Solution to Exercise 4.11.4

We first show the following:

(1) If $0 \neq \epsilon = \epsilon^2 \in FG$ and $N \trianglelefteq G$ with $h \cdot \epsilon = \epsilon$ for all $h \in N$ then $p \nmid |N|$.

Proof: Let $\epsilon := \sum_{g\in G} \alpha_g g$ with $\alpha_g \in F$. If $h \cdot \epsilon = \epsilon$ then $\alpha_g = \alpha_{h^{-1}g}$ for all $g \in G$. Thus by assumption α_g is constant on the cosets of N. Hence, writing $G = \dot{\bigcup}_{i=1}^{m} N t_i$ we get, using $N^+ \in \mathbf{Z}(FG)$

$$\epsilon = \sum_{i=1}^{m} \alpha_{t_i} N^+ t_i = \epsilon^2 = (N^+)^2 \cdot (\sum_{i=1}^{m} \alpha_{t_i} t_i)^2 = |N| \cdot N^+ (\sum_{i=1}^{m} \alpha_{t_i} t_i)^2.$$

This would be zero if $p \mid |N|$, which completes the proof.

(a) Let $B \in \mathrm{Bl}_p(G)$ and $N := \{g \in G \mid g\,\hat{\epsilon}_B = \hat{\epsilon}_B\}$. Clearly $\ker B \subseteq N$. For the converse observe that $\epsilon_N := \frac{1}{|N|} N^+$ is (by (1)) a central idempotent in RG and (since $h\,\epsilon_N = \epsilon_N$ for all $h \in N$)

$$\epsilon_N = \sum_{\substack{B' \in \mathrm{Bl}_p(G)\\ N \leq \ker B'}} \epsilon_{B'} \qquad \text{hence} \qquad \hat{\epsilon}_N = \sum_{\substack{B' \in \mathrm{Bl}_p(G)\\ N \leq \ker B'}} \hat{\epsilon}_{B'}.$$

Since $\hat{\epsilon}_N \hat{\epsilon}_B = \hat{\epsilon}_B$, the block idempotent $\hat{\epsilon}_B$ occurs in the last sum, thus $N \leq \ker B$.

(b) Using (a) we see that $\ker B \leq \mathrm{Inv}_G(V)$ for any FG-module V belonging to B. If V is a projective indecomposable FG-module belonging to B, say $V = FG\epsilon$ with $\epsilon = \epsilon^2 \in FG$, and $N := \mathrm{Inv}_G(V)$, then $p \nmid |N|$ by (1) and $\hat{\epsilon}_N\,\epsilon = \epsilon$. Thus $\hat{\epsilon}_B$ is a summand of $\hat{\epsilon}_N$ and consequently $N \leq \ker B$.

(c) Let $N := \bigcap\{\,\mathrm{Inv}_G(V) \mid V$ simple FG module belonging to $B\,\}$. By (a) and (1) we have $\ker B \leq \mathbf{O}_{p'}(N)$. Conversely, let $g \in N$. Then $(g-1)\,\epsilon \in \mathrm{J}(FG)\,\epsilon$ for every primitive idempotent $\epsilon \in FG$ with $\hat{\epsilon}_B\,\epsilon = \epsilon$. It follows that $(g-1)\,\hat{\epsilon}_B \in \mathrm{J}(FG)\,\hat{\epsilon}_B$. Since $\mathrm{J}(FG)$ is nilpotent, there is $n \in \mathbb{N}$ such that $0 = (g-1)^{p^n}\,\hat{\epsilon}_B = (g^{p^n} - 1)\,\hat{\epsilon}_B$ and consequently $g^{p^n} \in \ker B$. Thus $N/\ker B$ is a p-group.

Remark (i) Since every simple FG-module in B is the inflation of a simple $F(G/\ker B)$-module it follows from the proof of (c) and Corollary 3.6.3 that

$$N/\ker B = \mathbf{O}_p(G/\ker B).$$

(ii) With the notation of Definition 4.7.26 we have for $N \trianglelefteq G$ with $p \nmid |N|$

$$\mathrm{Bl}_p(G \mid B_0(N)) = \{B \in \mathrm{Bl}_p(G) \mid N \leq \ker B\}.$$

See also Exercise 4.7.6.