Solution to Exercise 4.6.2

Let $\mathcal{S} := (K, R, F, \eta)$ be a p-modular system for G and let $B \in \mathrm{Bl}_p(G)$ with $\mathrm{Irr}(B) = \{\chi\}$. By Theorem 4.6.6 (see also (4.20) on page 323) a class g^G is a defect class of B (with respect to $\mathcal{S}$) if and only if

$$\omega_\chi((g^G)^+) \cdot \frac{\chi(1)}{|G|}\chi(g^{-1}) = \frac{1}{|\mathbf{C}_G(g)|}\,\chi(g)\cdot\chi(g^{-1}) \notin \ker\eta. \tag{1}$$

Observe that $|\mathbf{C}_G(g)|$ is a unit in R if g^G is a defect class of B. So g^G is a defect class of B if and only if

$$p \nmid |\mathbf{C}_G(g)| \quad \text{and} \quad \chi(g)\cdot\chi(g^{-1}) \notin \ker\eta.$$

Now, let $G = \mathrm{U}_3(4)$ and $p := 3$. We choose the defect zero character $\chi := \chi_{19}$ and and $g \in$ `13a` in the notation of the Atlas (or GAP-library of character tables) and put $B \in \mathrm{Bl}_3(G)$ with $\mathrm{Irr}(B) = \{\chi\}$. We see that $3 \nmid |\mathbf{C}_G(g)|$ and

$$a := \chi(g)\cdot\chi(g^{-1}) = (-\boldsymbol{\zeta}_{13} - \boldsymbol{\zeta}_{13}^3 - \boldsymbol{\zeta}_{13}^9)\cdot(-\boldsymbol{\zeta}_{13}^4 - \boldsymbol{\zeta}_{13}^{10} - \boldsymbol{\zeta}_{13}^{12})$$

and

$$a_5 := \chi(g^5)\cdot\chi(g^{-5}) = (-\boldsymbol{\zeta}_{13}^5 - \boldsymbol{\zeta}_{13}^2 - \boldsymbol{\zeta}_{13}^6)\cdot(-\boldsymbol{\zeta}_{13}^7 - \boldsymbol{\zeta}_{13}^{8} - \boldsymbol{\zeta}_{13}^{11}).$$

If $\mathcal{S}$ is the standard p-modular splitting system then $\eta(\boldsymbol{\zeta}_{13}) = [X^2 + (f_{3,3})]_\sim \in \mathcal{F}_3$ because η extends the ring homomorphism

$$\mathbb{Z}[\boldsymbol{\zeta}_{26}] \to \mathbb{F}_3[X]/(f_{3,3}) \quad \text{with} \quad \boldsymbol{\zeta}_{26} \mapsto X + (f_{3,3})$$

$f_{3,3}$ being the Conway polynomial, see Example 4.2.8. We compute (e.g. with GAP, see below)

$$f_{3,3} \mid (-X^2 - X^6 - X^{18})\cdot(-X^8 - X^{20} - X^{24}) + 1 \in \mathbb{F}_3[X]$$

and

$$f_{3,3} \mid (-X^{10} - X^4 - X^{12})\cdot(-X^{14} - X^{16} - X^{22}) \in \mathbb{F}_3[X]$$

and conclude that $\eta(a) = -1$ and $\eta(a_5) = 0$. Thus g^G is a defect class of B and $(g^5)^G$ is not defect class of B (with respect to $\mathcal{S}$).

Now we choose $f := X^3 + X^2 - X + 1 \in \mathbb{F}_3[X]$ instead of the Conway polynomial $f_{3,3}$ and a p-modular system $\mathcal{S}' := (K', R', F', \eta')$ such that η' extends the ring homomorphism

$$\mathbb{Z}[\boldsymbol{\zeta}_{26}] \to \mathbb{F}_3[X]/(f) \quad \text{with} \quad \boldsymbol{\zeta}_{26} \mapsto X + (f).$$

Observe that f is primitive and $\mathcal{S}'$ exists by the proof of Theorem 4.2.7. Then we find that $\eta'(a) = 0$ and $\eta'(b) = -1$ showing that $(g^5)^G$ is a defect class of B (with respect to $\mathcal{S}'$) and g^G is not.

We add the GAP-code for doing the above computations:

```
gap> ct := CharacterTable("U3(4)");; PrimeBlocks(ct,3).defect;
[ 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0 ]
gap> List( Positions( PrimeBlocks(ct,3).defect, 0 ),
>                  i -> Position( PrimeBlocks(ct,3).block, i ) );
[ 2, 7, 8, 19, 20, 21, 22 ]
gap> Positions( OrdersClassRepresentatives(ct), 13 );
[ 15, 16, 17, 18 ]
gap> a := Irr(ct)[19][15]; aa := ComplexConjugate(a);
-E(13)-E(13)^3-E(13)^9
-E(13)^4-E(13)^10-E(13)^12
gap> b:=Irr(ct)[19][ComputedPowerMaps(ct)[5][15]]; bb:=ComplexConjugate(b);
-E(13)^2-E(13)^5-E(13)^6
-E(13)^7-E(13)^8-E(13)^11
gap> x := Indeterminate(GF(3),"x");;
gap> f33 := x^3 - x + Z(3)^0;; # the Conway polynomial
gap> g := (- x^2 - x^6 - x^18) * (- x^8 - x^20 - x^24);;
gap> g5 := (- x^10 - x^30 - x^90) * (- x^40 - x^100 - x^120);;
gap> IsPolynomial( (g + Z(3)^0)/f33 ); IsPolynomial( g5/f33 );
true
true
gap> f := x^3+x^2-x+ Z(3)^0;;
gap> IsPolynomial( g/f ); IsPolynomial( ( g5 + Z(3)^0 )/f );
true
true
```