Solution to Exercise 4.5.4

Let $G = \mathrm{L}_2(11)$. We first have a look at the ordinary character table in GAP:

```
gap> l211:=CharacterTable("L2(11)");
CharacterTable( "L2(11)" )
gap> Display(l211);
L2(11)

      2  2  2  1  .  .  1   .   .
      3  1  1  1  .  .  1   .   .
      5  1  .  .  1  1  .   .   .
     11  1  .  .  .  .  .   1   1

        1a 2a 3a 5a 5b 6a 11a 11b
     2P 1a 1a 3a 5b 5a 3a 11b 11a
     3P 1a 2a 1a 5b 5a 2a 11a 11b
     5P 1a 2a 3a 1a 1a 6a 11a 11b
    11P 1a 2a 3a 5a 5b 6a  1a  1a

X.1      1  1  1  1  1  1   1   1
X.2      5  1 -1  .  .  1   B  /B
X.3      5  1 -1  .  .  1  /B   B
X.4     10 -2  1  .  .  1  -1  -1
X.5     10  2  1  .  . -1  -1  -1
X.6     11 -1 -1  1  1 -1   .   .
X.7     12  .  .  A *A  .   1   1
X.8     12  .  . *A  A  .   1   1

A = E(5)+E(5)^4
  = (-1+ER(5))/2 = b5
B = E(11)+E(11)^3+E(11)^4+E(11)^5+E(11)^9
  = (-1+ER(-11))/2 = b11
```

We immediately detect that there are six 2-regular classes and so we are looking for 6 irreducible Brauer characters (and 6 indecomposable projective characters). Note further that two blocks , B_3 and B_4, are of of defect zero, containing χ_7 respectively χ_8. We check using the GAP-command `PrimeBlocks` that there are two further blocks, the principal block B_1 containing $\chi_1, \chi_2, \chi_3, \chi_6$ and a nonprincipal block B_2 containing χ_4, χ_5.

```
gap> bl:=PrimeBlocks(l211,2);
rec( block := [ 1, 1, 1, 2, 2, 1, 3, 4 ],
defect := [ 2, 1, 0, 0 ],
  height := [ 0, 0, 0, 0, 0, 0, 0, 0 ],
 relevant := [ 3, 4, 7 ],
  centralcharacter := [ [ ,, 110, 132,,, 60 ],
```

```
 [ ,, 11, 0,,, -6 ], [ ,, 0, 11*E(5)+11*E(5)^4,,, 5 ],
      [ ,, 0, 11*E(5)^2+11*E(5)^3,,, 5 ] ] )
```

Since the number of irreducible Brauer characters in a block not of defect zero has to be less than the number of ordinary irreducible characters in the block, we conclude that the principal block B_1 contains 3 irreducible Brauer characters and the nonprincipal block B_2 contains exactly one irreducible Brauer character. We also know that the trivial Brauer character is in B_1 and so we are looking for 2 irreducible Brauer characters in B_1 and one in B_2. It is straightforward to check that χ_1, χ_2, χ_3 restricted to the 2-regular classes are $\mathbb{Z}$-linear independent and χ_6 on the 2-regular classes is just the sum $\chi_1 + \chi_2 + \chi_3$. This implies that the restrictions of χ_1, χ_2, and χ_3 to the 2-regular classes are a basic set for B_1 and it is also clear that the restriction of χ_4 to the 2-regular classes is a basic set for B_2. We now follow the hint and induce 2-projective characters from a maximal subgroup H of type $11\colon 5$. Since H is of odd order, any ordinary irreducible character induced to G will give a 2-projective character of G. In the following lines of GAP-code, we get the character table of H, induce up all the irreducible characters, and display their decomposition into the irreducible characters of G, actually transposed.

```
gap> max:=Maxes(l211);
[ "A5", "A5", "11:5", "S3x2" ]
gap> h:=CharacterTable(max[3]);
CharacterTable( "11:5" )
gap> ind:=InducedClassFunctions(h,Irr(h),l211);;
gap> Display(TransposedMat(
      MatScalarProducts(l211,Irr(l211),ind)));
[ [  1,  0,  0,  0,  0,  0,  0 ],
  [  0,  0,  0,  0,  0,  1,  0 ],
  [  0,  0,  0,  0,  0,  0,  1 ],
  [  0,  0,  0,  0,  0,  1,  1 ],
  [  0,  0,  0,  0,  0,  1,  1 ],
  [  1,  0,  0,  0,  0,  1,  1 ],
  [  0,  1,  0,  0,  1,  1,  1 ],
  [  0,  0,  1,  1,  0,  1,  1 ] ]
```

The first column represents a projective indecomposable character for the principal block, and the last two columns represent the sums of four projective indecomposable characters, one from each of the blocks B_1, B_2, B_3, and B_4. This suffices to verify that the decomposition matrix is as stated:

	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6
χ_1	1	.	.	.	.	.
χ_2	.	1	.	.	.	.
χ_3	.	.	1	.	.	.
χ_4	.	.	.	1	.	.
χ_5	.	.	.	1	.	.
χ_6	1	1	1	.	.	.
χ_7	.	.	.	.	1	.
χ_8	.	.	.	.	.	1