Solution to Exercise 4.1.3

The claims follow directly from Lemma 1.1.9, which shows that the conjugacy class sums $(C^+)_{C\in \mathrm{cl}(G)}$ form an S-basis of $\mathbf{Z}(SG)$ for any commutative ring S.

Extending the canonical epimorphism $\eta\colon R \to F$ to

$$\eta'\colon\ \mathbf{Z}(RG) \to \mathbf{Z}(FG), \qquad \sum_{C\in \mathrm{cl}(G)} a_C C^+ \mapsto \sum_{C\in \mathrm{cl}(G)} \eta(a_C) C^+$$

we see that η' is an epimorphism, since it maps an R-basis of $\mathbf{Z}(RG)$ to an F–basis of $\mathbf{Z}(FG)$ and

$$\begin{aligned} \ker \eta' &= \{ \sum_{C\in \mathrm{cl}(G)} a_C C^+ \mid a_C \in \pi R \text{ for all } C \in \mathrm{cl}(G)\} \\ &= \pi\,\mathbf{Z}(RG). \end{aligned}$$