Solution to Exercise 3.6.9

Let $N \trianglelefteq G$ with $N \cong \mathrm{M}_{12}$ and $[G : N] = 2$. Looking at the character table of N we see that there are four possibilities:

(i) No classes of N fuse in G, then $|\operatorname{Irr}(G)| = 30$ and 15 classes in $G \setminus N$,

(ii) Only `11a/b` fuse in G, then $|\operatorname{Irr}(G)| = 27$ and 13 classes in $G \setminus N$,

(iiii) `4a/b` and `8a/b` fuse in G, then $|\operatorname{Irr}(G)| = 24$ and 11 classes in $G \setminus N$,

(iv) `4a/b`, `8a/b` and `11a/b` fuse in G, then $|\operatorname{Irr}(G)| = 21$ and 9 classes in $G \setminus N$.

In case (i) G has irreducible representations of degree 11 and we see as in Example 3.6.23 that $G = N \times \langle z \rangle$, where z is an involution in $\mathbf{C}_G(\texttt{11a})$. We will show that cases (ii) and (iii) are impossible and that there is a unique character table in case (iv).

For $g \in G$ let $\theta_2(g) = \sum_{\chi \in \operatorname{Irr}(G)} \nu_2(\chi)\, \chi(g)$. This is the number of square roots of g in G. Observe that $\nu_2(\psi) = 1$ for all $\psi \in \operatorname{Irr}(N)$ except for the pair (ψ_4, ψ_5) of complex conjugate characters of N of degree 16. By Exercise 3.1.4 the contribution of the two extensions of a real invariant $\psi \in \operatorname{Irr}(N)$ to $\theta_2(h)$ for $h \in N$ is zero or $2 \cdot \psi(h)$. For a pair of in G conjugate real $\psi, \psi' \in \operatorname{Irr}(N)$ the contribution of ψ^G to $\theta_2(h)$ is $\psi(h) + \psi'(h)$. Finally, the contribution of the extensions of ψ_4, ψ_5 is zero, if these characters are invarinant in G and the contribution of ψ_4^G is $\pm(\psi_4(h) + \psi_5(h))$ otherwise.

We will compute the hypothetical number $x_2(h)$ of square roots of any $h \in N$ in $G \setminus N$ in all cases — that is inserting the possible values of $\nu_2(\chi)$ in $\theta_2(h)$ — and test whether or not $x_2(h) \geq 0$ far all $h \in N$ and $\sum_{h \in N} x_2(h) = |N|$.

Using Lemma 2.9.8 we first compute for each $h^N \subset N$ the number of square roots of h in N, which we will have to subtract from $\theta_2(h)$:

```
gap> ct := CharacterTable("M12");;
gap> irrN := List( Irr(ct), ValuesOfClassFunction );;
gap> rootsinN := Indicator( ct, 2 ) * irrN;
[ 892, 0, 12, 10, 4, 4, 4, 2, 0, 0, 0, 0, 0, 1, 1 ]
```

Case (ii), all $\psi \in \operatorname{Irr}(N)$ invariant, except (ψ_4, ψ_5):

```
gap> posroots:= [];; cc := List( Irr(ct), x-> [2,0] );;
gap> cc{[1,4,5]} := [[2], [0], [0]];;
gap> ccc := Cartesian(cc);; Length(ccc);
4096
gap> for c in ccc do
>   x2 := c * irrN + Sum( irrN{[4,5]} ) - rootsinN;
>    if ForAll(x2, i -> i >= 0) and x2*SizesConjugacyClasses(ct)=Size(ct)
>       then Add(posroots, x2);
>    fi;
>   x2 := c * irrN - Sum(irrN{[4,5]}) - rootsinN;
>    if ForAll(x2, i -> i >= 0) and x2*SizesConjugacyClasses(ct)=Size(ct)
```

```
>       then Add( posroots, x2 );
>    fi;
> od;
gap> posroots;
[  ]
```

Case (iii), all $\psi \in \mathrm{Irr}(N)$ invariant, except (ψ_2, ψ_3) and (ψ_9, ψ_{10}):

```
gap> posroots := [];; cc := List( Irr(ct), x -> [2,0] );;
gap> cc{[1,2,3,4,5,9,10]} := [[2], [0], [0], [0], [0], [0], [0]];;
gap> ccc := Cartesian(cc);; Length(ccc);
256
gap> for c in ccc do
>   x2 := c * irrN + Sum( irrN{[2,3,9,10]} ) - rootsinN;
>    if ForAll(x2, i -> i >= 0) and x2*SizesConjugacyClasses(ct)=Size(ct)
>       then Add( posroots, x2 );
>    fi;
> od;
gap> posroots;
[ [ 760, 12, 8, 4, 4, 0, 0, 0, 0, 2, 0, 0, 2, 1, 1 ] ]
```

Since 760 is not a divisor of G, the involutions in $G \setminus N$ must split up into at least two classes. Likewise the roots of of an element of `3a` or `10a` in $G \setminus N$ must split up into at least two classes, because otherwise we would get centralizers in G of order 27 and 10 respectively, which is impossible. This would add up to at least 12 conjugacy classes in $G \setminus N$ wich is not compatible with case (iii).

Case (iv), all $\psi \in \mathrm{Irr}(N)$ invariant, except (ψ_2, ψ_3), (ψ_9, ψ_{10}) and (ψ_4, ψ_5):

```
gap> posroots := [];;
gap> for c in ccc do
>  x2 := c * irrN + Sum(irrN{[2,3,9,10]}) + Sum(irrN{[4,5]}) - rootsinN;
>   if ForAll(x2, i -> i >= 0) and x2*SizesConjugacyClasses(ct)=Size(ct)
>       then Add(posroots, x2);
>   fi;
>  x2 := c * irrN + Sum(irrN{[2,3,9,10]}) - Sum(irrN{[4,5]}) - rootsinN;
>   if ForAll(x2, i -> i >= 0) and x2*SizesConjugacyClasses(ct)=Size(ct)
>      then Add(posroots, x2);
>   fi;
> od;
gap> posroots;
[ [ 792, 20, 8, 0, 6, 0, 0, 2, 2, 2, 0, 0, 0, 0, 0 ] ]
```

We see that there is a unique solution (with $\nu_2(\chi) = 1$ for all $\chi \in \mathrm{Irr}(G)$) and we obtain the following information about the conjugacy classes in $G \setminus N$:

h^N	`1a`	`2b`	`2a`	`3b`	`5a`	`6a`	`6b`
$x_2(h)$	792	8	20	6	2	2	2
$\lvert\{g \in G \mid g^2 \in h^N\}\rvert$	$\frac{\lvert G\rvert}{240}$	$\frac{\lvert G\rvert}{48}$	$\frac{\lvert G\rvert}{24}$	$\frac{\lvert G\rvert}{12}$	$\frac{\lvert G\rvert}{10} = 2\,\frac{\lvert G\rvert}{20}$	$\frac{\lvert G\rvert}{12}$	$\frac{\lvert G\rvert}{6} = 2\,\frac{\lvert G\rvert}{12}$
g^G	`2c`	`4c`	`4d`	`6c`	`10b` `10c`	`12a`	`12b` `12c`

Observe that $\lvert \mathbf{C}_G(\mathtt{5a})\rvert = 20$ and $\lvert \mathbf{C}_G(\mathtt{6b})\rvert = 12$, so the roots of elements of `5a` and `6b` have to split up into two classes each.

Knowing the centralizer orders and the powermaps the construction of the character is not hard. We start with the column for `12a`. All entries are ± 1 and are determined by the congruence relations with `6a`. Note that we have a choice ordering the pairs of characters extending the invariant irreducible characters of N and we hereby make a definite choice for (χ_5, χ_6), (χ_9, χ_{10}), (χ_{14}, χ_{15}), (χ_{18}, χ_{19}) and (χ_{20}, χ_{21}). Clearly all these characters are rational valued. The character values on the class `4d` are then determined by the congruence relations modulo 6 and we have to take the smallest solution to achieve the correct centralizer order.

Similarly we obtain the values for `6c` up to some signs. Using Theorem 3.10.10 we deduce that for $i = 5, 6, 9, 10, 16, 17$ the characters χ_i vanish on elements of order 10 and we get for these i also $\chi_i(\texttt{2c})$ modulo 10, hence up some signs.

We also compute $\chi_3^{[1^2]}$ which is $\chi_{10} + \chi_{11} + \chi_{12}$ or $\chi_{10} + \chi_{11} + \chi_{13}$, and we choose the notation so that the second alternative holds. From this we get χ_{10} and χ_{13}. The status then is:

$\lvert \mathbf{C}_G(g) \rvert$	$\lvert G \rvert$	108	20	24	240	48	24	12	20	20	12	12	12
g^G	`1a`	`3b`	`5a`	`6a`	`2c`	`4c`	`4d`	`6c`	`10b`	`10c`	`12a`	`12b`	`12c`
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
χ_3	22	-2	2	-2	.	.	.	.	.	.	.	.	.
χ_4	32	2	2	2	.	.	.	.	.	.	.	.	.
χ_5	45	3	0	-1	± 5		1	∓ 1	.	.	1		
χ_6	45	3	0	-1	∓ 5		-1	± 1	.	.	-1		
χ_7	54	0	-1	0			.	.			.		
χ_8	54	0	-1	0			.	.			.		
χ_9	55	1	0	1	5	1	-1	-1	.	.	-1	1	1
χ_{10}	55	1	0	1	-5	-1	1	1	.	.	1	-1	-1
χ_{11}	110	2	0	2	.	.	.	.	.	.	.	.	.
χ_{12}	66	0	1	0	6	2	.	.	1	1	.	-1	-1
χ_{13}	66	0	1	0	-6	-2	.	.	-1	-1	.	1	1
χ_{14}	99	3	-1	-1			-1	± 1			-1		
χ_{15}	99	3	-1	-1			1	∓ 1			1		
χ_{16}	120	0	0	0	.		.	.	.	.	.		
χ_{17}	120	0	0	0	.		.	.	.	.	.		
χ_{18}	144	-3	-1	1			2	± 1			-1		
χ_{19}	144	-3	-1	1			-2	∓ 1			1		
χ_{20}	176	-1	1	-1			-2	± 1			1		
χ_{21}	176	-1	1	-1			2	∓ 1			-1		
$\chi_3^{[1^2]}$	231	6	1	3	-11	-3	1	1	-1	-1	1	0	0

The completion of the character table is now straightforward.