Faithfulness of a functor of Quillen

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Abstract

There exists a canonical functor from the category of fibrant objects of a model category modulo cylinder homotopy to its homotopy category. We show that this functor is faithful under certain conditions, but not in general.

1 Introduction

We let $\mathcal{M}$ be a model category. Quillen defines in [5, ch. I, §1] a homotopy relation on the full subcategory $\mathbf{Fib}(\mathcal{M})$ of fibrant objects, using cylinders. He obtains a quotient category $\mathbf{Fib}(\mathcal{M})/\overset{c}{\sim}$ and a canonical functor

$$\mathbf{Fib}(\mathcal{M})/\overset{c}{\sim} \to \operatorname{Ho}\mathbf{Fib}(\mathcal{M}).$$

The question occurs whether this functor is faithful.
We show that it is faithful if $\mathcal{M}$ is left proper and fulfills an additional technical condition. Moreover, we show by an example that it is not faithful in general.

Conventions and notations

- The composite of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is denoted by $fg\colon X \to Z$.
- Given $n \in \mathbb{N}_0$, we abbreviate $\mathbb{Z}/n := \mathbb{Z}/n\mathbb{Z}$. Given $k, m, n \in \mathbb{N}_0$, we write $k\colon \mathbb{Z}/m \to \mathbb{Z}/n, a + m\mathbb{Z} \mapsto ka + n\mathbb{Z}$, provided n divides km.
- Given a category $\mathcal{C}$ with finite coproducts and objects $X, Y \in \operatorname{Ob}\mathcal{C}$, we denote by $X \amalg Y$ a (chosen) coproduct. The embedding $X \to X \amalg Y$ is denoted by emb_0, the embedding $Y \to X \amalg Y$ by emb_1. Given morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ in $\mathcal{C}$, the induced morphism $X \amalg Y \to Z$ is denoted by $\binom{f}{g}$.
- Given a category $\mathcal{C}$ and an object $X \in \operatorname{Ob}\mathcal{C}$, the category of objects in $\mathcal{C}$ under X will be denoted by $(X \downarrow \mathcal{C})$. The objects in $(X \downarrow \mathcal{C})$ are denoted by (Y, f), where $Y \in \operatorname{Ob}\mathcal{C}$ and $f\colon X \to Y$ is a morphism in $\mathcal{C}$.

2 Preliminaries from homotopical algebra

We recall some basic facts from homotopical algebra. Our main reference is [5, ch. I, §1].

Model categories

Throughout this note, we let $\mathcal{M}$ be a model category, cf. [5, ch. I, §1, def. 1]. In $\mathcal{M}$, there are three kinds of distinguished morphisms, called *cofibrations*, *fibrations* and *weak equivalences*. Cofibrations are closed under pushouts. If weak equivalences in $\mathcal{M}$ are closed under pushouts along cofibrations, $\mathcal{M}$ is said to be *left proper*, cf. [3, def. 13.1.1(1)].

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An object $X \in \operatorname{Ob}\mathcal{M}$ is said to be *fibrant* if the unique morphism $\mathcal{M} \to *$ is a fibration, where $*$ is a (chosen) terminal object in $\mathcal{M}$. The full subcategory of $\mathcal{M}$ of fibrant objects is denoted by $\mathbf{Fib}(\mathcal{M})$.
The *homotopy category* of $\mathcal{C} = \mathcal{M}$ resp. $\mathcal{C} = \mathbf{Fib}(\mathcal{M})$ is a localisation of $\mathcal{C}$ with respect to the weak equivalences in $\mathcal{C}$ and is denoted by $\operatorname{Ho}\mathcal{C}$. The localisation functor of $\operatorname{Ho}\mathcal{C}$ is denoted by $\Gamma = \Gamma^{\operatorname{Ho}\mathcal{C}} \colon \mathcal{C} \to \operatorname{Ho}\mathcal{C}$.
Given an object $X \in \operatorname{Ob}\mathcal{M}$, the category $(X \downarrow \mathcal{M})$ of objects under X obtains a model category structure where a morphism in $(X \downarrow \mathcal{M})$ is a weak equivalence resp. a cofibration resp. a fibration if and only if it is one in $\mathcal{M}$.

Homotopies

A *cylinder* for an object $X \in \operatorname{Ob}\mathcal{M}$ consists of an object $Z \in \operatorname{Ob}\mathcal{M}$, a cofibration $\left(\begin{smallmatrix}\mathrm{ins}_0\\ \mathrm{ins}_1\end{smallmatrix}\right) = \mathrm{ins} = \mathrm{ins}^Z \colon X \amalg X \to Z$ and a weak equivalence $\mathrm{s} = \mathrm{s}^Z \colon Z \to X$ such that $\mathrm{ins}\,\mathrm{s} = \left(\begin{smallmatrix}1\\ 1\end{smallmatrix}\right)$.
Given parallel morphisms $f, g \colon X \to Y$ in $\mathcal{M}$, we say that f is *cylinder homotopic* to g, written $f \overset{\mathrm{c}}{\sim} g$, if there exists a cylinder Z for X and a morphism $H \colon Z \to Y$ with $\mathrm{ins}_0 H = f$ and $\mathrm{ins}_1 H = g$. In this case, H is said to be a *cylinder homotopy* from f to g. (In the literature, cylinder homotopy is also called left homotopy, cf. [5, ch. I, §1, def. 3, def. 4, lem. 1].) The relation $\overset{\mathrm{c}}{\sim}$ is reflexive and symmetric, but in general not transitive. Moreover, $\overset{\mathrm{c}}{\sim}$ is compatible with composition in $\mathbf{Fib}(\mathcal{M})$. We denote by $\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim}$ the quotient category of $\mathbf{Fib}(\mathcal{M})$ with respect to the congruence generated by $\overset{\mathrm{c}}{\sim}$.

Quillen's homotopy category theorem

There are dual notions to fibrant objects, cylinders, cylinder homotopic $\overset{\mathrm{c}}{\sim}$, the full subcategory of fibrant objects $\mathbf{Fib}(\mathcal{M})$, its quotient category $\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim}$ and its homotopy category $\operatorname{Ho}\mathbf{Fib}(\mathcal{M})$, namely *cofibrant objects*, *path objects*, *path homotopic* $\overset{\mathrm{p}}{\sim}$, the full subcategory of cofibrant objects $\mathbf{Cof}(\mathcal{M})$, its quotient category $\mathbf{Cof}(\mathcal{M})/\overset{\mathrm{p}}{\sim}$ and its homotopy category $\operatorname{Ho}\mathbf{Cof}(\mathcal{M})$, respectively. Moreover, an object $X \in \operatorname{Ob}\mathcal{M}$ is said to be bifibrant if it is cofibrant and fibrant. On the full subcategory of bifibrant objects $\mathbf{Bif}(\mathcal{M})$, the relations $\overset{\mathrm{c}}{\sim}$ and $\overset{\mathrm{p}}{\sim}$ coincide and yield a congruence. One writes $\sim \,:=\, \overset{\mathrm{c}}{\sim} \,=\, \overset{\mathrm{p}}{\sim}$ in this case, and the quotient category is denoted by $\mathbf{Bif}(\mathcal{M})/\sim$. Moreover, $\operatorname{Ho}\mathbf{Bif}(\mathcal{M})$ is a localisation of $\mathbf{Bif}(\mathcal{M})$ with respect to the weak equivalences in $\mathbf{Bif}(\mathcal{M})$.
Quillen's homotopy category theorem [5, ch. I, §1, th. 1] (cf. also [4, cor. 1.2.9, th. 1.2.10]) states that the various inclusion and localisation functors induce the following commutative diagram, where the functors labeled by $\simeq$ are equivalences and the functor labeled by $\cong$ is an isofunctor.

$$\begin{array}{ccccc}
\mathbf{Cof}(\mathcal{M})/\overset{\mathrm{p}}{\sim} & \longrightarrow & \operatorname{Ho}\mathbf{Cof}(\mathcal{M}) & & \\
\uparrow & & \uparrow{\scriptstyle\simeq} & \searrow{\scriptstyle\simeq} & \\
\mathbf{Bif}(\mathcal{M})/\sim & \xrightarrow{\cong} & \operatorname{Ho}\mathbf{Bif}(\mathcal{M}) & & \operatorname{Ho}\mathcal{M} \\
\downarrow & & \downarrow{\scriptstyle\simeq} & \nearrow{\scriptstyle\simeq} & \\
\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim} & \longrightarrow & \operatorname{Ho}\mathbf{Fib}(\mathcal{M}) & &
\end{array}$$

In this note, we treat the question whether the functors $\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim} \to \operatorname{Ho}\mathbf{Fib}(\mathcal{M})$ and $\mathbf{Cof}(\mathcal{M})/\overset{\mathrm{p}}{\sim} \to \operatorname{Ho}\mathbf{Cof}(\mathcal{M})$ are faithful. By duality, it suffices to consider the first functor.

The model category $\mathbf{mod}(\mathbb{Z}/4)$

The category $\mathbf{mod}(\mathbb{Z}/4)$ of finitely generated modules over $\mathbb{Z}/4$ is a Frobenius category (with respect to all short exact sequences), that is, there are enough projective and injective objects in $\mathbf{mod}(\mathbb{Z}/4)$ and, moreover, these objects coincide (we call such objects bijective). Therefore $\mathbf{mod}(\mathbb{Z}/4)$ carries a canonical model category structure (cf. also [4, sec. 2.2]): The cofibrations are the monomorphisms and the fibrations are the epimorphisms in $\mathbf{mod}(\mathbb{Z}/4)$. Every object in $\mathbf{mod}(\mathbb{Z}/4)$ is bifibrant, and the weak equivalences are precisely the homotopy equivalences, where parallel morphisms f and g are homotopic if $g - f$ factors over a bijective object in $\mathbf{mod}(\mathbb{Z}/4)$. That is, the weak equivalences in $\mathbf{mod}(\mathbb{Z}/4)$ are the stable isomorphisms and the homotopy category of $\mathbf{mod}(\mathbb{Z}/4)$ is isomorphic to the stable category of $\mathbf{mod}(\mathbb{Z}/4)$, cf. [2, ch. I, sec. 2.2].

We remark that every object in $\mathbf{mod}(\mathbb{Z}/4)$ is isomorphic to $(\mathbb{Z}/4)^{\oplus k} \oplus (\mathbb{Z}/2)^{\oplus l}$ for some $k, l \in \mathbb{N}_0$, and every bijective object is isomorphic to $(\mathbb{Z}/4)^{\oplus k}$ for some $k \in \mathbb{N}_0$.

3 Faithfulness of the functor $\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim} \to \operatorname{Ho}\mathbf{Fib}(\mathcal{M})$

We give a sufficient criterion for the functor under consideration to be faithful.

Proposition. If the model category $\mathcal{M}$ is left proper and if $w \amalg w$ is a weak equivalence for every weak equivalence w in $\mathcal{M}$, then $\overset{\mathrm{c}}{\sim}$ is a congruence on $\mathbf{Fib}(\mathcal{M})$ and the canonical functor $\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim} \to \operatorname{Ho}\mathbf{Fib}(\mathcal{M})$ is faithful.

Proof. We suppose given fibrant objects X and Y and morphisms $f, g\colon X \to Y$ with $\Gamma f = \Gamma g$ in $\operatorname{Ho}\mathbf{Fib}(\mathcal{M})$. By [1, th. 1(ii)], there exists a weak equivalence $w\colon X' \to X$ such that $wf \overset{\mathrm{p}}{\sim} wg$. It follows that $wf \overset{\mathrm{c}}{\sim} wg$ by [5, ch. I, §1, dual of lem. 5], that is, there exists a cylinder Z' for X' and a cylinder homotopy $H'\colon Z' \to Y$ from wf to wg. We let

$$\begin{array}{ccc}
X' \amalg X' & \xrightarrow[\approx]{w \amalg w} & X \amalg X \\
{\scriptstyle \mathrm{ins}^{Z'}}\downarrow & & \downarrow{\scriptstyle i} \\
Z' & \xrightarrow[\approx]{w'} & Z
\end{array}$$

be a pushout of $w \amalg w$ along $\mathrm{ins}^{Z'}$. By assumption, $w \amalg w$ and w' are weak equivalences. Since $(w \amalg w)\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = \mathrm{ins}^{Z'}\mathrm{s}^{Z'}w$, there exists a unique morphism $s\colon Z \to X$ with $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right) = is$ and $\mathrm{s}^{Z'}w = w's$. Then s is a weak equivalence since $\mathrm{s}^{Z'}$, w and w' are weak equivalences and therefore Z becomes a cylinder for X with $\mathrm{ins}^{Z} := i$ and $\mathrm{s}^{Z} := s$. Moreover, $(w \amalg w)\left(\begin{smallmatrix}f\\g\end{smallmatrix}\right) = \mathrm{ins}^{Z'}H'$ implies that there exists a unique morphism $H\colon Z \to Y$ with $\left(\begin{smallmatrix}f\\g\end{smallmatrix}\right) = \mathrm{ins}^{Z}H$ and $H' = w'H$. So in particular $f \overset{\mathrm{c}}{\sim} g$.

$$\begin{array}{ccccc}
X' \amalg X' & \xrightarrow[\approx]{w \amalg w} & X \amalg X & \xrightarrow{\left(\begin{smallmatrix}f\\g\end{smallmatrix}\right)} & Y \\
{\scriptstyle \mathrm{ins}^{Z'}}\downarrow & & \downarrow{\scriptstyle \mathrm{ins}^{Z}} & & \| \\
Z' & \xrightarrow[\approx]{w'} & Z & \xrightarrow{H} & Y \\
{\scriptstyle \mathrm{s}^{Z'}}\downarrow\wr & & \wr\downarrow{\scriptstyle \mathrm{s}^{Z}} & & \\
X' & \xrightarrow[\approx]{w} & X & &
\end{array}$$

Altogether, we have shown that morphisms in $\mathbf{Fib}(\mathcal{M})$ represent the same morphism in $\operatorname{Ho}\mathbf{Fib}(\mathcal{M})$ if and only if they are cylinder homotopic. In particular, $\overset{\mathrm{c}}{\sim}$ is a congruence on $\mathbf{Fib}(\mathcal{M})$. $\square$

The following counterexample shows that the canonical functor $\mathbf{Fib}(\mathcal{M})/\overset{\mathrm{c}}{\sim} \to \operatorname{Ho}\mathbf{Fib}(\mathcal{M})$ is not faithful in general.

Example. We consider the category $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ of finitely generated $\mathbb{Z}/4$-modules under $\mathbb{Z}/4$ with the model category structure inherited from $\mathbf{mod}(\mathbb{Z}/4)$, cf. section 2. All objects of $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ are fibrant since all objects in $\mathbf{mod}(\mathbb{Z}/4)$ are fibrant.
We study morphisms $(\mathbb{Z}/4, 2) \to (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (\begin{smallmatrix}2 & 0\end{smallmatrix}))$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$. We let (Z, t) be a cylinder of $(\mathbb{Z}/4, 2)$ and we let $H\colon (Z, t) \to (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (\begin{smallmatrix}2 & 0\end{smallmatrix}))$ be a cylinder homotopy (from $\mathrm{ins}_0 H$ to $\mathrm{ins}_1 H$). Then we have a weak equivalence $(Z, t) \to (\mathbb{Z}/4, 2)$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ and hence a weak equivalence $Z \to \mathbb{Z}/4$ in $\mathbf{mod}(\mathbb{Z}/4)$. Thus Z is bijective and therefore we may assume that $Z = (\mathbb{Z}/4)^{\oplus k}$. Since ins_0 and ins_1 are morphisms from $(\mathbb{Z}/4, 2)$ to (Z, t), we have $2\mathrm{ins}_0 = t = 2\mathrm{ins}_1$ and hence $\mathrm{ins}_0 \equiv_2 \mathrm{ins}_1$ as morphisms from $\mathbb{Z}/4$ to Z. But this implies that the second components of $\mathrm{ins}_0 H$ and $\mathrm{ins}_1 H$ are the same. In other words, we have shown that cylinder homotopic morphisms from $(\mathbb{Z}/4, 2)$ to $(\mathbb{Z}/4 \oplus \mathbb{Z}/2, (\begin{smallmatrix}2 & 0\end{smallmatrix}))$ coincide in the second component. It follows that the morphisms $(\begin{smallmatrix}1 & 0\end{smallmatrix})\colon (\mathbb{Z}/4, 2) \to (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (\begin{smallmatrix}2 & 0\end{smallmatrix}))$ and $(\begin{smallmatrix}1 & 1\end{smallmatrix})\colon (\mathbb{Z}/4, 2) \to (\mathbb{Z}/4 \oplus \mathbb{Z}/2, (\begin{smallmatrix}2 & 0\end{smallmatrix}))$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$ represent different morphisms in the quotient category $\mathbf{Fib}((\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4)))/\overset{\mathrm{c}}{\sim}$.

On the other hand, since $\mathbb{Z}/4$ is bijective, the morphism $2\colon \mathbb{Z}/4 \to \mathbb{Z}/4$ is a weak equivalence in $\mathbf{mod}(\mathbb{Z}/4)$, and therefore $2\colon (\mathbb{Z}/4, 1) \to (\mathbb{Z}/4, 2)$ is a weak equivalence in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$. But $2\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right) = 2\left(\begin{smallmatrix}1 & 1\end{smallmatrix}\right)$ as morphisms from $(\mathbb{Z}/4, 1)$ to $(\mathbb{Z}/4 \oplus \mathbb{Z}/2, \left(\begin{smallmatrix}2 & 0\end{smallmatrix}\right))$ in $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$, so in particular $\Gamma(2\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right)) = \Gamma(2\left(\begin{smallmatrix}1 & 1\end{smallmatrix}\right))$ and hence $\Gamma\left(\begin{smallmatrix}1 & 0\end{smallmatrix}\right) = \Gamma\left(\begin{smallmatrix}1 & 1\end{smallmatrix}\right)$.

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