Solution to Exercise 4.2.2

Part a). According to Exercise 1.3.4 the group $\mathrm{SL}_2(p)$ has p irreducible representations of dimension $1, 2, \ldots, p$ over $\mathbb{F}_p$ and, since theses representations are absolutely irreducible, over any field of characteristic p. The ordinary character table of $\mathrm{SL}_2(7)$ is available in GAP as $2.L2(7)$:

```
gap> c:=CharacterTable("2.L2(7)");;
gap> Display(c);
2.L3(2)

      2  4  4  3  1  1  3  3   1   1   1   1
      3  1  1  .  1  1  .  .   .   .   .   .
      7  1  1  .  .  .  .  .   1   1   1   1

        1a 2a 4a 3a 6a 8a 8b  7a 14a  7b 14b
     2P 1a 1a 2a 3a 3a 4a 4a  7a  7a  7b  7b
     3P 1a 2a 4a 1a 2a 8b 8a  7b 14b  7a 14a
     7P 1a 2a 4a 3a 6a 8a 8b  1a  2a  1a  2a

X.1      1  1  1  1  1  1  1   1   1   1   1
X.2      3  3 -1  .  .  1  1   B   B  /B  /B
X.3      3  3 -1  .  .  1  1  /B  /B   B   B
X.4      6  6  2  .  .  .  .  -1  -1  -1  -1
X.5      7  7 -1  1  1 -1 -1   .   .   .   .
X.6      8  8  . -1 -1  .  .   1   1   1   1
X.7      4 -4  .  1 -1  .  .  -B   B -/B  /B
X.8      4 -4  .  1 -1  .  . -/B  /B  -B   B
X.9      6 -6  .  .  .  A -A  -1   1  -1   1
X.10     6 -6  .  .  . -A  A  -1   1  -1   1
X.11     8 -8  . -1  1  .  .   1  -1   1  -1

A = E(8)-E(8)^3
  = ER(2) = r2
B = E(7)+E(7)^2+E(7)^4
  = (-1+ER(-7))/2 = b7
```

In particular the number of 7-regular classes is 7 and therefore the representations constructed in Exercise 1.3.4 cover all the irreducible representations up to isomorphism. In order to minimize the amount of work we have to do, observe that the action on the homogeneous polynomials of degree 1 is equivalent to the natural representation (actually the contragredient, but the natural representation is selfdual) and the action on the homogeneous polynomials of degree $n, n \geq 2$, is equivalent to the action of $\mathrm{SL}_2(7)$ on the n-th symmetric power of the natural representation. Therefore we will only determine the Brauer character φ_2 of the natural representation and then use the GAP function `SymmetricParts` to complete the Brauer character table. We stick to the list of conjugacy classes as given above. The values of the Brauer character φ_2 are then given as follows:

$$\begin{array}{ccccccc} 1a & 2a & 3a & 4a & 6a & 8a & 8b \\ 2 & -2 & -1 & 0 & 1 & \boldsymbol{\zeta}_8^3+\boldsymbol{\zeta}_8^5 & \boldsymbol{\zeta}_8+\boldsymbol{\zeta}_8^{-1} \end{array}$$

First note that $\varphi_2(g)$ for an element of order $m \geq 2$ is of the form $\varphi_2(g) = \zeta + \zeta^{-1}$, where ζ is **some** primitive m-th root of unity. This observation uniquely determines the first 5 character values. For the classes with elements of order 8 we choose without loss of generality the class $8b$ to be the one with $\varphi_2(8a) = \boldsymbol{\zeta}_8 + \boldsymbol{\zeta}_8^{-1}$, where $\boldsymbol{\zeta}_8 := \exp(2\pi i/8)$.

In the following GAP session, we fetch the 7-modular table of $\mathrm{SL}_2(7)$ (only using the conjugacy class information), type in φ_2 and then compute and display the symmetric powers of φ_2 for $n = 1, \ldots, 6$.

```
gap> c := CharacterTable( "2.L2(7)mod7" );;
gap> phi2 := [ 2, -2, 0, -1, 1, -E(8)+E(8)^3, E(8)-E(8)^3 ];;
gap> irr := List([1..6], x-> SymmetricParts( c, [phi2], x )[1] );;
gap> Display(c, rec( chars := irr, powermap := false ));
2.L3(2)mod7

     2  4  4  3  1  1  3  3
     3  1  1  .  1  1  .  .
     7  1  1  .  .  .  .  .

       1a 2a 4a 3a 6a 8a 8b

Y.1     2 -2  . -1  1  A -A
Y.2     3  3 -1  .  .  1  1
Y.3     4 -4  .  1 -1  .  .
Y.4     5  5  1 -1 -1 -1 -1
Y.5     6 -6  .  .  . -A  A
Y.6     7  7 -1  1  1 -1 -1

A = -E(8)+E(8)^3
  = -ER(2) = -r2
```

Part b): We consider an element g of order 8 in class $8b$ of $\mathrm{SL}_2(7)$ with Brauer character value $\varphi_2(g) = \boldsymbol{\zeta}_8 + \boldsymbol{\zeta}_8^{-1}$. If we apply σ_2 to φ_2 we get a class function φ_2^σ on the 7-regular classes with $\varphi_2^\sigma(1) = 2$ and $\varphi_2^\sigma(g) = \boldsymbol{\zeta}_8^3 + \boldsymbol{\zeta}_8^{-3} \neq \boldsymbol{\zeta}_8 + \boldsymbol{\zeta}_8^{-1}$. Since there is a unique irreducible Brauer character of degree 2 namely φ_2 it follows that φ_2^σ is not an irreducible Brauer character. Since it is not the sum of two trivial characters either, it follows that φ_2^σ is not a Brauer character.

Part c): First note that the number of p-regular classes in $\mathrm{SL}_2(p)$ is exactly p, so again the representations constructed in Exercise 1.3.4 give all the irreducible representations over any field of characteristic p. For the case of a prime $p > 5$

consider an element of order $p+1$ in $\mathrm{SL}_2(p)$ with Brauer character value $\varphi_2(g) = \boldsymbol{\zeta}_{p+1} + \boldsymbol{\zeta}_{p+1}^{-1}$. Choose a prime q not dividing $q+1$ and $q \not\equiv 1, -1 \mod (p+1)$ and define an automorphism σ of $\mathbb{Q}(\boldsymbol{\zeta}_{p+1})$ as $\sigma(\boldsymbol{\zeta}_{p+1}) = \boldsymbol{\zeta}_{p+1}^{q}$ and extend to the algebraic closure. The the same reasoning as in part b) shows that φ_2^{σ} is not a Brauer character.