$$EV = \psi_1 \otimes V \oplus\ \cdots\ \oplus \psi_t \otimes V$$

as L-vectorspaces, but the summands on the right hand side are in fact EN-modules and $\psi_1 \otimes V$ may be identified with the absolutely simple EN-module EV_N. Comparing with Theorem 1.8.4 (with $A := LN$ and $V = LV$) we see that

$$EV \cong_{EN} {}^{\gamma_1}V_N \oplus\ \cdots\ \oplus {}^{\gamma_t}V_N$$

where V_N has to be considered as an EN-module and $\{\gamma_1, \ldots, \gamma_t\} = \operatorname{Gal}(E/L)$.