(Co)homology of crossed modules

Sebastian Thomas

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Rheinisch-Westfälisch Technische Hochschule Aachen
Lehrstuhl D für Mathematik
Prof. Dr. Gerhard Hiß

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Contents

Introduction

In the 1940s, EILENBERG and MACLANE developed a homology theory for groups (see [10], [11] for example). By definition, the homology of a group G is the (singular) homology of a connected CW-space T, called its classifying space, whose fundamental group $\pi_1(T)$ is isomorphic to the given group G and whose higher homotopy groups $\pi_n(T)$ for $n \geq 2$ are all trivial. Previously, they and HUREWICZ (cf. [14] resp. [18]) had independently recognised that the homotopy type and hence the homology of such a topological space is uniquely determined by its fundamental group. In fact, the achievement of EILENBERG and MACLANE was their purely algebraic approach to the homology of groups, circumventing topological spaces. To this end, they implicitly used a combinatorial model for the topological space in question, namely the classifying simplicial set $\mathrm{B}G$ of the group G. The calculation of its homology leads to the homological algebra description of the homology of G via a projective resolution.

So groups determine connected homotopy types T with only $\pi_1(T)$ as non-vanishing homotopy group. The homology of T is algebraically calculable starting from this group. Later, in 1949, WHITEHEAD introduced crossed modules (cf. [30]), which determine connected homotopy types T with only $\pi_1(T)$ and $\pi_2(T)$ as non-vanishing homotopy groups (cf. [25]), also known as 2-types. Examples of crossed modules comprise inclusions of normal subgroups in a group, the inner automorphism homomorphism from a group to its automorphism group, or any surjection from a central extension of a group to this group.

In this work, we want to calculate the homology of T algebraically. Therefore we can now proceed as follows. Given a crossed module V corresponding to T, we attach a classifying simplicial set $\mathrm{B}V$ to V, combinatorially modelling T. The homology of $\mathrm{B}V$ is given in algebraic terms and calculates the homology of T. In analogy to the definition of the homology of groups, we may now define the homology of V as the homology of T, or, what amounts to the same, of $\mathrm{B}V$.

The construction of the classifying simplicial set $\mathrm{B}V$ is done in two steps. First, we associate to V a simplicial group, its coskeleton $\operatorname{Cosk} V$. Second, we construct a classifying simplical set $\mathrm{B}G$ for a general simplicial group G. Thereafter we may define $\mathrm{B}V := \mathrm{B}\operatorname{Cosk} V$.

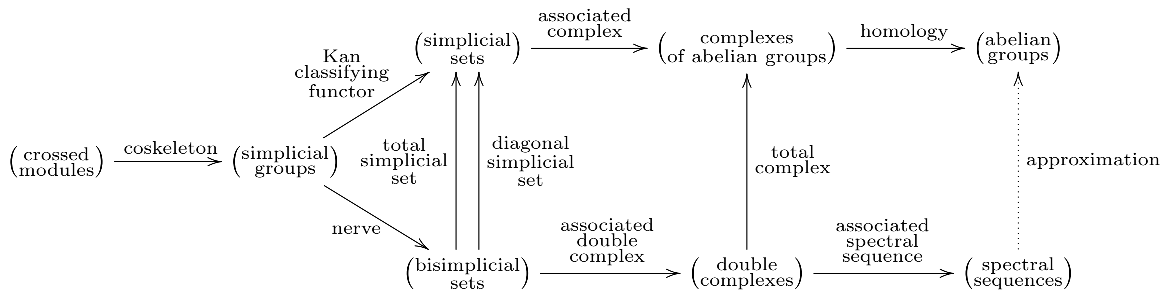
To motivate this construction, firstly, we can mention that to a simplicial group G, we can associate a crossed module $\operatorname{Trunc} G$. If only the first two homotopy groups of G are nontrivial, then there is a weak homotopy equivalence $\operatorname{Cosk}\operatorname{Trunc} G \simeq G$, so that each such simplicial group is modelled by a crossed module via Cosk. Secondly, simplicial groups model all connected homotopy types via B.

To construct the classifying simplicial set $\mathrm{B}G$ of a simplicial group G, there are two possibilities.

First, KAN introduced in [20] the Kan classifying functor $\overline{\mathrm{W}}$. This functor is the right adjoint and actually, the homotopy inverse to the Kan loop group functor, which is a combinatorial analogon to the topological loop space functor. This justifies calling $\overline{\mathrm{W}}G$ a classifying simplicial set of G.

Second, we can invoke bisimplicial sets in the following way. As mentioned in the beginning, we can attach a classifying simplicial set to a group, which in this context shall be called its nerve. A simplicial group is a sequence of groups, together with so-called face and degeneracy morphisms between them. So we can apply this nerve functor to the groups occuring in this sequence. We end up with two simplicial directions, one from the simplicial group and one from the nerve construction; i.e. we end up with a bisimplicial set $\mathrm{B}^{(2)}G$. Reading off the diagonal simplicial set of this bisimplicial set, we obtain the second variant for the classifying simplicial set of G.

The second variant is more common; the first, however, yields smaller objects in a certain sense, which is more convenient for direct calculations.



It is well-known that these two variants for the classifying simplicial set of G are indeed homotopy equivalent after topological realisation ([1]). Better still, the Kan classifying functor $\overline{\mathrm{W}}$ can be obtained as the composite of the nerve functor with a so-called total simplicial set functor introduced by Artin and Mazur [1] ([2]); and Cegarra and Remedios [7] showed that already the total simplicial set functor and the diagonal functor yield homotopy equivalent results after topological realisation ([3]).

Here, we will give an algebraic proof by constructing a simplicial homotopy equivalence between both simplicial sets. In other words, we show that the triangle in the diagram above commutes up to simplicial homotopy equivalence. This confirms in an algebraic way that both variants for the classifying simplicial set of G essentially coincide. As far as the author is aware, so far, this has only been known in a topological way.

The homology of a crossed module V is defined to be the homology of its classifying simplicial set $\mathrm{B}V$. More generally, to calculate the homology of a simplicial set, one associates a complex to it and takes its homology. Similarly, we can attach a double complex to a bisimplicial set. To a double complex in turn, we can attach its total complex. The generalised Eilenberg-Zilber theorem (due to Dold, Puppe and Cartier [9]) states that the total complex of the double complex associated to a bisimplicial set is homotopy equivalent to the complex associated to its diagonal simplicial set; i.e. the quadrangle in the middle of the diagram above commutes up to homotopy equivalence of complexes.

We solve the exercise of constructing an explicit homotopy equivalence to prove Eilenberg-Zilber, adapting the arguments of Eilenberg and Mac Lane in [12] and [13].

The homology of the total complex of a double complex can be approximated by means of a spectral sequence. Its starting terms are the horizontally taken homology groups of the vertically taken homology; it converges to the homology of the total complex. In our case of the classifying bisimplicial set $\mathrm{B}^{(2)}G$ of a simplicial group G, this yields the Jardine spectral sequence [19], whose starting terms involve ordinary group homology, and which converges to the homology of G. So the second variant of the classifying simplicial set of G enables us to use a spectral sequence. In particular, taking $G = \operatorname{Cosk} V$ for a crossed module V, this yields a spectral sequence converging to the homology of V.

To obtain results in cohomology instead of homology, we have to apply the duality functor ${}_{\mathbb{Z}}(-, \mathbb{Z})$ to the associated complex resp. to the associated double complex in the procedure described above.

Finally, we show by an example that the Jardine spectral sequence does not degenerate in the case of crossed modules.

In Ellis' approach to the (co)homology of crossed modules via quadratic modules, he develops a (co)homology theory for crossed modules that yields the (co)homology groups of its classifying space in dimensions less or equal than 4 [15]. Moreover, Carrasco, Cegarra and Grandjéan in [6] develop still another (co)homology theory of crossed modules, and Grandjean, Ladra and Pirashvili established a long exact sequence relating this homology theory with the homology of crossed modules via classifying sets as considered here. Moreover, this alternative (co)homology theory was extended by Paoli in [27] to the case, where the coefficients are in a π_1-module. These alternative (co)homology theories will not be dealt with here.

[1] Addendum (December 19, 2011): The fact that $\overline{\mathrm{W}}G$ and $\operatorname{Diag} \mathrm{N}G$ are weakly homotopy equivalent has been shown by Zisman [31, sec. 3.3.4, cf. sec. 1.3.3, rem. 1]. He shows that a morphism $\operatorname{Diag} \mathrm{N}G \to \overline{\mathrm{W}}G$, which is essentially the same as the morphism D_G we consider in chapter IV, §5, induces an isomorphism on the fundamental groups as well as isomorphisms on the homology groups of their universal coverings.

[2] This is not the total simplicial set as used by Bousfield and Friedlander [2, appendix B, p. 118].

[3] Addendum (December 19, 2011): To this end, Cegarra and Remedios consider a morphism $\operatorname{Diag} X \to \operatorname{Tot} X$, which is essentially the same as the morphism ϕ_X we consider in proposition (3.15).

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Sebastian Thomas

Conventions and notations

We use the following conventions and notations.

- The composite of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is denoted by $X \xrightarrow{fg} Z$. The composite of functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$ is denoted by $\mathcal{C} \xrightarrow{G \circ F} \mathcal{E}$.
- Isomorphy of objects X and Y in any category is denoted by $X \cong Y$.
- If $\mathcal{C}$ is a category and $X, Y \in \operatorname{Ob}\mathcal{C}$ are objects in $\mathcal{C}$, we write ${}_{\mathcal{C}}(X, Y) = \operatorname{Mor}_{\mathcal{C}}(X, Y)$ for the set of morphisms between X and Y. In particular, we write ${}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D})$ for the set of functors between (small) categories $\mathcal{C}$ and $\mathcal{D}$. To distinguish this notation from the functor category of functors between $\mathcal{C}$ and $\mathcal{D}$ as objects and natural transformations between functors as morphisms, we write $\boldsymbol{(}\mathcal{C}, \mathcal{D}\boldsymbol{)}$ in the latter case.
- We suppose given categories $\mathcal{C}$ and $\mathcal{D}$. A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is said to be an isofunctor if there exists a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be isomorphic, written $\mathcal{C} \cong \mathcal{D}$, if an isofunctor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ exists.

  A functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is said to be an (category) equivalence if there exists a functor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ such that $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$. The categories $\mathcal{C}$ and $\mathcal{D}$ are said to be equivalent, written $\mathcal{C} \simeq \mathcal{D}$, if a category equivalence $\mathcal{C} \xrightarrow{F} \mathcal{D}$ exists.
- Given a functor $I \xrightarrow{X} \mathcal{C}$, we sometimes denote the image of a morphism $i \xrightarrow{\theta} j$ in I by $X_i \xrightarrow{X_\theta} X_j$. This applies in particular if $I = \mathbf{\Delta}^{\mathrm{op}}$ or $I = \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$.
- In certain standard categories like **Set**, **Grp**, **Top**, etc., we also use the common notation for the set of morphisms between two objects, for example, we write $\operatorname{Map}(X, Y)$ for the set of maps between sets X and Y, we write $\operatorname{Hom}(G, H)$ for the set of group homomorphisms between groups G and H, and we write $\mathrm{C}(T, U)$ for the set of continuous maps between topological spaces T and U.
- The category associated to a poset P is denoted by $\mathtt{Cat}(P)$. Similarly, given a group G, we write $\mathtt{Cat}(G)$ for the associated category with one object.
- Products of objects X_1 and X_2 in arbitrary categories are denoted as $X_1 \ \mathbf{\Pi}\ X_2$. Pullbacks of morphisms $X_1 \longrightarrow Y$, $X_2 \longrightarrow Y$ are denoted as $X_1 \ {}_{\varphi_1}\mathbf{\Pi}_{\varphi_2}\ X_2 = X_1 \ {}_{\varphi_1}\mathbf{\Pi}^{Y}_{\varphi_2}\ X_2$. The diagonal morphism is written $X \xrightarrow{\Delta} X \ \mathbf{\Pi}\ X$.
- Given an index set I and a family of groups $(G_i)_{i \in I}$, we denote the direct product by $\bigtimes_{i \in I} G_i$. Similarly for morphisms.
- Projections are denoted as pr, embeddings as emb.
- A subobject B of an object A in an abelian category is denoted as $B \preceq A$.
- Given an additive category $\mathcal{A}$, the additive category of complexes resp. double complexes in $\mathcal{A}$ is denoted by $\mathbf{C}(\mathcal{A})$ resp. $\mathbf{C}^2(\mathcal{A})$. The full subcategory of $\mathbf{C}^2(\mathcal{A})$ with objects C such that $C_{p,q} \cong 0$ for $p < 0$ or $q < 0$ is denoted by $\mathbf{C}^2_{\urcorner}(\mathcal{A})$
- If we have a complex C in an additive category $\mathcal{A}$ such that $C_n \cong 0$ for $n < 0$, we usually omit to denote these zero objects. Similarly for morphisms, complex homotopies, etc. and for the dual situation if $C^n = C_{-n} \cong 0$ for $n < 0$.

- In any complex C with differentials ∂, we write $\mathrm{Z}_n C := \mathrm{Ker}(C_n \xrightarrow{\partial} C_{n-1})$ and $\mathrm{B}_n C := \mathrm{Im}(C_{n+1} \xrightarrow{\partial} C_n)$.
- Homotopy equivalence of complexes C and D in an additive category $\mathcal{A}$ is denoted by $C \simeq D$.
- We use the notations $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- The Kronecker delta is defined by

  $$\delta_{x,y} = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y, \end{cases}$$

  where x and y are elements of some set.
- Given a map $f\colon X \to Y$ and subsets $X' \subseteq X$, $Y' \subseteq Y$ with $X'f \subseteq Y'$, we let $f|_{X'}^{Y'}$ the map $X' \to Y', x' \mapsto x'f$. In the special cases, where $Y' = Y$ resp. $X' = X$, we also write $f|_{X'} := f|_{X'}^{Y}$ resp. $f|^{Y'} := f|_{X}^{Y'}$.
- Given integers $a, b, c \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ for the set of integers lying between a and b. Furthermore, we write $[a, b] \wedge c := [a, b] \setminus \{c\}$ to omit elements in the interval.

  Sometimes, we need some specified orientation, then we write $\lceil a, b \rceil := (z \in \mathbb{Z} \mid a \leq z \leq b)$ for the *ascending interval* and $\lfloor a, b \rfloor = (z \in \mathbb{Z} \mid a \geq z \geq b)$ for the *descending interval.* Likewise $\lceil a, b \rceil \wedge c$, etc. Whereas we formally deal with tuples, we use the element notation, for example we write

  $$\prod_{i \in \lceil 1,3 \rceil} g_i = g_1 g_2 g_3 \text{ and } \prod_{i \in \lfloor 3,1 \rfloor} g_i = g_3 g_2 g_1$$

  or

  $$(g_i)_{i \in \lfloor 3,1 \rfloor} = (g_3, g_2, g_1)$$

  for group elements g_1, g_2, g_3.
- If we have tuples $(x_j)_{j \in A}$ and $(x_j)_{j \in B}$ with disjoint index sets A and B, then we write $(x_j)_{j \in A} \cup (x_j)_{j \in B}$ for their concatenation.
- A composite of zero morphisms is stipulated to be an identity. For instance, $f_1 \dots f_k = \mathrm{id}$ if $k = 0$.

Chapter I

Simplicial objects

In this chapter, we recall the standard facts about simplicial sets or, more generally, simplicial objects in an arbitrary category. For further information, the reader is referred for example to [17], [23], [26], [29, §8].

§1 The category of simplex types

Before we can introduce simplicial sets, we have to study the following category.

(1.1) Definition (category of simplex types)**.**

(a) For $n \in \mathbb{N}_0$ we let $[n] := \texttt{Cat}([0,n])$ be the category with objects $[0,n]$ and exactly one morphism $i \longrightarrow j$ for $i, j \in [0,n]$ if and only if $i \leq j$.

(b) The full subcategory $\mathbf{\Delta}$ in **Cat** with objects $\operatorname{Ob} \mathbf{\Delta} := \{[n] \mid n \in \mathbb{N}_0\}$ is called the *category of simplex types.*

Hence, if we disregard the category aspect of an object $[n]$, the category $\mathbf{\Delta}$ consists of linearly ordered sets $[n]$ as objects and monotonically increasing maps as morphisms.

(1.2) Example (embedding of $\mathbf{\Delta}$ in **Top**)**.** For every $n \in \mathbb{N}_0$ we define the *topological standard n-simplex* $|\Delta^n|$ to be

$$|\Delta^n| := \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{j \in [0,n]} x_j = 1 \text{ and } x_j \geq 0 \text{ for all } j \in [0,n] \right\},$$

equipped with the relative topology. We consider for any morphism $[m] \xrightarrow{\theta} [n]$ the induced map $\theta_* \colon |\Delta^m| \to |\Delta^n|$ defined by

$$(x_i)_{i \in [0,m]} \theta_* := (\sum_{\substack{i \in [0,m] \\ i\theta = j}} x_i)_{j \in [0,n]} \text{ for all } x = (x_i)_{i \in [0,m]} \in |\Delta^m|.$$

Since $|\Delta^m|$ and $|\Delta^n|$ carry the relative topologies of $\mathbb{R}^{m+1}$ resp. $\mathbb{R}^{n+1}$, the map θ_* is continuous. If we have morphisms $[m] \xrightarrow{\theta} [n]$ and $[n] \xrightarrow{\rho} [p]$ in $\mathbf{\Delta}$, this yields

$$(x_i)_{i \in [0,m]} \theta_* \rho_* = (\sum_{\substack{i \in [0,m] \\ i\theta = j}} x_i)_{j \in [0,n]} \, \rho_* = (\sum_{\substack{j \in [0,n] \\ j\rho = k}} \sum_{\substack{i \in [0,m] \\ i\theta = j}} x_i)_{k \in [0,p]} = (\sum_{\substack{i \in [0,m] \\ i(\theta\rho) = k}} x_i)_{k \in [0,p]} = (x_i)_{i \in [0,m]} (\theta\rho)_*$$

and

$$(x_i)_{i \in [0,m]} (\mathrm{id}_{[m]})_* = (\sum_{\substack{i \in [0,m] \\ i\mathrm{id}_{[m]} = j}} x_i)_{j \in [0,m]} = (x_j)_{j \in [0,m]} = (x_i)_{i \in [0,m]} \mathrm{id}_{|\Delta^m|}$$

for all $(x_i)_{i\in[0,m]} \in |\Delta^m|$. Hence

$$\mathbf{\Delta} \xrightarrow{|\Delta^-|} \mathbf{Top}, ([m] \xrightarrow{\theta} [n]) \mapsto (|\Delta^m| \xrightarrow{\theta_*} |\Delta^n|)$$

is well defined as a functor from the category of simplex types $\mathbf{\Delta}$ to the category **Top** of topological spaces. In order to prove that $|\Delta^-|$ is faithful, we let $[m] \xrightarrow{\theta} [n]$ be an arbitrary morphism in $\mathbf{\Delta}$. Further we let $\{e_i \mid i \in [0,m]\}$ resp. $\{e_j \mid j \in [0,n]\}$ denote the standard basis of $\mathbb{R}^{m+1}$ resp. $\mathbb{R}^{n+1}$. Then we get

$$e_i\theta_* = (\sum_{\substack{i'\in[0,m]\\ i'\theta=j}} (e_i)_{i'})_{j\in[0,n]} = (\sum_{\substack{i'\in[0,m]\\ i'\theta=j}} \delta_{i,i'})_{j\in[0,n]} = (\delta_{i\theta,j})_{j\in[0,n]} = e_{i\theta}$$

for every $i \in [0,m]$. Thus if we have morphisms $[m] \xrightarrow{\theta} [n]$ and $[m] \xrightarrow{\rho} [n]$ in $\mathbf{\Delta}$ with $\theta_* = \rho_*$, then in particular we have $e_{i\theta} = e_i\theta_* = e_i\rho_* = e_{i\rho}$ and therefore $i\theta = i\rho$ for all $i \in [0,m]$. Hence $\theta = \rho$, and consequently $|\Delta^-|$ is a faithful functor.

We aim to distinguish generators for $\mathbf{\Delta}$, which we will define now.

(1.3) Definition (cofaces and codegeneracies)**.**

(a) For $n \in \mathbb{N}$, $k \in [0,n]$, the morphism $[n-1] \xrightarrow{\delta^k} [n]$ defined by

$$i\delta^k := \begin{cases} i & \text{for } i \in [0,k-1], \\ i+1 & \text{for } i \in [k,n-1] \end{cases}$$

is called the *k-th coface* of $[n]$.

(b) For $n \in \mathbb{N}_0$, $k \in [0,n]$, the morphism $[n+1] \xrightarrow{\sigma^k} [n]$ defined by

$$i\sigma^k := \begin{cases} i & \text{for } i \in [0,k], \\ i-1 & \text{for } i \in [k+1,n+1] \end{cases}$$

is called the *k-th codegeneracy* of $[n]$.

(1.4) Proposition (cosimplicial identities)**.** We let $n \in \mathbb{N}$ be a natural number. For the cofaces and codegeneracies the following identities hold:

$$\delta^k\delta^l = \delta^{l-1}\delta^k \text{ for } 0 \le k < l \le n+1 \text{ as morphisms } [n-1] \longrightarrow [n+1],$$
$$\sigma^k\sigma^l = \sigma^{l+1}\sigma^k \text{ for } 0 \le k \le l \le n-1 \text{ as morphisms } [n+1] \longrightarrow [n-1],$$
$$\delta^k\sigma^l = \begin{Bmatrix} \sigma^{l-1}\delta^k & \text{for } k < l, \\ \mathrm{id}_{[n-1]} & \text{for } l \le k \le l+1, \\ \sigma^l\delta^{k-1} & \text{for } k > l+1 \end{Bmatrix} \text{ as morphisms } [n-1] \longrightarrow [n-1], \text{ where } k \in [0,n], l \in [0,n-1].$$

Proof.

(a) If $k < l$, then

$$i\delta^k\delta^l = \begin{Bmatrix} i\delta^l & \text{for } i \in [0,k-1], \\ (i+1)\delta^l & \text{for } i \in [k,n-1] \end{Bmatrix} = \begin{Bmatrix} i & \text{for } i \in [0,k-1], \\ i+1 & \text{for } i \in [k,l-2], \\ i+2 & \text{for } i \in [l-1,n-1] \end{Bmatrix}$$
$$= \begin{Bmatrix} i\delta^k & \text{for } i \in [0,l-2], \\ (i+1)\delta^k & \text{for } i \in [l-1,n-1] \end{Bmatrix} = i\delta^{l-1}\delta^k$$

for all $i \in [0,n-1]$.

(b) For $k \leq l$ we calculate

$$i\sigma^k\sigma^l = \left.\begin{cases} i\sigma^l & \text{for } i \in [0,k], \\ (i-1)\sigma^l & \text{for } i \in [k+1,n+1] \end{cases}\right\} = \left\{\begin{matrix} i & \text{for } i \in [0,k], \\ i-1 & \text{for } i \in [k+1,l+1], \\ i-2 & \text{for } i \in [l+2,n+1] \end{matrix}\right\}$$
$$= \left.\begin{cases} i\sigma^k & \text{for } i \in [0,l+1], \\ (i-1)\sigma^k & \text{for } i \in [l+2,n+1] \end{cases}\right\} = i\sigma^{l+1}\sigma^k$$

for every $i \in [0, n+1]$.

(c) Furthermore:

$$i\delta^k\sigma^l = \left.\begin{cases} i\sigma^l & \text{for } i \in [0,k-1], \\ (i+1)\sigma^l & \text{for } i \in [k,n-1] \end{cases}\right\} = \left\{\begin{matrix} i & \text{for } i \in [0,k-1], k \leq l+1, \\ i+1 & \text{for } i \in [k,l-1], k \leq l+1, \\ i & \text{for } i \in [l,n-1], k \leq l+1, \\ i & \text{for } i \in [0,l], k > l+1, \\ i-1 & \text{for } i \in [l+1,k-1], k > l+1, \\ i & \text{for } i \in [k,n-1], k > l+1 \end{matrix}\right\}$$
$$= \left.\begin{cases} i\delta^k & \text{for } i \in [0,l-1], k < l, \\ (i-1)\delta^k & \text{for } i \in [l,n-1], k < l, \\ i & \text{for } l \leq k \leq l+1, \\ i\delta^{k-1} & \text{for } i \in [0,l], k > l+1, \\ (i-1)\delta^{k-1} & \text{for } i \in [l+1,n-1], k > l+1 \end{cases}\right\} = \begin{cases} i\sigma^{l-1}\delta^k & \text{for } k < l, \\ i & \text{for } l \leq k \leq l+1, \\ i\sigma^l\delta^{k-1} & \text{for } k > l+1. \end{cases}$$

□

Our next aim is to show that, in some sense, the cofaces and codegeneracies generate the category of simplex types $\boldsymbol{\Delta}$ and that the cosimplicial identities of the preceding proposition yield a set of relations defining $\boldsymbol{\Delta}$.

(1.5) Notation. Given $m, n \in \mathbb{N}_0$ and $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u \leq n$ for some $t, u \in \mathbb{N}_0$, we write

$$\sigma^{m_{\lfloor t,1 \rfloor}} := \sigma^{m_t} \cdots \sigma^{m_1} \text{ as morphism } [m] \longrightarrow [m-t]$$

and

$$\delta^{n_{\lceil 1,u \rceil}} := \delta^{n_1} \cdots \delta^{n_u} \text{ as morphism } [n-u] \longrightarrow [n].$$

(1.6) Remark. We let $[m] \xrightarrow{\theta} [n]$ in $\boldsymbol{\Delta}$ be defined by

$$\theta := \sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u \rceil}},$$

where $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u \leq n$, and where $t, u \in \mathbb{N}_0$ such that $m - t = n - u$. Furthermore, we let $k \in [0,t]$ and $l \in [0,u]$ be the unique elements such that $i \in [m_k + 1, m_{k+1}]$ and $i\theta \in [n_l, n_{l+1} - 1]$, where $m_0 := -1$, $m_{t+1} := m$, $n_0 := 0$ and $n_{u+1} := n + 1$. Then we have

$$i\theta = i - k + l \text{ for every } i \in [0,m].$$

Proof. By induction on t, the case $t = 0$ being trivial, we have

$$i\sigma^{m_{\lfloor t,1 \rfloor}} = \left.\begin{cases} i\sigma^{m_{\lfloor t-1,1 \rfloor}} & \text{for } i \in [0,m_t], \text{i.e. } k \in [0,t-1], \\ (i-1)\sigma^{m_{\lfloor t-1,1 \rfloor}} & \text{for } i \in [m_t+1,m], \text{i.e. } k = t \end{cases}\right\}$$
$$= \left.\begin{cases} i-k & \text{for } k \in [0,t-1], \\ (i-1)-(t-1) & \text{for } k = t \end{cases}\right\} = i - k$$

for all $i \in [0,m]$ with $i \in [m_k+1, m_{k+1}]$, $k \in [0,t]$. Furthermore, by induction on u, the case $u = 0$ being trivial, we have

$$\begin{aligned} i\sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u \rceil}} &= \left\{ \begin{array}{ll} i\sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u-1 \rceil}} & \text{for } i\sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u-1 \rceil}} \in [0, n_u - 1], \text{i.e. } l \in [0, u-1], \\ i\sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u-1 \rceil}} + 1 & \text{for } i\sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u-1 \rceil}} \in [n_u, n-1], \text{i.e. } l = u \end{array} \right\} \\ &= \left\{ \begin{array}{ll} i\sigma^{m_{\lfloor t,1 \rfloor}} + l & \text{for } l \in [0, u-1], \\ i\sigma^{m_{\lfloor t,1 \rfloor}} + (u-1) + 1 & \text{for } l = u \end{array} \right\} = i\sigma^{m_{\lfloor t,1 \rfloor}} + l \end{aligned}$$

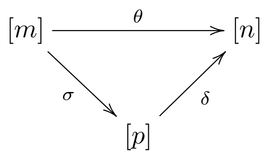
for all $i \in [0,m]$. Finally, we get $i\theta = i\sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u \rceil}} = (i-k) + l$. □

(1.7) Theorem. Every morphism $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$ can uniquely be written as

$$\theta = \sigma^{m_{\lfloor t,1 \rfloor}}\delta^{n_{\lceil 1,u \rceil}},$$

where $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u \leq n$, and where $t, u \in \mathbb{N}_0$.

Proof. We begin by showing the existence of a factorisation. We let $m_1 < \cdots < m_t$ be the elements of $[0,m]$ such that $m_k\theta = (m_k+1)\theta$ for every $k \in [1,t]$ and we let $n_1 < \cdots < n_u$ be the elements of $[0,n]$, that do not lie in $[0,m]\theta$. Setting $p := m - t = n - u$ as well as $\sigma := \sigma^{m_{\lfloor t,1 \rfloor}}$ and $\delta := \delta^{n_{\lceil 1,u \rceil}}$, we have to show that we get the factorisation $\theta = \sigma\delta$.



Thereto we proceed by induction on $i \in [0,m]$.
If $i = 0$ and $l := 0\theta \in [0,n]$, then, due to the monotony of θ, we have $[0, l-1] \cap [0,m]\theta = \emptyset$. Since $i \in [0, m_1]$, remark (1.6) yields $0\sigma\delta = 0 - 0 + l = l = 0\theta$.
If $i \in [1,m]$, we choose $k \in [0,t]$ and $l \in [0,u]$ such that $i \in [m_k+1, m_{k+1}]$ and $i\theta \in [n_l, n_{l+1}-1]$. We distinguish the following two cases: If $i\theta = (i-1)\theta$, then by the choice of $m_1, \ldots, m_t$ we get $i - 1 = m_k \in [m_{k-1}+1, m_k]$. Using the induction hypothesis and remark (1.6), this yields

$$i\theta = (i-1)\theta = (i-1)\sigma\delta = ((i-1) - (k-1))\delta = (i-k)\delta = i\sigma\delta.$$

Otherwise, $i\theta > (i-1)\theta$ and $i - 1 \in [m_k+1, m_{k+1}]$. We let $l' \in [0,u]$ be such that $(i-1)\theta \in [n_{l'}, n_{l'+1} - 1]$. If $l' = l$, then, by the induction hypothesis and remark (1.6),

$$i\theta = (i-1)\theta + 1 = (i-1)\sigma\delta + 1 = ((i-1) - k + l) + 1 = i - k + l = i\sigma\delta.$$

If $l' < l$, we must have $(i-1)\theta = n_{l'+1} - 1$ and $i\theta = n_l + 1$. Further we have $n_l - n_{l'+1} = l - (l'+1) = l - l' - 1$ since $[n_{l'+1}, n_l] \subseteq [0,n] \setminus ([0,m]\theta)$. By induction hypothesis and remark (1.6), we obtain

$$\begin{aligned} i\theta &= (i-1)\theta + (i\theta - (i-1)\theta) = (i-1)\sigma\delta + ((n_l + 1) - (n_{l'+1} - 1)) \\ &= ((i-1) - k + l') + (n_l - n_{l'+1} + 2) = i - 1 - k + l' + l - l' - 1 + 2 = i - k + l = i\sigma\delta. \end{aligned}$$

Thus we have $\theta = \sigma\delta$.
Now, we show the uniqueness of the factorisation. We suppose $\theta = \sigma\delta$ with $\sigma = \sigma^{m_{\lfloor t,1 \rfloor}}$ and $\delta = \delta^{n_{\lceil 1,u \rceil}}$, where $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u \leq n$ with $t, u \in \mathbb{N}_0$.
We claim that $m_1, \ldots, m_t$ consists of exactly those elements $i \in [0,m]$ with $i\theta = (i+1)\theta$. To this end, we let $k \in [1,t]$ be such that $i \in [m_{k-1}+1, m_k]$. Then the injectivity of δ yields the equivalence of $(i+1)\theta = i\theta$ and $(i+1)\sigma = i\sigma$. But, by remark (1.6), this is equivalent to $(i+1)\sigma = i - (k-1) = (i+1) - k$, that is, to $i + 1 \in [m_k+1, m_{k+1}]$. Since $i \in [m_{k-1}+1, m_k]$, saying $i + 1 \in [m_k+1, m_{k+1}]$ is the same as saying $i = m_k$. This proves the claim.
Further, surjectivity of σ and $\delta = \delta^{n_1} \cdots \delta^{n_u}$ show that $[0,n] \setminus ([0,m]\theta) = \{n_1, \ldots, n_u\}$. Therefore, the morphism θ determines the numbers $m_1, \ldots, m_t$ and $n_1, \ldots, n_u$. This shows the uniqueness of the representation. □

§2 Simplicial objects in arbitrary categories

(1.8) Definition (simplicial objects and their morphisms)**.**

(a) We let $\mathcal{C}$ be an arbitrary category. The *category of simplicial objects* in $\mathcal{C}$ is defined to be the functor category

$$\mathbf{s}\mathcal{C} := (\boldsymbol{\Delta}^{\text{op}}, \mathcal{C}).$$

An object in $\mathbf{s}\mathcal{C}$ is called a *simplicial object* in $\mathcal{C}$, a morphism in $\mathbf{s}\mathcal{C}$ is called *morphism of simplicial objects* in $\mathcal{C}$ or a *simplicial morphism* in $\mathcal{C}$.

  (i) A simplicial object in **Set** is called a *simplicial set*, a morphism is called a *simplicial map*.

  (ii) A simplicial object in **Grp** is called a *simplicial group*, a morphism is called a *simplicial group homomorphism*.

  (iii) A simplicial object in **AbGrp** is called a *simplicial abelian group*, a morphism is called a *simplicial homomorphism of abelian groups*.

  (iv) We let R be a ring. A simplicial object in R-**Mod** is called a *simplicial R-module*, a morphism is called a *simplicial R-module homomorphism*.

  (v) A simplicial object in **Top** is called a *simplicial topological space*, a morphism is called a *simplicial continuous map*.

(b) Dually, we define for every category $\mathcal{C}$ the *category of cosimplicial objects* in $\mathcal{C}$ by

$$\mathbf{cs}\mathcal{C} := (\boldsymbol{\Delta}, \mathcal{C}).$$

(1.9) Example (constant simplicial object)**.** We let $\mathcal{C}$ be a category and $X \in \operatorname{Ob}\mathcal{C}$ an object in $\mathcal{C}$. Then the constant functor

$$\boldsymbol{\Delta}^{\text{op}} \xrightarrow{\text{Const}\, X} \mathcal{C}$$

with $(\text{Const}\, X)_{[n]} = X$ for $n \in \mathbb{N}_0$ and $(\text{Const}\, X)_\theta = \text{id}_X$ for $\theta \in \operatorname{Mor}\boldsymbol{\Delta}$ is a simplicial object in $\mathcal{C}$, the *constant simplicial object*.
This yields a functor $\mathcal{C} \xrightarrow{\text{Const}} \mathbf{s}\mathcal{C}$ by letting $(\text{Const}\, f)_{[n]} := f$ for $n \in \mathbb{N}_0$, $f \in {}_{\mathcal{C}}(X, Y)$, $X, Y \in \operatorname{Ob}\mathcal{C}$.

(1.10) Example (singular simplicial set)**.**

(a) We let $n \in \mathbb{N}$ be a natural number. Concerning example (1.2), the topological standard simplex functor $|\Delta^-|$ is a (covariant) functor $\boldsymbol{\Delta} \longrightarrow \mathbf{Top}$, that is, a cosimplicial topological space.

(b) For an arbitrary topological space T, we let $\boldsymbol{\Delta}^{\text{op}} \xrightarrow{\text{S}T} \mathbf{Set}$ be the contravariant functor given by

$$\text{S}T := \text{C}(|\Delta^-|, T).$$

This is a simplicial set, which is called the *singular simplicial set* to the topological space T. In fact, we have a functor

$$\mathbf{Top} \xrightarrow{\text{S}} \mathbf{sSet}$$

given by $\text{S}(=) = \text{C}(|\Delta^-|, =)$.

(1.11) Example. For any commutative ring R we let $\mathbf{Set} \xrightarrow{R-} R\text{-}\mathbf{Mod}$ be the functor that assigns to every set M the free R-left-module RM on the set M and to every map $f\colon M \to N$ for sets M and N the R-module homomorphism $Rf\colon RM \to RN$, which is defined by the operation of f on the basis M. Since **sSet** and $\mathbf{s}R\text{-}\mathbf{Mod}$ are functor categories, this functor $R-$ lifts to a functor $\mathbf{sSet} \xrightarrow{R-} \mathbf{s}R\text{-}\mathbf{Mod}$. If we have an arbitrary simplicial set X, then RX is per definitionem the simplicial R-module with $(RX)_{[n]} = RX_{[n]}$, that is, the set of n-simplices $(RX)_{[n]}$ of RX is a free R-module on the set $X_{[n]}$.

(1.12) Definition (faces and degeneracies)**.** For a simplicial object X in a category $\mathcal{C}$, we define morphisms

$$X_{[n]} \xrightarrow{\mathrm{d}_k} X_{[n-1]}$$

by $\mathrm{d}_k := \mathrm{d}_k^X := X_{\delta^k}$ for $k \in [0, n]$, $n \in \mathbb{N}$, called *faces*, and morphisms

$$X_{[n]} \xrightarrow{\mathrm{s}_k} X_{[n+1]}$$

by $\mathrm{s}_k := \mathrm{s}_k^X := X_{\sigma^k}$ for $k \in [0, n]$, $n \in \mathbb{N}_0$, called *degeneracies*.

(1.13) Notation. We let X be a simplicial object in a category $\mathcal{C}$. Given $m, n \in \mathbb{N}_0$ and $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u \leq n$ for some $t, u \in \mathbb{N}_0$, we write

$$\mathrm{s}_{m_{\lceil 1,t \rceil}} := \mathrm{s}_{m_1} \cdots \mathrm{s}_{m_t} \text{ as morphism } X_{[m-t]} \longrightarrow X_{[m]}$$

and

$$\mathrm{d}_{n_{\lfloor u,1 \rfloor}} := \mathrm{d}_{n_u} \cdots \mathrm{d}_{n_1} \text{ as morphism } X_{[n]} \longrightarrow X_{[n-u]}.$$

Furthermore, we use the interval notations

$$\mathrm{s}_{\lceil k-t+1,k \rceil} := \mathrm{s}_{k-t+1} \cdots \mathrm{s}_k \text{ as morphism } X_{m-t} \longrightarrow X_m$$

and

$$\mathrm{d}_{\lfloor l,l-u+1 \rfloor} = \mathrm{d}_l \cdots \mathrm{d}_{l-u+1} \text{ as morphisms } X_n \longrightarrow X_{n-u}$$

for $k \in [t-1, m-1]$, $l \in [u-1, n]$, $t \in [0, m-1]$, $u \in [0, n]$.

(1.14) Proposition (simplicial identities)**.** We let X be a simplicial object in a category $\mathcal{C}$. The faces and degeneracies satisfy the following identities:

$$\mathrm{d}_l \mathrm{d}_k = \mathrm{d}_k \mathrm{d}_{l-1} \text{ for } 0 \leq k < l \leq n+1 \text{ as morphisms } X_{[n+1]} \longrightarrow X_{[n-1]},$$
$$\mathrm{s}_l \mathrm{s}_k = \mathrm{s}_k \mathrm{s}_{l+1} \text{ for } 0 \leq k \leq l \leq n \text{ as morphisms } X_{[n+1]} \longrightarrow X_{[n-1]},$$
$$\mathrm{s}_l \mathrm{d}_k = \left\{ \begin{array}{ll} \mathrm{d}_k \mathrm{s}_{l-1}, & \text{for } k < l, \\ \mathrm{id}_{X_{[n-1]}}, & \text{for } l \leq k \leq l+1, \\ \mathrm{d}_{k-1} \mathrm{s}_l, & \text{for } k > l+1 \end{array} \right\} \text{ as morphisms } X_{[n-1]} \longrightarrow X_{[n-1]}, \text{ where } k \in [0, n], l \in [0, n-1].$$

In particular, every face is a retraction and every degeneracy is a coretraction.

Proof. The required identities result from proposition (1.4). □

The identities in the previous proposition are even characterising a simplicial object, as we will see now.

(1.15) Theorem (classical definition of a simplicial object)**.** We let $(X_n)_{n \in \mathbb{N}_0}$ be a sequence of objects in a category $\mathcal{C}$ and we suppose given morphisms

$$X_n \xrightarrow{d_k} X_{n-1} \text{ for } k \in [0, n], n \in \mathbb{N},$$

and

$$X_n \xrightarrow{s_k} X_{n+1} \text{ für } k \in [0, n], n \in \mathbb{N}_0,$$

which satisfy the simplicial identities

$$d_l d_k = d_k d_{l-1} \text{ for } 0 \leq k < l \leq n+1 \text{ as morphisms } X_{[n+1]} \longrightarrow X_{[n-1]},$$
$$s_l s_k = s_k s_{l+1} \text{ for } 0 \leq k \leq l \leq n \text{ as morphisms } X_{[n+1]} \longrightarrow X_{[n-1]},$$
$$s_l d_k = \left\{ \begin{array}{ll} d_k s_{l-1}, & \text{for } k < l, \\ \mathrm{id}_{X_{[n-1]}}, & \text{for } l \leq k \leq l+1, \\ d_{k-1} s_l, & \text{for } k > l+1 \end{array} \right\} \text{ as morphisms } X_{[n-1]} \longrightarrow X_{[n-1]}, \text{ where } k \in [0, n], l \in [0, n-1].$$

Then there exists a simplicial object X in $\mathcal{C}$ with $X_{[n]} = X_n$ for all $n \in \mathbb{N}_0$ and $\mathrm{d}_k^X = d_k$ for $k \in [0, n]$, $n \in \mathbb{N}$, as well as $\mathrm{s}_k^X = s_k$ for $k \in [0, n]$, $n \in \mathbb{N}_0$.

Proof. We define X on the objects of $\mathbf{\Delta}$ by $X_{[n]} := X_n$ for $n \in \mathbb{N}_0$. On the morphisms in the category of simplex types $\mathbf{\Delta}$, we define X as follows: Given a morphism $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$, then according to theorem (1.7), there is a unique representation of θ as a composite of codegeneracies and cofaces, $\theta = \sigma^{m_{\lfloor t,1 \rfloor}} \delta^{n_{\lceil 1,u \rceil}}$ with $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u \leq n$. We let $X_\theta := d_{n_{\lfloor u,1 \rfloor}} s_{m_{\lceil 1,t \rceil}} := d_{n_u} \cdots d_{n_1} s_{m_1} \cdots s_{m_t}$. In particular, we have $X_{\delta^k} = d_k$ for $k \in [0,n]$, $n \in \mathbb{N}$, and $X_{\sigma^k} = s_k$ for $k \in [0,n]$, $n \in \mathbb{N}_0$. Since the morphisms d_k for $k \in [0,n]$, $n \in \mathbb{N}$, and s_k for $k \in [0,n]$, $n \in \mathbb{N}_0$, satisfy the simplicial identities, while the corresponding cofaces and codegeneracies in $\mathbf{\Delta}$ satisfy the cosimplicial identities, X is compatible with the composition of morphisms and therefore a well defined functor

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathcal{C},$$

that is, a simplicial object in $\mathcal{C}$. □

(1.16) Proposition (classical definition of a simplicial morphism)**.** We let X and Y be simplicial objects in a category $\mathcal{C}$ and we suppose given morphisms $X_n \xrightarrow{f_n} Y_n$ for $n \in \mathbb{N}_0$. If these morphisms commute with the faces and degeneracies of X and Y, that is, if

$$f_n \mathrm{d}_k = \mathrm{d}_k f_{n-1} \text{ for } k \in [0,n], n \in \mathbb{N},$$

and

$$f_n \mathrm{s}_k = \mathrm{s}_k f_{n+1} \text{ for } k \in [0,n], n \in \mathbb{N}_0,$$

then there exists a simplicial morphism $X \xrightarrow{f} Y$ with $f_{[n]} = f_n$ for all $n \in \mathbb{N}_0$.

Proof. Follows from theorem (1.7). □

At the end of this section, we want to fix some notions.

(1.17) Definition (n-simplices)**.** We let X, Y be simplicial objects in a category $\mathcal{C}$ and $X \xrightarrow{f} Y$ a simplicial morphism. We set $X_n := X_{[n]}$ and $f_n := f_{[n]}$ for all $n \in \mathbb{N}_0$. If X_n is a set or has a set as underlying structure, then the elements of X_n are called *n-simplices*. The 0-simplices are also called *vertices* and the 1-simplices are also called *edges* of X. The n-simplices of the form $x_{n-1}\mathrm{s}_k$ for $x_{n-1} \in X_{n-1}$, $k \in [0, n-1]$, $n \in \mathbb{N}$, are said to be *degenerate*.

(1.18) Definition (reduced simplicial set)**.** A simplicial set X is called *reduced*, if it has exactly one vertex, i.e. if $|X_0| = 1$. The full subcategory of reduced simplicial sets in $\mathbf{sSet}$ is denoted by $\mathbf{sSet}_0$.

(1.19) Definition (cartesian product of simplicial sets)**.**

(a) Given simplicial sets X, Y, we define their (*cartesian*) *product* $X \times Y$ by $(X \times Y)_n := X_n \times Y_n$ for all $n \in \mathbb{N}_0$ and $(X \times Y)_\theta := X_\theta \times Y_\theta$ for all morphisms $[m] \xrightarrow{\theta} [n]$ in the category of simplex types $\mathbf{\Delta}$.

(b) Given simplicial sets X, Y, X', Y' and simplicial maps $X \xrightarrow{f} X'$, $Y \xrightarrow{g} Y'$, the simplicial map $X \times Y \xrightarrow{f \times g} X' \times Y'$ is defined by $(f \times g)_n := f_n \times g_n$ for every $n \in \mathbb{N}_0$.

§3 The standard n-simplex

We consider a standard example of a family of simplicial sets which will be needed later.

(1.20) Definition (standard n-simplex)**.** We let $n \in \mathbb{N}_0$ be a non-negative integer. The *standard n-simplex* Δ^n in the category $\mathbf{sSet}$ is defined by

$$\Delta^n := {}_{\mathbf{\Delta}}(\bullet, [n]),$$

that is, Δ^n is the functor $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ represented by $[n]$. For the set of m-simplices of Δ^n, we write $\Delta^n_m := (\Delta^n)_m = {}_{\mathbf{\Delta}}([m],[n])$.

(1.21) Lemma. The standard n-simplices Δ^n with $n \in \mathbb{N}_0$ form a cosimplicial object Δ^- in the category of simplicial sets $\mathbf{sSet}$, that is, $\Delta^- \in \operatorname{Ob} \mathbf{cs}(\mathbf{sSet})$.

Proof. The homfunctor ${}_{\mathbf{\Delta}}(\bullet, -) = \Delta^{-}$ is a functor with two arguments, contravariant in the first argument and covariant in the second one. □

(1.22) Lemma. We let X be a simplicial set and $n \in \mathbb{N}_0$. Then we have a bijective correspondence between the n-simplices in X and the simplicial maps $\Delta^n \longrightarrow X$. It is given by

$$X_n \to {}_{\mathbf{sSet}}(\Delta^n, X), x_n \mapsto (\theta \mapsto x_n X_\theta \text{ for every } \theta \in \Delta^n_m)$$

with inverse

$${}_{\mathbf{sSet}}(\Delta^n, X) \to X_n, f \mapsto (\mathrm{id}_{[n]})f_n.$$

Proof. This is a consequence from the Yoneda lemma. □

The next corollary justifies the name *category of simplex types* for the category $\mathbf{\Delta}$.

(1.23) Corollary. For all non-negative integers $m, n \in \mathbb{N}_0$, we have

$${}_{\mathbf{sSet}}(\Delta^m, \Delta^n) \cong {}_{\mathbf{\Delta}}([m], [n]).$$

Proof. Lemma (1.22) implies ${}_{\mathbf{sSet}}(\Delta^m, \Delta^n) \cong \Delta^n_m = {}_{\mathbf{\Delta}}([m], [n])$ for all $m, n \in \mathbb{N}_0$. □

§4 The nerve

In this section, we study an example of a simplicial set, which is going to be the most important one for our purposes - the nerve $\mathrm{N}\mathcal{C}$ of a given category $\mathcal{C}$. Intuitively explained, the nerve $\mathrm{N}\mathcal{C}$ of a given category $\mathcal{C}$ has the objects of $\mathcal{C}$ as vertices, while the morphisms are the edges. Furthermore, the 2-simplices are exactly the pairs of composable morphisms, the 3-simplices are the triples, and so on. The kth face is given by "deleting" the object number k, that is, the two morphisms that end resp. start with this object are composed, and the kth degeneracy inserts an identity morphism for the object number k.
Since any group can be regarded as a category with a single object, we also obtain a nerve functor for groups. We need the notion of a nerve for a category object and a group object in an arbitrary category (under certain technical conditions).
Throughout this section, we assume given a category $\mathcal{C}$, in which pullbacks and a terminal object exist. A terminal object in $\mathcal{C}$ is denoted by T and the unique morphism from an object $X \in \operatorname{Ob}\mathcal{C}$ is written as $X \xrightarrow{*} T$.

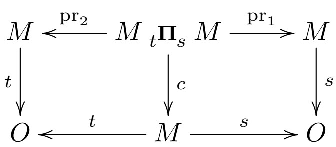
Examples of algebraic structures within arbitrary categories

An introduction to category objects and group objects can be found in [16].

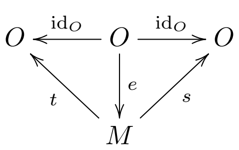
(1.24) Definition (category objects and functors)**.**

(a) A *category object* (or *internal category*) in $\mathcal{C}$ consists of objects $O, M \in \operatorname{Ob}\mathcal{C}$ and morphisms $M \xrightarrow{s} O$, $M \xrightarrow{t} O$, $O \xrightarrow{e} M$ and $M \,{}_t\mathbf{\Pi}_s\, M \xrightarrow{c} M$, where $M \,{}_t\mathbf{\Pi}_s\, M$ is a pullback of the morphisms t and s, such that the following four diagrams commute.

(STC) Source and target of the composition morphism:



(STI) Source and target of the identity morphism:



(AC) Associativity of the composition morphism:

$$\begin{array}{ccc} M\,{}_t\mathbf{\Pi}_s\,M\,{}_t\mathbf{\Pi}_s\,M & \xrightarrow{c\,{}_t\mathbf{\Pi}_s\,\mathrm{id}_M} & M\,{}_t\mathbf{\Pi}_s\,M \\ \Big\downarrow{\scriptstyle \mathrm{id}_M\,{}_t\mathbf{\Pi}_s\,c} & & \Big\downarrow{\scriptstyle c} \\ M\,{}_t\mathbf{\Pi}_s\,M & \xrightarrow{\quad c \quad} & M \end{array}$$

(CI) Composition of the identity morphism:

$$\begin{array}{ccccc} O\,{}_{\mathrm{id}_O}\mathbf{\Pi}_s\,M & \xrightarrow{e\mathbf{\Pi}\mathrm{id}_M} & M\,{}_t\mathbf{\Pi}_s\,M & \xleftarrow{\mathrm{id}_M\mathbf{\Pi}e} & M\,{}_t\mathbf{\Pi}_{\mathrm{id}_O}\,O \\ & {\scriptstyle \mathrm{pr}_2}\searrow & \Big\downarrow{\scriptstyle c} & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & & M & & \end{array}$$

We call O the *object of objects* and M the *object of morphisms* in the category object, the morphisms s, t, e and c are called *source* (*morphism*), *target* (*morphism*), *identity* (*morphism*) and *composition* (*morphism*), respectively.

Given a category object C in $\mathcal{C}$ with object of objects O, object of morphisms M, source s, target t, identity e and composition c, we write $\mathrm{Ob}\,C := O$, $\mathrm{Mor}\,C := M$, $\mathrm{s} := \mathrm{s}^C := s$, $\mathrm{t} := \mathrm{t}^C := t$, $\mathrm{e} := \mathrm{e}^C := e$ and $\mathrm{c} := \mathrm{c}^C := c$.

(b) We let C, D be category objects in $\mathcal{C}$. A *functor* from C to D in $\mathcal{C}$ consists of two morphisms $\mathrm{Ob}\,C \xrightarrow{o} \mathrm{Ob}\,D$ and $\mathrm{Mor}\,C \xrightarrow{m} \mathrm{Mor}\,D$, that are compatible with the categorical structure morphisms, that is,

$$\mathrm{s}^C o = m\mathrm{s}^D, \mathrm{t}^C o = m\mathrm{t}^D, \mathrm{e}^C m = o\mathrm{e}^D \text{ and } \mathrm{c}^C m = (m\,\mathbf{\Pi}\,m)\mathrm{c}^D.$$

We call o the *morphism on the objects* and m the *morphism on the morphisms* of the functor.

Given a functor f from C to D consisting of a morphism on the objects o and a morphism on the morphisms m, we write $\mathrm{Ob}\,f := o$, $\mathrm{Mor}\,f := m$ and $C \xrightarrow{f} D$.

Composition of functors is defined by the composition on the objects and on the morphisms.

(c) The *category of category objects in* $\mathcal{C}$, where the objects are the category objects in $\mathcal{C}$ and the morphisms are the functors in $\mathcal{C}$, is denoted by $\mathbf{Cat}(\mathcal{C})$.

(1.25) Example (category objects in **Set**)**.** A category object in **Set** is just an arbitrary (small) category.

(1.26) Definition (group objects and group homomorphisms)**.**

(a) A *group* (*object*) in $\mathcal{C}$ consists of an object G in $\mathcal{C}$ and morphisms $G\,\mathbf{\Pi}\,G \xrightarrow{m} G$, $T \xrightarrow{n} G$ and $G \xrightarrow{i} G$, such that the following diagrams commute.

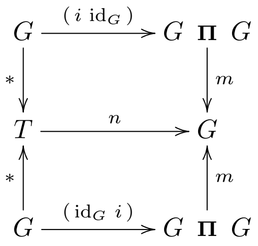
(AM) Associativity of the multiplication:

$$\begin{array}{ccc} G\,\mathbf{\Pi}\,G\,\mathbf{\Pi}\,G & \xrightarrow{\mathrm{id}_G\,\mathbf{\Pi}\,m} & G\,\mathbf{\Pi}\,G \\ \Big\downarrow{\scriptstyle m\,\mathbf{\Pi}\,\mathrm{id}_G} & & \Big\downarrow{\scriptstyle m} \\ G\,\mathbf{\Pi}\,G & \xrightarrow{\quad m \quad} & G \end{array}$$

(MN) Multiplication with identity:

$$\begin{array}{ccccc} T\,\mathbf{\Pi}\,G & \xrightarrow{n\,\mathbf{\Pi}\,\mathrm{id}_G} & G\,\mathbf{\Pi}\,G & \xleftarrow{\mathrm{id}_G\,\mathbf{\Pi}\,n} & G\,\mathbf{\Pi}\,T \\ & {\scriptstyle \mathrm{pr}_2}\searrow & \Big\downarrow{\scriptstyle m} & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & & G & & \end{array}$$

(MI) Multiplication with inverse:



We call m, n and i the *multiplication* (*operation*), *identity* or *neutral* (*operation*) and *inversion* (*operation*), respectively.

Given a group object G in $\mathcal{C}$ with multiplication m, identity n and inversion i, then we write $\mathrm{m} := \mathrm{m}^G := m$, $\mathrm{n} := \mathrm{n}^G := n$ and $\mathrm{i} := \mathrm{i}^G := i$.

(b) We let G, H be group objects in $\mathcal{C}$. A *group homomorphism* from G to H in $\mathcal{C}$ is a morphism $G \xrightarrow{\varphi} H$, that is compatible with multiplication, neutral operation and inversion, that is

$$\mathrm{m}^G\varphi = (\varphi \;\Pi\; \varphi)\mathrm{m}^H, \mathrm{n}^G\varphi = \mathrm{n}^H \text{ and } \mathrm{i}^G\varphi = \varphi\,\mathrm{i}^H.$$

Composition of group homomorphisms in $\mathcal{C}$ is given just by the ordinary composition in $\mathcal{C}$.

(c) The *category of group objects* in $\mathcal{C}$, where the objects are the group objects in $\mathcal{C}$ and the morphisms are the group homomorphisms in $\mathcal{C}$, is denoted by $\mathbf{Grp}(\mathcal{C})$.

(1.27) Example (group objects in **Set**, **Top** and **sSet**).

(a) The group objects in **Set** are just ordinary groups.

(b) In the category of topological spaces **Top**, the group objects are the topological groups. These are topological spaces whose underlying sets are endowed with a group structure such that the multiplication map and the inversion map are continuous.

(c) The group objects in the category of simplicial sets **sSet** are the simplicial objects in **Grp** and hence simplicial groups (more precisely, there is an equivalence $\mathbf{Grp}(\mathbf{sSet}) \longrightarrow \mathbf{sGrp}$).

(1.28) Lemma.

(a) We suppose given a group object G in $\mathcal{C}$. Then ${}_{\mathcal{C}}(X, G)$ is a group for every $X \in \operatorname{Ob}\mathcal{C}$ with $\mathrm{m}^{{}_{\mathcal{C}}(X,G)} = {}_{\mathcal{C}}(X, \mathrm{m}^G)$, $\mathrm{n}^{{}_{\mathcal{C}}(X,G)} = {}_{\mathcal{C}}(X, \mathrm{n}^G)$ and $\mathrm{i}^{{}_{\mathcal{C}}(X,G)} = {}_{\mathcal{C}}(X, \mathrm{i}^G)$.

(b) We suppose given a group homomorphism $G \xrightarrow{\varphi} H$ in $\mathcal{C}$, where $G, H \in \operatorname{Ob}\mathbf{Grp}(\mathcal{C})$. Then ${}_{\mathcal{C}}(X, \varphi)$ is a group homomorphism for every $X \in \operatorname{Ob}\mathcal{C}$.

Proof. Follows from definition (1.26) and the fact that the hom functor ${}_{\mathcal{C}}(X, -)$ commutes with products. □

As an application, we show by an example how results ordinary group theory (proven by calculations with elements) can be used to obtain results for category objects in $\mathcal{C}$.

(1.29) Proposition. We suppose given group objects G and H in $\mathcal{C}$ and a morphism $G \xrightarrow{\varphi} H$. Then φ is a group homomorphism in $\mathcal{C}$ if and only if $\mathrm{m}^G\varphi = (\varphi \;\Pi\; \varphi)\mathrm{m}^H$.

Proof. If φ is a group homomorphism in $\mathcal{C}$, then in particular $\mathrm{m}^G\varphi = (\varphi \;\Pi\; \varphi)\mathrm{m}^H$.
So let us conversely assume that φ is a morphism with $\mathrm{m}^G\varphi = (\varphi \;\Pi\; \varphi)\mathrm{m}^H$. Then we have

$$\begin{aligned}\mathrm{m}^{{}_{\mathcal{C}}(X,G)}\,{}_{\mathcal{C}}(X,\varphi) &= {}_{\mathcal{C}}(X,\mathrm{m}^G)\,{}_{\mathcal{C}}(X,\varphi) = {}_{\mathcal{C}}(X,\mathrm{m}^G\varphi) = {}_{\mathcal{C}}(X,(\varphi \;\Pi\; \varphi)\mathrm{m}^H) = {}_{\mathcal{C}}(X,\varphi \;\Pi\; \varphi)\,{}_{\mathcal{C}}(X,\mathrm{m}^H)\\ &= ({}_{\mathcal{C}}(X,\varphi) \times {}_{\mathcal{C}}(X,\varphi))\mathrm{m}^{{}_{\mathcal{C}}(X,H)}\end{aligned}$$

for every $X \in \operatorname{Ob} \mathcal{C}$. Hence ${}_{\mathcal{C}}(X, \varphi)$ is a semigroup homomorphism and thus, by ordinary group theory, already a group homomorphism, that is, we have $\mathrm{n}^{\mathcal{C}(X,G)} {}_{\mathcal{C}}(X, \varphi) = \mathrm{n}^{\mathcal{C}(X,H)}$ and $\mathrm{i}^{\mathcal{C}(X,G)} {}_{\mathcal{C}}(X, \varphi) = {}_{\mathcal{C}}(X, \varphi) \mathrm{i}^{\mathcal{C}(X,H)}$ for every $X \in \operatorname{Ob} \mathcal{C}$. In particular, we obtain

$$\mathrm{n}^G \varphi = \mathrm{n}^G {}_{\mathcal{C}}(T, \varphi) = (\mathrm{id}_T) {}_{\mathcal{C}}(T, \mathrm{n}^G) {}_{\mathcal{C}}(T, \varphi) = (\mathrm{id}_T) \mathrm{n}^{\mathcal{C}(T,G)} {}_{\mathcal{C}}(T, \varphi) = (\mathrm{id}_T) \mathrm{n}^{\mathcal{C}(T,H)} = (\mathrm{id}_T) {}_{\mathcal{C}}(T, \mathrm{n}^H) = \mathrm{n}^H$$

and

$$\begin{aligned} \mathrm{i}^G \varphi &= \mathrm{i}^G {}_{\mathcal{C}}(G, \varphi) = (\mathrm{id}_G) {}_{\mathcal{C}}(G, \mathrm{i}^G) {}_{\mathcal{C}}(G, \varphi) = (\mathrm{id}_G) \mathrm{i}^{\mathcal{C}(G,G)} {}_{\mathcal{C}}(G, \varphi) = (\mathrm{id}_G) {}_{\mathcal{C}}(G, \varphi) \mathrm{i}^{\mathcal{C}(G,H)} \\ &= (\mathrm{id}_G) {}_{\mathcal{C}}(G, \varphi) {}_{\mathcal{C}}(G, \mathrm{i}^H) = \varphi {}_{\mathcal{C}}(G, \mathrm{i}^H) = \varphi \, \mathrm{i}^H \end{aligned}$$

Hence φ is a group homomorphism in $\mathcal{C}$. □

The nerve of a category object

Since we need the notion of a nerve for a category object in an arbitrary category, existing in every category with pullbacks, we have to introduce some notation.

(1.30) Definition. We let C, D be category objects in $\mathcal{C}$ and $C \xrightarrow{f} D$ be a functor. We set

$$(\operatorname{Mor} C)^{{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}} n} := \begin{cases} \operatorname{Ob} C & \text{if } n = 0, \\ \operatorname{Mor} C & \text{if } n = 1, \\ \operatorname{Mor} C \;{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}}^{\operatorname{Ob} C} \; (\operatorname{Mor} C)^{{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}} (n-1)} & \text{if } n > 1, \end{cases}$$

and analogously

$$(\operatorname{Mor} f)^{\boldsymbol{\sqcap} n} := \begin{cases} \operatorname{Ob} f & \text{if } n = 0, \\ \operatorname{Mor} f & \text{if } n = 1, \\ \operatorname{Mor} f \; \boldsymbol{\sqcap} \; (\operatorname{Mor} f)^{\boldsymbol{\sqcap}(n-1)} & \text{if } n > 1. \end{cases}$$

A morphism $X \xrightarrow{f} (\operatorname{Mor} C)^{{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}} n}$ can be denoted as the tuple $(f \mathrm{pr}_j)_{j \in \lfloor n-1, 0 \rfloor}$.
Furthermore, we define morphisms $(\operatorname{Mor} C)^{{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}} n} \xrightarrow{\mathrm{t}_j} \operatorname{Ob} C$ and $(\operatorname{Mor} C)^{{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}} n} \xrightarrow{\mathrm{c}_{\lfloor j_1, j_0 \rfloor}} \operatorname{Mor} C$ by

$$\mathrm{t}_j := \begin{cases} \mathrm{pr}_j \mathrm{t} & \text{if } j < n, \\ \mathrm{pr}_{n-1} \mathrm{s} & \text{if } j = n \end{cases}$$

for $j \in [0, n]$, $n \in \mathbb{N}$, resp. $\mathrm{t}_0 := \mathrm{id}_{\operatorname{Ob} C}$ for $n = 0$, and

$$\mathrm{c}_{\lfloor j_1, j_0 \rfloor} := \begin{cases} \mathrm{t}_{j_0} \mathrm{e} & \text{if } j_1 = j_0, \\ (\mathrm{pr}_{j_1 - 1}, \mathrm{c}_{\lfloor j_1 - 1, j_0 \rfloor}) \mathrm{c} & \text{if } j_1 > j_0 \end{cases}$$

for $j_0, j_1 \in [0, n]$ with $j_1 \geq j_0$, $n \in \mathbb{N}_0$.

Thinking in elements, the notation $\mathrm{c}_{\lfloor j_1, j_0 \rfloor}$ should simply express that the morphisms which start with object number j_1 and end with the object number j_0 are composed. Similarly, the morphism t_j picks the object with number j.

(1.31) Remark. We let C be a category object in $\mathcal{C}$. There is a simplicial object $\mathrm{N}C$ in $\mathcal{C}$ given by

$$\mathrm{N}_n C := (\mathrm{N}C)_n := (\operatorname{Mor} C)^{{}_{\mathrm{t}}\boldsymbol{\sqcap}_{\mathrm{s}}^{\operatorname{Ob} C} n} \text{ for } n \in \mathbb{N}_0$$

and

$$\mathrm{N}_\theta C := (\mathrm{N}C)_\theta := \left\{ \begin{matrix} \mathrm{t}_{0\theta} & \text{if } m = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0 \end{matrix} \right\} \text{ as morphisms } \mathrm{N}_n C \longrightarrow \mathrm{N}_m C,$$

for all morphisms $[m] \xrightarrow{\theta} [n]$ in $\boldsymbol{\Delta}$, $m, n \in \mathbb{N}_0$.

Proof. For all non-negative integers $m, n, p \in \mathbb{N}_0$ and all morphisms $[m] \xrightarrow{\theta} [n]$, $[n] \xrightarrow{\rho} [p]$ in $\mathbf{\Delta}$ we have

$$(\mathrm{N}_\rho C)(\mathrm{N}_\theta C) = \begin{Bmatrix} \mathrm{t}_{0\rho}\mathrm{t}_{0\theta} & \text{if } m = 0, n = 0, \\ (\mathrm{c}_{\lfloor (j+1)\rho, j\rho \rfloor})_{j \in \lfloor n-1, 0 \rfloor} \mathrm{t}_{0\theta} & \text{if } m = 0, n > 0, \\ \mathrm{t}_{0\rho}(\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n = 0, \\ (\mathrm{c}_{\lfloor (j+1)\rho, j\rho \rfloor})_{j \in \lfloor n-1, 0 \rfloor}(\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n > 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathrm{t}_{0\rho}\mathrm{t}_0 & \text{if } m = 0, n = 0, \\ \mathrm{t}_{0\theta\rho} & \text{if } m = 0, n > 0, \\ \mathrm{t}_{0\rho}(\mathrm{c}_{\lfloor 0,0 \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta\rho, i\theta\rho \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n > 0 \end{Bmatrix} = \begin{Bmatrix} \mathrm{t}_{0\rho}\mathrm{id}_{\operatorname{Ob} C} & \text{if } m = 0, n = 0, \\ \mathrm{t}_{0\theta\rho} & \text{if } m = 0, n > 0, \\ \mathrm{t}_{0\rho}(\mathrm{t}_0\mathrm{e})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta\rho, i\theta\rho \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n > 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathrm{t}_{0\rho} & \text{if } m = 0, n = 0, \\ \mathrm{t}_{0\theta\rho} & \text{if } m = 0, n > 0, \\ (\mathrm{t}_{0\rho}\mathrm{e})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta\rho, i\theta\rho \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n > 0 \end{Bmatrix} = \begin{Bmatrix} \mathrm{t}_{0\theta\rho} & \text{if } m = 0, n = 0, \\ \mathrm{t}_{0\theta\rho} & \text{if } m = 0, n > 0, \\ (\mathrm{c}_{\lfloor 0\rho, 0\rho \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta\rho, i\theta\rho \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, n > 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} \mathrm{t}_{0\theta\rho} & \text{if } m = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta\rho, i\theta\rho \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0 \end{Bmatrix} = \mathrm{N}_{\theta\rho} C$$

as well as

$$\mathrm{N}_{\mathrm{id}_{[n]}} C = \begin{Bmatrix} \mathrm{t}_{0\mathrm{id}_{[n]}} & \text{if } n = 0, \\ (\mathrm{c}_{\lfloor (j+1)\mathrm{id}_{[n]}, j\mathrm{id}_{[n]} \rfloor})_{j \in \lfloor n-1, 0 \rfloor} & \text{if } n > 0 \end{Bmatrix} = \begin{Bmatrix} \mathrm{t}_0 & \text{if } n = 0, \\ (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor n-1, 0 \rfloor} & \text{if } n > 0 \end{Bmatrix}$$
$$= \begin{Bmatrix} \mathrm{id}_{\operatorname{Ob} C} & \text{if } n = 0, \\ (\mathrm{pr}_j)_{j \in \lfloor n-1, 0 \rfloor} & \text{if } n > 0 \end{Bmatrix} = \mathrm{id}_{(\mathrm{N}C)_n},$$

that is, $\mathrm{N}C$ is a simplicial object in $\mathcal{C}$. □

(1.32) Definition (nerve)**.** We let C be a category object in $\mathcal{C}$. The simplicial object $\mathrm{N}C$ in $\mathcal{C}$ given as in remark (1.31) by

$$\mathrm{N}_n C = (\mathrm{N}C)_n = (\operatorname{Mor} C)^{{}_{\mathrm{t}}\Pi_{\mathrm{s}} n} \text{ for all } n \in \mathbb{N}_0$$

and

$$\mathrm{N}_\theta C = \begin{Bmatrix} \mathrm{t}_{0\theta} & \text{if } m = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0, \end{Bmatrix}$$

for all morphisms $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$, $m, n \in \mathbb{N}_0$, is called the *nerve* of the category object C.

(1.33) Proposition. The faces $\mathrm{N}_n C \xrightarrow{\mathrm{d}_k} \mathrm{N}_{n-1} C$ and degeneracies $\mathrm{N}_n C \xrightarrow{\mathrm{s}_k} \mathrm{N}_{n+1} C$ for a category object C in $\mathcal{C}$ are given by

$$\mathrm{d}_k = \begin{Bmatrix} (\mathrm{pr}_j)_{j \in \lfloor n-1, 1 \rfloor} & \text{if } k = 0, \\ (\mathrm{pr}_j)_{j \in \lfloor n-1, k+1 \rfloor} \cup (\mathrm{c}_{\lfloor k+1, k-1 \rfloor}) \cup (\mathrm{pr}_j)_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{pr}_j)_{j \in \lfloor n-2, 0 \rfloor} & \text{if } k = n \end{Bmatrix} \text{ for all } k \in [0, n], n \in \mathbb{N}, n \geq 2,$$

resp.

$$\mathrm{d}_0 = \mathrm{s}, \mathrm{d}_1 = \mathrm{t} \text{ for } n = 1,$$

and

$$\mathrm{s}_k = (\mathrm{pr}_j)_{j \in \lfloor n-1, k \rfloor} \cup (\mathrm{t}_k \mathrm{e}) \cup (\mathrm{pr}_j)_{j \in \lfloor k-1, 0 \rfloor} \text{ for all } k \in [0, n], n \in \mathbb{N}_0.$$

Proof. We have

$$\begin{aligned}
\mathrm{d}_k &= \mathrm{N}_{\delta^k} C = (\mathrm{c}_{\lfloor (j+1)\delta^k, j\delta^k \rfloor})_{j \in \lfloor n-2, 0 \rfloor} \\
&= \begin{Bmatrix} (\mathrm{c}_{\lfloor (j+1)\delta^k, j\delta^k \rfloor})_{j \in \lfloor n-2, k \rfloor} & \text{if } k = 0, \\ (\mathrm{c}_{\lfloor (j+1)\delta^k, j\delta^k \rfloor})_{j \in \lfloor n-2, k \rfloor} \cup (\mathrm{c}_{\lfloor k\delta^k, (k-1)\delta^k \rfloor}) \cup (\mathrm{c}_{\lfloor (j+1)\delta^k, j\delta^k \rfloor})_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{c}_{\lfloor (j+1)\delta^k, j\delta^k \rfloor})_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k = n \end{Bmatrix} \\
&= \begin{Bmatrix} (\mathrm{c}_{\lfloor j+2, j+1 \rfloor})_{j \in \lfloor n-2, k \rfloor} & \text{if } k = 0, \\ (\mathrm{c}_{\lfloor j+2, j+1 \rfloor})_{j \in \lfloor n-2, k \rfloor} \cup (\mathrm{c}_{\lfloor k+1, k-1 \rfloor}) \cup (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k = n \end{Bmatrix} \\
&= \begin{Bmatrix} (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor n-1, k+1 \rfloor} & \text{if } k = 0, \\ (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor n-1, k+1 \rfloor} \cup (\mathrm{c}_{\lfloor k+1, k-1 \rfloor}) \cup (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k = n, \end{Bmatrix} \\
&= \begin{cases} (\mathrm{pr}_j)_{j \in \lfloor n-1, 1 \rfloor} & \text{if } k = 0, \\ (\mathrm{pr}_j)_{j \in \lfloor n-1, k+1 \rfloor} \cup (\mathrm{c}_{\lfloor k+1, k-1 \rfloor}) \cup (\mathrm{pr}_j)_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{pr}_j)_{j \in \lfloor n-2, 0 \rfloor} & \text{if } k = n \end{cases}
\end{aligned}$$

for $k \in [0, n]$, $n \in \mathbb{N}$, $n \geq 2$,

$$\mathrm{d}_0 = \mathrm{N}_{\delta^0} C = \mathrm{t}_{0\delta^0} = \mathrm{t}_1 = \mathrm{s} \text{ and } \mathrm{d}_1 = \mathrm{N}_{\delta^1} C = \mathrm{t}_{0\delta^1} = \mathrm{t}_0 = \mathrm{t}$$

for $n = 1$, and finally

$$\begin{aligned}
\mathrm{s}_k &= \mathrm{N}_{\sigma^k} C = (\mathrm{c}_{\lfloor (j+1)\sigma^k, j\sigma^k \rfloor})_{j \in \lfloor n, 0 \rfloor} \\
&= (\mathrm{c}_{\lfloor (j+1)\sigma^k, j\sigma^k \rfloor})_{j \in \lfloor n, k+1 \rfloor} \cup (\mathrm{c}_{\lfloor (k+1)\sigma^k, k\sigma^k \rfloor}) \cup (\mathrm{c}_{\lfloor (j+1)\sigma^k, j\sigma^k \rfloor})_{j \in \lfloor k-1, 0 \rfloor} \\
&= (\mathrm{c}_{\lfloor j, j-1 \rfloor})_{j \in \lfloor n, k+1 \rfloor} \cup (\mathrm{c}_{\lfloor k, k \rfloor}) \cup (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor k-1, 0 \rfloor} = (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor n-1, k \rfloor} \cup (\mathrm{c}_{\lfloor k, k \rfloor}) \cup (\mathrm{c}_{\lfloor j+1, j \rfloor})_{j \in \lfloor k-1, 0 \rfloor} \\
&= (\mathrm{pr}_j)_{j \in \lfloor n-1, k \rfloor} \cup (\mathrm{t}_k \mathrm{e}) \cup (\mathrm{pr}_j)_{j \in \lfloor k-1, 0 \rfloor}
\end{aligned}$$

for $k \in [0, n]$, $n \in \mathbb{N}_0$. □

(1.34) Proposition.

(a) If C and D are category objects in $\mathcal{C}$ and if $C \xrightarrow{f} D$ is a functor in $\mathcal{C}$, then we get an induced morphism

$$\mathrm{N}C \xrightarrow{\mathrm{N}f} \mathrm{N}D$$

with

$$\mathrm{N}_n f = (\mathrm{Mor}\, f)^{\sqcap n}.$$

(b) The construction in (a) yields a functor

$$\mathbf{Cat}(\mathcal{C}) \xrightarrow{\mathrm{N}} \mathbf{s}\mathcal{C}.$$

Proof.

(a) We have

$$\begin{aligned}
(\mathrm{N}_\theta C)(\mathrm{N}_m f) &= \begin{Bmatrix} \mathrm{t}_{0\theta}(\mathrm{Ob}\, f) & \text{if } m = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} (\mathrm{Mor}\, f)^{\sqcap m} & \text{if } m > 0 \end{Bmatrix} \\
&= \begin{Bmatrix} \mathrm{t}_{0\theta}(\mathrm{Ob}\, f) & \text{if } m = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor}(\mathrm{Mor}\, f))_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0 \end{Bmatrix}
\end{aligned}$$

$$= \begin{cases} (\mathrm{Mor}\, f)^{\sqcap n}\mathrm{t}_{0\theta} & \text{if } m = 0, \\ (\mathrm{Mor}\, f)^{\sqcap n}(\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0 \end{cases}$$
$$= (\mathrm{N}_n f)(\mathrm{N}_\theta D)$$

for all morphisms $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$, $m, n \in \mathbb{N}_0$, that is

$$\mathrm{N}C \xrightarrow{\mathrm{N}f} \mathrm{N}D$$

is a morphism of simplicial objects in $\mathcal{C}$.

(b) If C, D, E are category objects in $\mathcal{C}$ and $C \xrightarrow{f} D$, $D \xrightarrow{g} E$ are functors in $\mathcal{C}$, then we have

$$(\mathrm{N}_n f)(\mathrm{N}_n g) = (\mathrm{Mor}\, f)^{\sqcap n}(\mathrm{Mor}\, g)^{\sqcap n} = ((\mathrm{Mor}\, f)(\mathrm{Mor}\, g))^{\sqcap n} = (\mathrm{Mor}(fg))^{\mathrm{t}\sqcap_{\mathrm{s}} n} = \mathrm{N}_n(fg)$$

and

$$(\mathrm{N}_n \mathrm{id}_C) = (\mathrm{Mor}\, \mathrm{id}_C)^{\sqcap n} = (\mathrm{id}_{\mathrm{Mor}\, C})^{\sqcap n} = \mathrm{id}_{(\mathrm{Mor}\, C)^{\sqcap n}} = \mathrm{id}_{\mathrm{N}_n C} = (\mathrm{id}_{\mathrm{N}C})_n$$

for all $n \in \mathbb{N}_0$, that is, $(\mathrm{N}f)(\mathrm{N}g) = \mathrm{N}(fg)$ and $\mathrm{N}\mathrm{id}_C = \mathrm{id}_{\mathrm{N}C}$. Hence

$$\mathbf{Cat}(\mathcal{C}) \xrightarrow{\mathrm{N}} \mathbf{s}\mathcal{C}.$$

is a functor. □

We remark that the nerve of a category $\mathcal{K}$, that is a category object in **Set**, can also be defined by $\mathrm{N}\mathcal{K} := \mathrm{Mor}_{\mathbf{Cat}}(-, \mathcal{K})|_{\mathbf{\Delta}^{\mathrm{op}}}$. With this description, we obtain the functor

$$\mathbf{Cat} \xrightarrow{\mathrm{N}} \mathbf{sSet}$$

for free.

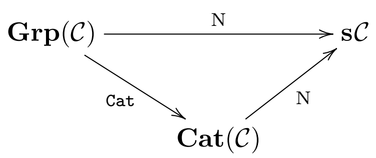
The nerve of a group object

Every group G can be considered as a category with a single object, the morphisms given by the elements of G. The construction is functorial and thus we can define a nerve functor for groups. In the following, we proceed in a more general manner for arbitrary group objects.

(1.35) Definition (nerve for group objects)**.** The composition of functors

$$\mathbf{Grp}(\mathcal{C}) \xrightarrow{\mathrm{N} \circ \mathtt{Cat}} \mathbf{s}\mathcal{C}$$

is called the *nerve functor* for group objects in $\mathcal{C}$ and will be denoted by N, too.



(1.36) Proposition. We let G be a group object in $\mathcal{C}$. Then the nerve of G is given by

$$\mathrm{N}_n G = G^{\sqcap n} \text{ for all } n \in \mathbb{N}_0,$$

while the faces $\mathrm{N}_n G \xrightarrow{\mathrm{d}_k} \mathrm{N}_{n-1} G$ and degeneracies $\mathrm{N}_n G \xrightarrow{\mathrm{s}_k} \mathrm{N}_{n+1} G$ are

$$\mathrm{d}_k = \begin{cases} (\mathrm{pr}_i)_{i \in \lfloor n-1, 1 \rfloor} & \text{if } k = 0, \\ (\mathrm{pr}_i)_{i \in \lfloor n-1, k+1 \rfloor} \cup ((\mathrm{pr}_k, \mathrm{pr}_{k-1})\mathrm{m}) \cup (\mathrm{pr}_i)_{i \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{pr}_i)_{i \in \lfloor n-2, 0 \rfloor} & \text{if } k = n \end{cases} \quad \text{for } k \in [0, n], n \in \mathbb{N}, n \geq 2,$$

resp.

$$\mathrm{d}_0 = \mathrm{d}_1 = 1 \text{ for } n = 1,$$

and

$$\mathrm{s}_k = (\mathrm{pr}_i)_{i \in \lfloor n-1, k \rfloor} \cup (1\mathrm{n}) \cup (\mathrm{pr}_i)_{i \in \lfloor k-1, 0 \rfloor} \text{ for } k \in [0, n], n \in \mathbb{N}_0,$$

where $G^{\sqcap n} \xrightarrow{1} T$ denotes the unique morphism to the terminal object.

Proof. This follows from proposition (1.33). □

Chapter II

Simplicial homotopies and simplicial homology

Our overall aim is to compute the homology of the simplicial set associated to a given crossed module. As an intermediate step, we attach a complex to such a simplicial set. Since we want to invoke Eilenberg-Zilber, we consider both ways to attach such a complex, the *associated complex* and the *Moore complex*.
Moreover, we intend to show further down that certain diagrams commute up to simplicial homotopy, a notion which we consider here.
Finally, in this chapter we consider the path object of a simplicial object, which then will appear in our approach to Eilenberg-Zilber.
For more details, the reader is referred to [8], [17] or the original articles of KAN, for example [20].

§1 Simplicial homotopies

Recall that two continuous maps $f, g\colon T \to U$ between topological spaces T and U are homotopic if there exists a continuous map $H\colon T \times [0,1] \to U$ such that $(-,0)H = f$ and $(-,1)H = g$. A similar notion can be obtained for simplicial sets. However, we begin with a more general definition of homotopies between simplicial morphisms in an arbitrary category, which does not necessarily contain an analogue of the operation $- \times [0,1]$. The similarity to the topological definition in the case of simplicial sets will be shown thereafter.

(2.1) Definition (simplicial homotopy)**.** We let X and Y be simplicial objects in a category $\mathcal{C}$. Simplicial morphisms $X \xrightarrow{\varphi} Y$ and $X \xrightarrow{\psi} Y$ are said to be *simplicially homotopic*, if for every $n \in \mathbb{N}_0$ there are morphisms $X_n \xrightarrow{h_k} Y_{n+1}$ for $k \in [0,n]$ in $\mathcal{C}$ such that $h_n \mathrm{d}_{n+1} = \varphi_n$, $h_0 \mathrm{d}_0 = \psi_n$ and

$$h_l \mathrm{d}_k = \begin{cases} \mathrm{d}_k h_{l-1} & \text{for } k < l, \\ h_{k-1} \mathrm{d}_k & \text{for } k = l, k \neq 0, \\ h_k \mathrm{d}_k & \text{for } k = l+1, k \neq n+1, \\ \mathrm{d}_{k-1} h_l & \text{for } k > l+1, \end{cases} \qquad h_l \mathrm{s}_k = \begin{cases} \mathrm{s}_k h_{l+1} & \text{for } k \leq l, \\ \mathrm{s}_{k-1} h_l & \text{for } k > l, \end{cases}$$

for all $k \in [0, n+1]$, $l \in [0,n]$. In this case, we write $\varphi \sim \psi$ and $(h_k \in {}_{\mathcal{C}}(X_n, Y_{n+1}) \mid k \in [0,n], n \in \mathbb{N}_0)$ is called a *simplicial homotopy* from φ to ψ.

(2.2) Definition. For a simplicial set X we define ins_0 and ins_1 to be the composite morphisms

$$X \xrightarrow{\cong} X \times \Delta^0 \xrightarrow{\mathrm{id} \times \mathrm{d}^1} X \times \Delta^1 \text{ resp. } X \xrightarrow{\cong} X \times \Delta^0 \xrightarrow{\mathrm{id} \times \mathrm{d}^0} X \times \Delta^1.$$

Here $\mathrm{d}^l = \Delta^{\delta^l}$ for $l \in [0,1]$ denotes the l-th coface in the cosimplicial set Δ^-.

(2.3) Definition. For $k \in [0, n+1]$, $n \in \mathbb{N}_0$, we let $\tau^k \in \Delta^1_n = {}_{\mathbf{\Delta}}([n],[1])$ be the functor with

$$[0, n-k]\tau^k = \{0\} \text{ and } [n-k+1, n]\tau^k = \{1\}.$$

Note that, with the definitions from above, we have

$$(x_n)(\mathrm{ins}_0)_n = (x_n, \tau^0) \text{ and } (x_n)(\mathrm{ins}_1)_n = (x_n, \tau^{n+1})$$

for all n-simplices $x_n \in X_n$ in a given simplicial set X and all $n \in \mathbb{N}_0$.

(2.4) Proposition. The composite morphisms

$$[n-1] \xrightarrow{\delta^k} [n] \xrightarrow{\tau^l} [1] \text{ and } [n+1] \xrightarrow{\sigma^k} [n] \xrightarrow{\tau^l} [1]$$

are given as follows: We have

$$\delta^k \tau^l = \begin{cases} \tau^l & \text{if } k \leq n-l, \\ \tau^{l-1} & \text{if } k > n-l \end{cases}$$

for $k \in [0, n]$, $l \in [0, n+1]$, $n \in \mathbb{N}$, and

$$\sigma^k \tau^l = \begin{cases} \tau^l & \text{if } k \leq n-l, \\ \tau^{l+1} & \text{if } k > n-l \end{cases}$$

for $k \in [0, n]$, $l \in [0, n+1]$, $n \in \mathbb{N}_0$.

Proof. If $k \leq n-l$, we compute

$$\begin{aligned} i\delta^k\tau^l &= \left.\begin{cases} i\tau^l & \text{for } i \in [0, k-1], \\ (i+1)\tau^l & \text{for } i \in [k, n-1] \end{cases}\right\} = \left.\begin{cases} 0 & \text{for } i \in [0, k-1], \\ 0 & \text{for } i \in [k, n-1-l], \\ 1 & \text{for } i \in [n-l, n-1] \end{cases}\right\} = \left.\begin{cases} 0 & \text{for } i \in [0, n-1-l], \\ 1 & \text{for } i \in [n-1-l, n-1] \end{cases}\right\} \\ &= i\tau^l, \end{aligned}$$

and if $k > n-l$, we have

$$\begin{aligned} i\delta^k\tau^l &= \left.\begin{cases} i\tau^l & \text{for } i \in [0, k-1], \\ (i+1)\tau^l & \text{for } i \in [k, n-1] \end{cases}\right\} = \left.\begin{cases} 0 & \text{for } i \in [0, n-l], \\ 1 & \text{for } i \in [n-l+1, k-1], \\ 1 & \text{for } i \in [k, n-1] \end{cases}\right\} \\ &= \left.\begin{cases} 0 & \text{for } i \in [0, n-l], \\ 1 & \text{for } i \in [n-l+1, n-1] \end{cases}\right\} = i\tau^{l-1}. \end{aligned}$$

Furthermore, if $k \leq n-l$, we have

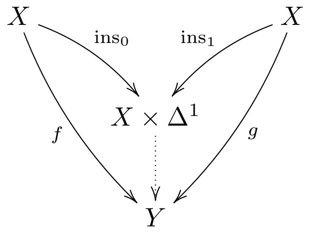
$$\begin{aligned} i\sigma^k\tau^l &= \left.\begin{cases} i\tau^l & \text{for } i \in [0, k], \\ (i-1)\tau^l & \text{for } i \in [k+1, n+1] \end{cases}\right\} = \left.\begin{cases} 0 & \text{for } i \in [0, k], \\ 0 & \text{for } i \in [k, n+1-l], \\ 1 & \text{for } i \in [n+2-l, n+1] \end{cases}\right\} \\ &= \left.\begin{cases} 0 & \text{for } i \in [0, n+1-l], \\ 1 & \text{for } i \in [n+2-l, n+1] \end{cases}\right\} = i\sigma^l, \end{aligned}$$

and if $k > n-l$, we compute

$$\begin{aligned} i\sigma^k\tau^l &= \left.\begin{cases} i\tau^l & \text{for } i \in [0, k], \\ (i-1)\tau^l & \text{for } i \in [k+1, n+1] \end{cases}\right\} = \left.\begin{cases} 0 & \text{for } i \in [0, n-l], \\ 1 & \text{for } i \in [n+1-l, k], \\ 1 & \text{for } i \in [k+1, n+1] \end{cases}\right\} \\ &= \left.\begin{cases} 0 & \text{for } i \in [0, n-l], \\ 1 & \text{for } i \in [n+1-l, n+1] \end{cases}\right\} = i\sigma^l. \end{aligned}$$

□

(2.5) Proposition. We let X and Y be simplicial sets and $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$ be simplicial maps. There is a bijective correspondence between the simplicial homotopies from f to g and the simplicial maps $X \times \Delta^1 \longrightarrow Y$ that let the following diagram commute:



Proof. First, we let $(h_k \in {}_{\mathbf{Set}}(X_n, Y_{n+1}) \mid k \in [0,n], n \in \mathbb{N}_0)$ be a simplicial homotopy from f to g. For each $n \in \mathbb{N}_0$, we define $H_n \colon X_n \times \Delta^1_n \to Y_n$ by $(x_n, \tau^0)H_n := x_n f_n$ and $(x_n, \tau^{n+1-k})H_n := x_n h_k \mathrm{d}_k$ for $k \in [0,n]$, $x_n \in X_n$. Using proposition (2.4), this implies

$$
\begin{aligned}
(x_n, \tau^{n+1-l})H_n \mathrm{d}_k &= x_n h_l \mathrm{d}_l \mathrm{d}_k = \begin{Bmatrix} x_n h_l \mathrm{d}_k \mathrm{d}_{l-1} & \text{for } k < l, \\ x_n h_l \mathrm{d}_{k+1} \mathrm{d}_l & \text{for } k \geq l \end{Bmatrix} = \begin{Bmatrix} x_n \mathrm{d}_k h_{l-1} \mathrm{d}_{l-1} & \text{for } k < l, \\ x_n h_{k+1} \mathrm{d}_{k+1} \mathrm{d}_k & \text{for } k = l \neq n, \\ x_n f_n \mathrm{d}_n & \text{for } k = l = n, \\ x_n \mathrm{d}_k h_l \mathrm{d}_l & \text{for } k > l \end{Bmatrix} \\
&= \begin{Bmatrix} x_n \mathrm{d}_k h_{l-1} \mathrm{d}_{l-1} & \text{for } k < l, \\ x_n h_{k+1} \mathrm{d}_k \mathrm{d}_k & \text{for } k = l \neq n, \\ x_n \mathrm{d}_n f_{n-1} & \text{for } k = l = n, \\ x_n \mathrm{d}_k h_l \mathrm{d}_l & \text{for } k > l \end{Bmatrix} = \begin{Bmatrix} x_n \mathrm{d}_k h_{l-1} \mathrm{d}_{l-1} & \text{for } k < l, \\ x_n \mathrm{d}_k h_k \mathrm{d}_k & \text{for } k = l \neq n, \\ x_n \mathrm{d}_k f_{n-1} & \text{for } k = l = n, \\ x_n \mathrm{d}_k h_l \mathrm{d}_l & \text{for } k > l \end{Bmatrix} \\
&= \begin{Bmatrix} (x_n \mathrm{d}_k, \tau^{n-(l-1)})H_{n-1} & \text{for } k < l, \\ (x_n \mathrm{d}_k, \tau^{n-k})H_{n-1} & \text{for } k = l \neq n, \\ (x_n \mathrm{d}_k, \tau^{n-k})H_{n-1} & \text{for } k = l = n, \\ (x_n \mathrm{d}_k, \tau^{n-l})H_{n-1} & \text{for } k > l \end{Bmatrix} = \begin{Bmatrix} (x_n \mathrm{d}_k, \tau^{n+1-l})H_{n-1} & \text{for } k < l, \\ (x_n \mathrm{d}_k, \tau^{n-l})H_{n-1} & \text{for } k \geq l \end{Bmatrix} \\
&= (x_n \mathrm{d}_k, \delta^k \tau^{n+1-l})H_{n-1} = (x_n \mathrm{d}_k, \tau^{n+1-l} \mathrm{d}_k)H_{n-1} = (x_n, \tau^{n+1-l})\mathrm{d}_k H_{n-1}
\end{aligned}
$$

for all $x_n \in X_n$, $k \in [0,n]$, $l \in [0,n]$, $n \in \mathbb{N}$, and

$$
\begin{aligned}
(x_n, \tau^{n+1-l})H_n \mathrm{s}_k &= x_n h_l \mathrm{d}_l \mathrm{s}_k = \begin{Bmatrix} x_n h_l \mathrm{s}_k \mathrm{d}_{l+1} & \text{for } k < l, \\ x_n h_l \mathrm{s}_{k+1} \mathrm{d}_l & \text{for } k \geq l \end{Bmatrix} = \begin{Bmatrix} x_n \mathrm{s}_k h_{l+1} \mathrm{d}_{l+1} & \text{for } k < l, \\ x_n \mathrm{s}_k h_l \mathrm{d}_l & \text{for } k \geq l \end{Bmatrix} \\
&= \begin{Bmatrix} (x_n \mathrm{s}_k, \tau^{n+2-(l+1)})H_{n+1} & \text{for } k < l, \\ (x_n \mathrm{s}_k, \tau^{n+2-l})H_{n+1} & \text{for } k \geq l \end{Bmatrix} = \begin{Bmatrix} (x_n \mathrm{s}_k, \tau^{n+1-l})H_{n+1} & \text{for } k < l, \\ (x_n \mathrm{s}_k, \tau^{n+2-l})H_{n+1} & \text{for } k \geq l \end{Bmatrix} \\
&= (x_n \mathrm{s}_k, \sigma^k \tau^{n+1-l})H_{n+1} = (x_n \mathrm{s}_k, \tau^{n+1-l} \mathrm{s}^k)H_{n+1} \\
&= (x_n, \tau^{n+1-l})\mathrm{s}_k H_{n+1}
\end{aligned}
$$

for all $x_n \in X_n$, $k \in [0,n]$, $l \in [0,n]$, $n \in \mathbb{N}_0$. Since additionally

$$
\begin{aligned}
(x_n, \tau^0)H_n \mathrm{d}_k &= x_n f_n \mathrm{d}_k = x_n \mathrm{d}_k f_{n-1} = (x_n \mathrm{d}_k, \tau^0)H_{n-1} = (x_n \mathrm{d}_k, \delta^k \tau^0)H_{n-1} = (x_n \mathrm{d}_k, \tau^0 \mathrm{d}_k)H_{n-1} \\
&= (x_n, \tau^0)\mathrm{d}_k H_{n-1}
\end{aligned}
$$

for all $x_n \in X_n$, $k \in [0,n]$, $n \in \mathbb{N}$, and

$$
\begin{aligned}
(x_n, \tau^0)H_n \mathrm{s}_k &= x_n f_n \mathrm{s}_k = x_n \mathrm{s}_k f_{n-1} = (x_n \mathrm{s}_k, \tau^0)H_{n-1} = (x_n \mathrm{s}_k, \sigma^k \tau^0)H_{n-1} = (x_n \mathrm{s}_k, \tau^0 \mathrm{s}_k)H_{n-1} \\
&= (x_n, \tau^0)\mathrm{s}_k H_{n-1}
\end{aligned}
$$

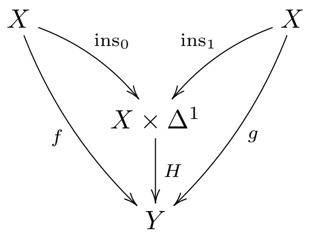
for all $x_n \in X_n$, $k \in [0,n]$, $n \in \mathbb{N}_0$, we know that the maps $H_n \colon X_n \times \Delta^1_n \to Y_n$ yield a simplicial map $X \times \Delta^1 \xrightarrow{H} Y$. Furthermore, we get

$$x_n(\mathrm{ins}_0)_n H_n = (x_n, \tau^0)H_n = x_n f_n$$

as well as

$$x_n(\mathrm{ins}_1)_n H_n = (x_n, \tau^{n+1})H_n = x_n h_0 \mathrm{d}_0 = x_n g_n$$

for all $x_n \in X_n$, but this shows the $\mathrm{ins}_0 H = f$ and $\mathrm{ins}_1 H = g$.
Conversely, we let $X \times \Delta^1 \xrightarrow{H} Y$ be a simplicial map such that



commutes. For $n \in \mathbb{N}_0$, $k \in [0, n]$, we define $h_k \colon X_n \to Y_{n+1}, x_n \mapsto (x_n \mathrm{s}_k, \tau^{n+1-k})H_{n+1}$. According to proposition (2.4) we get

$$\begin{aligned}
x_n h_l \mathrm{d}_k &= (x_n \mathrm{s}_l, \tau^{n+1-l})H_{n+1}\mathrm{d}_k = (x_n \mathrm{s}_l, \tau^{n+1-l})\mathrm{d}_k H_n = (x_n \mathrm{s}_l \mathrm{d}_k, \tau^{n+1-l}\mathrm{d}_k)H_n \\
&= \left\{\begin{matrix} (x_n \mathrm{d}_k \mathrm{s}_{l-1}, \delta^k \tau^{n+1-l})H_n & \text{if } k < l, \\ (x_n, \delta^k \tau^{n+1-l})H_n & \text{if } k = l, \\ (x_n, \delta^k \tau^{n+1-l})H_n & \text{if } k = l+1, \\ (x_n \mathrm{d}_{k-1} \mathrm{s}_l, \delta^k \tau^{n+1-l})H_n & \text{if } k > l+1 \end{matrix}\right\} = \left\{\begin{matrix} (x_n \mathrm{d}_k \mathrm{s}_{l-1}, \tau^{n+1-l})H_n & \text{if } k < l, \\ (x_n, \tau^{n+1-l})H_n & \text{if } k = l, \\ (x_n, \tau^{n-l})H_n & \text{if } k = l+1, \\ (x_n \mathrm{d}_{k-1} \mathrm{s}_l, \tau^{n-l})H_n & \text{if } k > l+1 \end{matrix}\right\} \\
&= \left\{\begin{matrix} (x_n \mathrm{d}_k \mathrm{s}_{l-1}, \tau^{n-(l-1)})H_n & \text{if } k < l, \\ (x_n, \tau^{n-(l-1)})H_n & \text{if } k = l, \\ (x_n, \tau^{n-l})H_n & \text{if } k = l+1, \\ (x_n \mathrm{d}_{k-1} \mathrm{s}_l, \tau^{n-l})H_n & \text{if } k > l+1 \end{matrix}\right\} = \left\{\begin{matrix} x_n \mathrm{d}_k h_{l-1} & \text{if } k < l, \\ x_n h_{k-1} \mathrm{d}_k & \text{if } k = l, k \neq 0, \\ x_n h_k \mathrm{d}_k & \text{if } k = l+1, k \neq n+1, \\ x_n \mathrm{d}_{k-1} h_l & \text{if } k > l+1 \end{matrix}\right\}
\end{aligned}$$

and

$$\begin{aligned}
x_n h_l \mathrm{s}_k &= (x_n \mathrm{s}_l, \tau^{n+1-l})H_{n+1}\mathrm{s}_k = (x_n \mathrm{s}_l, \tau^{n+1-l})\mathrm{s}_k H_{n+2} = (x_n \mathrm{s}_l \mathrm{s}_k, \tau^{n+1-l}\mathrm{s}_k)H_{n+2} \\
&= \left\{\begin{matrix} (x_n \mathrm{s}_k \mathrm{s}_{l+1}, \sigma^k \tau^{n+1-l})H_{n+2} & \text{if } k \leq l, \\ (x_n \mathrm{s}_{k-1} \mathrm{s}_l, \sigma^k \tau^{n+1-l})H_{n+2} & \text{if } k > l \end{matrix}\right\} = \left\{\begin{matrix} (x_n \mathrm{s}_k \mathrm{s}_{l+1}, \tau^{n+1-l})H_{n+2} & \text{if } k \leq l, \\ (x_n \mathrm{s}_{k-1} \mathrm{s}_l, \tau^{n+2-l})H_{n+2} & \text{if } k > l \end{matrix}\right\} \\
&= \left\{\begin{matrix} (x_n \mathrm{s}_k \mathrm{s}_{l+1}, \tau^{n+2-(l+1)})H_{n+2} & \text{if } k \leq l, \\ (x_n \mathrm{s}_{k-1} \mathrm{s}_l, \tau^{n+2-l})H_{n+2} & \text{if } k > l \end{matrix}\right\} = \left\{\begin{matrix} x_n \mathrm{s}_k h_{l+1} & \text{if } k \leq l, \\ x_n \mathrm{s}_{k-1} h_l & \text{if } k > l \end{matrix}\right\}
\end{aligned}$$

for all $x_n \in X_n$, $k \in [0, n+1]$, $l \in [0, n]$, $n \in \mathbb{N}_0$. Moreover, we have

$$x_n h_n \mathrm{d}_{n+1} = (x_n \mathrm{s}_n \mathrm{d}_{n+1}, \tau^1 \mathrm{d}_{n+1})H_n = (x_n, \delta^{n+1}\tau^1)H_n = (x_n, \tau^0)H_n = x_n(\mathrm{ins}_0)_n H_n = x_n f_n$$

and

$$x_n h_0 \mathrm{d}_0 = (x_n \mathrm{s}_0 \mathrm{d}_0, \tau^{n+1}\mathrm{d}_0) = (x_n \mathrm{s}_0 \mathrm{d}_0, \delta^0 \tau^{n+1}) = (x_n, \tau^{n+1})H_n = x_n(\mathrm{ins}_1)_n H_n = x_n g_n$$

for all $x_n \in X_n$, $n \in \mathbb{N}_0$, that is, $(h_k \in {}_{\mathbf{Set}}(X_n, Y_{n+1}) \mid k \in [0, n], n \in \mathbb{N}_0)$ is a simplicial homotopy from f to g.
At last, it remains to show the bijectivity. Thereto, we let $(h_k \in {}_{\mathbf{Set}}(X_n, Y_{n+1}) \mid k \in [0, n], n \in \mathbb{N}_0)$ be a simplicial homotopy from f to g. We define $H_n \colon X_n \times \Delta^1_n \to Y_n$ by $(x_n, \tau^0)H_n := x_n f_n$ and $(x_n, \tau^{n+1-k})H_n := x_n h_k \mathrm{d}_k$ for $x_n \in X_n$, $k \in [0, n]$, $n \in \mathbb{N}_0$, and we define $h'_k \colon X_n \to Y_{n+1}, x_n \mapsto (x_n \mathrm{s}_k, \tau^{n+1-k})H_{n+1}$ for $k \in [0, n]$, $n \in \mathbb{N}_0$. Then we get

$$x_n h'_k = (x_n \mathrm{s}_k, \tau^{n+1-k})H_{n+1} = (x_n \mathrm{s}_k, \tau^{n+2-(k+1)})H_{n+1} = x_n \mathrm{s}_k h_{k+1} \mathrm{d}_{k+1} = x_n h_k \mathrm{s}_k \mathrm{d}_{k+1} = x_n h_k$$

for all $x_n \in X_n$, that is, $h'_k = h_k$ for all $k \in [0, n]$, $n \in \mathbb{N}_0$.

Conversely, we let $X \times \Delta^1 \xrightarrow{H} Y$ be a simplicial map such that $\mathrm{ins}_0 H = f$ and $\mathrm{ins}_1 H = g$ and define maps $h_k \colon X_n \to Y_{n+1}, x_n \mapsto (x_n \mathrm{s}_k, \tau^{n+1-k})H_{n+1}$ for $k \in [0,n]$, $n \in \mathbb{N}_0$ and maps $H'_n \colon X_n \times \Delta^1_n \to Y_n$ by $(x_n, \tau^0)H'_n := x_n f_n$ and $(x_n, \tau^{n+1-k})H'_n := x_n h_k \mathrm{d}_k$ for $x_n \in X_n$, $k \in [0,n]$, $n \in \mathbb{N}_0$. This implies

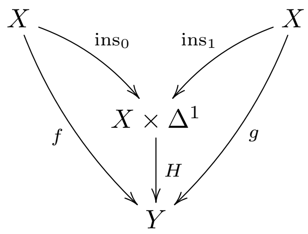
$$(x_n, \tau^0)H'_n = x_n f_n = x_n(\mathrm{ins}_0)_n H_n = (x_n, \tau^0)H_n$$

and

$$\begin{aligned}(x_n, \tau^{n+1-k})H'_n = x_n h_k \mathrm{d}_k = (x_n \mathrm{s}_k, \tau^{n+1-k})H_{n+1}\mathrm{d}_k = (x_n \mathrm{s}_k, \tau^{n+1-k})\mathrm{d}_k H_n = (x_n \mathrm{s}_k \mathrm{d}_k, \tau^{n+1-k}\mathrm{d}_k)H_n \\ = (x_n, \delta^k \tau^{n+1-k})H_n = (x_n, \tau^{n+1-k})H_n\end{aligned}$$

for $k \in [0,n]$, $x_n \in X_n$, $n \in \mathbb{N}_0$, and hence $H' = H$. □

(2.6) Definition (simplicial homotopy)**.** We let X and Y be simplicial sets and $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$ be simplicially homotopic simplicial maps. A simplicial map $X \times \Delta^1 \xrightarrow{H} Y$ such that



commutes is also called a *simplicial homotopy* from f to g. We write $f \xrightarrow{H} g$ for a simplicial homotopy from f to g.

Hence the standard 1-simplex Δ^1 plays the role that the real interval $[0,1]$ has in the category of topological spaces.
Next, we show how simplicial homotopy in one category can be transferred to simplicial homotopy in another category and introduce the related notion of a homotopy equivalence.

(2.7) Proposition. We let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor. Further, we let X and Y be simplicial objects in $\mathcal{C}$. If $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Y$ are simplicially homotopic morphisms in $\mathcal{C}$, then $FX \xrightarrow{Ff} FY$ and $FX \xrightarrow{Fg} FY$ are simplicially homotopic in $\mathcal{D}$.

Proof. We suppose that $f \sim g$ by a simplicial homotopy $(h_k \in {}_{\mathcal{C}}(X_n, Y_{n+1}) \mid k \in [0,n], n \in \mathbb{N}_0)$. Then we have

$$(Fh_n)\mathrm{d}^{FY}_{n+1} = (Fh_n)(F\mathrm{d}^Y_{n+1}) = F(h_n \mathrm{d}^Y_{n+1}) = Ff_n = (Ff)_n$$

and

$$(Fh_0)\mathrm{d}^{FY}_0 = (Fh_0)(F\mathrm{d}^Y_0) = F(h_0 \mathrm{d}^Y_0) = Fg_n = (Fg)_n$$

as well as

$$\begin{aligned}(Fh_l)\mathrm{d}^{FY}_k = (Fh_l)(F\mathrm{d}^Y_k) = F(h_l \mathrm{d}^Y_k) = \left\{\begin{array}{ll} F(\mathrm{d}^X_k h_{l-1}) & \text{for } k < l, \\ F(h_{k-1}\mathrm{d}^Y_k) & \text{for } k = l, k \neq 0, \\ F(h_k \mathrm{d}^Y_k) & \text{for } k = l+1, k \neq n+1, \\ F(\mathrm{d}^X_{k-1} h_l) & \text{for } k > l+1 \end{array}\right\} \\ = \left\{\begin{array}{ll} (F\mathrm{d}^X_k)(Fh_{l-1}) & \text{for } k < l, \\ (Fh_{k-1})(F\mathrm{d}^Y_k) & \text{for } k = l, k \neq 0, \\ (Fh_k)(F\mathrm{d}^Y_k) & \text{for } k = l+1, k \neq n+1, \\ (F\mathrm{d}^X_{k-1})(Fh_l) & \text{for } k > l+1 \end{array}\right\} = \left\{\begin{array}{ll} \mathrm{d}^{FX}_k(Fh_{l-1}) & \text{for } k < l, \\ (Fh_{k-1})\mathrm{d}^{FY}_k & \text{for } k = l, k \neq 0, \\ (Fh_k)\mathrm{d}^{FY}_k & \text{for } k = l+1, k \neq n+1, \\ \mathrm{d}^{FX}_{k-1}(Fh_l) & \text{for } k > l+1 \end{array}\right\}\end{aligned}$$

and

$$(Fh_l)\mathrm{s}_k^{FY} = (Fh_l)(F\mathrm{s}_k^Y) = F(h_l\mathrm{s}_k^Y) = \begin{Bmatrix} F(\mathrm{s}_k^X h_{l+1}) & \text{for } k \leq l, \\ F(\mathrm{s}_{k-1}^X h_l) & \text{for } k > l, \end{Bmatrix} = \begin{Bmatrix} (F\mathrm{s}_k^X)(Fh_{l+1}) & \text{for } k \leq l, \\ (F\mathrm{s}_{k-1}^X)(Fh_l) & \text{for } k > l \end{Bmatrix}$$
$$= \begin{Bmatrix} \mathrm{s}_k^{FX}(Fh_{l+1}) & \text{for } k \leq l, \\ \mathrm{s}_{k-1}^{FX}(Fh_l) & \text{for } k > l \end{Bmatrix}$$

for all $k \in [0, n+1]$, $l \in [0, n]$, $n \in \mathbb{N}_0$. Thus $(Fh_k \in {}_{\mathcal{D}}(FX_n, FY_{n+1}) \mid k \in [0, n], n \in \mathbb{N}_0)$ is a simplicial homotopy from Ff to Fg and we have $Ff \sim Fg$. □

(2.8) Definition (simplicial homotopy equivalence)**.** We let $\mathcal{C}$ be a category. Simplicial objects X and Y in $\mathcal{C}$ are said to be *simplicially homotopy equivalent* if there are simplicial morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ in $\mathcal{C}$ such that $fg \sim \mathrm{id}_X$ and $gf \sim \mathrm{id}_Y$. In this case we write $X \simeq Y$ and we call f and g mutually inverse *simplicial homotopy equivalences.*

(2.9) Proposition. We let $\mathcal{C}$ and $\mathcal{D}$ be categories and $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a functor. If there are simplicially homotopy equivalent simplicial objects X and Y in $\mathcal{C}$, then the simplicial objects FX and FY in $\mathcal{D}$ are simplicially homotopy equivalent, too.

Proof. Suppose there is a homotopy equivalence $X \xrightarrow{f} Y$ with inverse $Y \xrightarrow{g} X$. Then we have $fg \sim \mathrm{id}_X$ and $gf \sim \mathrm{id}_Y$. By proposition (2.7), we get

$$(Ff)(Fg) = F(fg) \sim F(\mathrm{id}_X) = \mathrm{id}_{FX} \text{ and } (Fg)(Ff) = F(gf) \sim F(\mathrm{id}_Y) = \mathrm{id}_{FY}.$$

Thus Ff is a simplicial homotopy equivalence between FX and FY with inverse Fg. □

We introduce some notation needed later.

(2.10) Definition (simplicially contractible simplicial set)**.** A simplicial set is called *simplicially contractible* if it is simplicially homotopy equivalent to Const $*$, where $*$ denotes a set with a single point.

(2.11) Definition (simplicial deformation retract)**.** We let X, Y be simplicial sets and $Y \xrightarrow{i} X$ be a dimensionwise injective simplicial map, that is, i_n is assumed to be injective for all $n \in \mathbb{N}_0$.

(a) We call Y a *simplicial deformation retract* of X, if there exists a simplicial map $X \xrightarrow{r} Y$ such that $ir = \mathrm{id}_Y$ and $ri \sim \mathrm{id}_X$. In this case, r is called a *simplicial deformation retraction.*

(b) We suppose given a simplicial set X', simplicial maps $X \xrightarrow{f} X'$ and $X \xrightarrow{g} X'$ and a simplicial homotopy $f \xrightarrow{H} g$. Then H is said to be *constant along* i, if $(y_n i_n, \tau^k)H_n = y_n i_n f_n = y_n i_n g_n$ for all $y_n \in Y_n$, $k \in [0, n+1]$, $n \in \mathbb{N}_0$.

(c) If there exists a homotopy $ri \xrightarrow{H} \mathrm{id}_X$ which is constant along i, we call Y a *strong simplicial deformation retract* of X and r a *strong simplicial deformation retraction.*

§2 Simplicial homology

In this section, we introduce simplicial homology for simplicial sets. To do this, we have to construct from a given simplicial set a complex in a module category, so that we can take its homology to be the homology of the given simplicial set.
We begin now with a little bit more general framework, introducing a complex associated to every simplicial object in an additive category.
Throughout this section, we let $\mathcal{A}$ be an additive category.

(2.12) Remark. We let A be a simplicial object in $\mathcal{A}$. Then the objects A_n together with the morphisms $A_n \xrightarrow{\partial} A_{n-1}$ defined by

$$\partial := \sum_{k \in [0,n]} (-1)^k \mathrm{d}_k$$

form a complex with entries in $\mathcal{A}$.

$$\ldots \xrightarrow{\partial} A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0.$$

Proof. Applying the simplicial identities, we compute

$$\begin{aligned}
\partial\partial &= \Big(\sum_{l\in[0,n+1]} (-1)^l \mathrm{d}_l\Big)\Big(\sum_{k\in[0,n]} (-1)^k \mathrm{d}_k\Big) = \sum_{l\in[0,n+1]} \sum_{k\in[0,n]} (-1)^{l+k} \mathrm{d}_l \mathrm{d}_k \\
&= \sum_{l\in[0,n+1]} \sum_{k\in[0,l-1]} (-1)^{l+k} \mathrm{d}_l \mathrm{d}_k + \sum_{l\in[0,n+1]} \sum_{k\in[l,n]} (-1)^{l+k} \mathrm{d}_l \mathrm{d}_k \\
&= \sum_{k\in[0,n+1]} \sum_{l\in[0,k-1]} (-1)^{l+k} \mathrm{d}_k \mathrm{d}_l + \sum_{k\in[0,n]} \sum_{l\in[0,k]} (-1)^{l+k} \mathrm{d}_l \mathrm{d}_k \\
&= \sum_{k\in[1,n+1]} \sum_{l\in[0,k-1]} (-1)^{l+k} \mathrm{d}_k \mathrm{d}_l + \sum_{k\in[1,n+1]} \sum_{l\in[0,k-1]} (-1)^{l+k-1} \mathrm{d}_l \mathrm{d}_{k-1} \\
&= \sum_{k\in[1,n+1]} \sum_{l\in[0,k-1]} (-1)^{l+k} (\mathrm{d}_k \mathrm{d}_l - \mathrm{d}_l \mathrm{d}_{k-1}) = 0
\end{aligned}$$

for all $n \in \mathbb{N}$. □

(2.13) Definition (associated complex)**.** For a simplicial object A in $\mathcal{A}$, the complex

$$\mathrm{C}A := (\ldots \xrightarrow{\partial} A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0)$$

with entries $\mathrm{C}_n A := (\mathrm{C}A)_n = A_n$ for $n \in \mathbb{N}_0$ and differentials

$$\partial := \sum_{k\in[0,n]} (-1)^k \mathrm{d}_k$$

for $n \in \mathbb{N}$ is said to be the *complex associated to* A.

(2.14) Proposition.

(a) Given simplicial objects A, B in $\mathcal{A}$ and a simplicial morphism $A \xrightarrow{\varphi} B$, there exists an induced complex morphism

$$\mathrm{C}A \xrightarrow{\mathrm{C}\varphi} \mathrm{C}B,$$

which is given by $\mathrm{C}_n\varphi := (\mathrm{C}\varphi)_n = \varphi_n$ for all $n \in \mathbb{N}_0$.

(b) The construction in (a) yields a functor

$$\mathbf{s}\mathcal{A} \xrightarrow{\mathrm{C}} \mathbf{C}(\mathcal{A}).$$

Proof.

(a) Since the morphisms $A_n \xrightarrow{\varphi_n} B_n$ for $n \in \mathbb{N}_0$ commute with all face maps, they commute with the differentials of $\mathrm{C}A$.

(b) We let A, B, C be simplicial objects in $\mathcal{A}$ and $A \xrightarrow{\varphi} B$, $B \xrightarrow{\varphi} C$ be simplicial morphisms. Then we obtain

$$(\mathrm{C}_n f)(\mathrm{C}_n g) = f_n g_n = (fg)_n = \mathrm{C}_n(fg)$$

and

$$\mathrm{C}_n \mathrm{id}_A = (\mathrm{id}_A)_n = \mathrm{id}_{A_n} = \mathrm{id}_{\mathrm{C}_n A} = (\mathrm{id}_{\mathrm{C}A})_n$$

for all $n \in \mathbb{N}_0$, that is, we have a functor

$$\mathbf{s}\mathcal{A} \xrightarrow{\mathrm{C}} \mathbf{C}(\mathcal{A}).$$

□

Simplicial homotopy induces complex homotopy. This will be shown now.

(2.15) Proposition. We let A and B be simplicial objects in $\mathcal{A}$. If $A \xrightarrow{f} B$ and $A \xrightarrow{g} B$ are simplicially homotopic morphisms in $\mathcal{A}$, then $\mathrm{C}A \xrightarrow{\mathrm{C}f} \mathrm{C}B$ and $\mathrm{C}A \xrightarrow{\mathrm{C}g} \mathrm{C}B$ are complex homotopic morphisms between the corresponding associated complexes.

Proof. We suppose $f \sim g$ by a simplicial homotopy $(h_k \in {}_{\mathcal{A}}(A_n, B_{n+1}) \mid k \in [0,n], n \in \mathbb{N}_0)$. For each $n \in \mathbb{N}_0$, we define a morphism $A_n \xrightarrow{h'_n} B_{n+1}$ by

$$h'_n := \sum_{k\in[0,n]} (-1)^k h_k.$$

This implies

$$\begin{aligned}
h'_n\partial + \partial h'_{n-1} &= (\sum_{l\in[0,n]} (-1)^l h_l)(\sum_{k\in[0,n+1]} (-1)^k \mathrm{d}_k) + (\sum_{k\in[0,n]} (-1)^k \mathrm{d}_k)(\sum_{l\in[0,n-1]} (-1)^l h_l) \\
&= \sum_{l\in[0,n]} \sum_{k\in[0,n+1]} (-1)^{l+k} h_l \mathrm{d}_k + \sum_{l\in[0,n-1]} \sum_{k\in[0,n]} (-1)^{l+k} \mathrm{d}_k h_l \\
&= \sum_{l\in[0,n]} \sum_{k\in[0,l-1]} (-1)^{l+k} h_l \mathrm{d}_k + \sum_{l\in[0,n]} (h_l \mathrm{d}_l - h_l \mathrm{d}_{l+1}) + \sum_{l\in[0,n]} \sum_{k\in[l+2,n+1]} (-1)^{l+k} h_l \mathrm{d}_k \\
&\quad + \sum_{l\in[0,n-1]} \sum_{k\in[0,n]} (-1)^{l+k} \mathrm{d}_k h_l \\
&= \sum_{l\in[1,n]} \sum_{k\in[0,l-1]} (-1)^{l+k} \mathrm{d}_k h_{l-1} + (h_0 \mathrm{d}_0 - h_n \mathrm{d}_{n+1}) + \sum_{l\in[0,n-1]} \sum_{k\in[l+2,n+1]} (-1)^{l+k} \mathrm{d}_{k-1} h_l \\
&\quad + \sum_{l\in[0,n-1]} \sum_{k\in[0,n]} (-1)^{l+k} \mathrm{d}_k h_l \\
&= g_n - f_n - \sum_{l\in[0,n-1]} \sum_{k\in[0,l]} (-1)^{l+k} \mathrm{d}_k h_l - \sum_{l\in[0,n-1]} \sum_{k\in[l+1,n]} (-1)^{l+k} \mathrm{d}_k h_l \\
&\quad + \sum_{l\in[0,n-1]} \sum_{k\in[0,n]} (-1)^{l+k} \mathrm{d}_k h_l \\
&= g_n - f_n
\end{aligned}$$

for all non-negative integer $n \in \mathbb{N}_0$ and hence $(h'_n \in {}_{\mathcal{A}}(A_n, B_{n+1}) \mid n \in \mathbb{N}_0)$ is a complex homotopy between $\mathrm{C}f$ and $\mathrm{C}g$. □

(2.16) Proposition. We let A and B be simplicial objects in $\mathcal{A}$. If A and B are simplicially homotopy equivalent simplicial objects in $\mathcal{A}$, then the associated complexes $\mathrm{C}A$ and $\mathrm{C}B$ are homotopy equivalent complexes with entries in $\mathcal{A}$.

Proof. We let $A \xrightarrow{f} B$ be a simplicial homotopy equivalence with inverse $B \xrightarrow{g} A$, that is, $fg \sim \mathrm{id}_A$ and $gf \sim \mathrm{id}_B$. By proposition (2.15), we get

$$(\mathrm{C}f)(\mathrm{C}g) = \mathrm{C}(fg) \sim \mathrm{C}\mathrm{id}_A = \mathrm{id}_{\mathrm{C}A} \text{ and } (\mathrm{C}g)(\mathrm{C}f) = \mathrm{C}(gf) \sim \mathrm{C}\mathrm{id}_B = \mathrm{id}_{\mathrm{C}B}.$$

Hence $\mathrm{C}A$ is homotopy equivalent to $\mathrm{C}B$ since $\mathrm{C}f$ is a homotopy equivalence with inverse $\mathrm{C}g$. □

Now we apply definition (2.13) to the context we have in mind.

(2.17) Definition (associated complex to a simplicial set)**.** We let R be a commutative ring. For a simplicial set X, the complex $\mathrm{C}(X; R) := \mathrm{C}RX$ is called the *complex associated to* X *over* R. If $R = \mathbb{Z}$, we will just speak of the *complex associated to* X and we write $\mathrm{C}(X) := \mathrm{C}(X; \mathbb{Z})$.

(2.18) Definition (simplicial homology and cohomology of a simplicial set)**.** We let R be a commutative ring, M an R-module and X a simplicial set. The *nth* (*simplicial*) *homology group* of X with coefficients in M over R is defined to be

$$\mathrm{H}_n(X, M; R) := \mathrm{H}_n(\mathrm{C}(X; R) \otimes_R M).$$

The *nth* (*simplicial*) *cohomology group* of X with coefficients in M over R is defined to be

$$\mathrm{H}^n(X, M; R) := \mathrm{H}^n({}_R(\mathrm{C}(X; R), M)).$$

Moreover, we abbreviate

$$\mathrm{H}_n(X; R) := \mathrm{H}_n(X, R; R),$$
$$\mathrm{H}_n(X, M) := \mathrm{H}_n(X, M; \mathbb{Z}),$$
$$\mathrm{H}_n(X) := \mathrm{H}_n(X, \mathbb{Z}; \mathbb{Z})$$

and

$$\mathrm{H}^n(X; R) := \mathrm{H}^n(X, R; R),$$
$$\mathrm{H}^n(X, M) := \mathrm{H}^n(X, M; \mathbb{Z}),$$
$$\mathrm{H}^n(X) := \mathrm{H}^n(X, \mathbb{Z}; \mathbb{Z}).$$

(2.19) Example (singular homology of a topological space)**.** Given a topological space T, then the integral singular homology groups (cf. example (1.10)) are defined to be

$$\mathrm{H}_n(T; \mathbb{Z}) := \mathrm{H}_n(\mathbb{Z}\mathrm{S}T) \text{ for } n \in \mathbb{N}_0.$$

(2.20) Proposition. Given simplicially homotopy equivalent simplicial sets X and Y, we have

$$\mathrm{H}_n(X, M; R) \cong \mathrm{H}_n(Y, M; R) \text{ and } \mathrm{H}^n(X, M; R) \cong \mathrm{H}^n(Y, M; R)$$

for all $n \in \mathbb{N}_0$ and every module M over a commutative ring R.

Proof. We let $X, Y \in \mathrm{Ob}\,\mathbf{sSet}$ with $X \simeq Y$ and we let R be a commutative ring and M be an R-module. It follows from proposition (2.9) and proposition (2.16) that $\mathrm{C}(X; R) \simeq \mathrm{C}(Y; R)$. Since the functors $- \otimes_R M$ and ${}_R(-, M)$ preserve homotopy equivalences, the complex homotopy invariance of the homology functor implies the asserted statements. □

§3 The Moore complex

In the last section we attached a complex C to a given simplicial object A in an additive category $\mathcal{A}$. Here we will introduce a subcomplex $\mathrm{M}A$ of $\mathrm{C}A$ if $\mathcal{A}$ is an abelian category. At the end, we will show that both complexes are homotopy equivalent and so deliver the same homology.
We suppose given an abelian category $\mathcal{A}$.

(2.21) Remark. We let A be a simplicial object in $\mathcal{A}$. We let

$$\mathrm{M}_n A := \bigcap_{k \in [1,n]} \mathrm{Ker}\, \mathrm{d}_k \text{ for all } n \in \mathbb{N}_0.$$

Then we may let

$$\mathrm{M}_n A \xrightarrow{\partial} \mathrm{M}_{n-1} A$$

be induced by $A_n \xrightarrow{\mathrm{d}_0} A_{n-1}$ for all $n \in \mathbb{N}$, forming a complex

$$\mathrm{M}A := (\ldots \xrightarrow{\partial} \mathrm{M}_2 A \xrightarrow{\partial} \mathrm{M}_1 A \xrightarrow{\partial} \mathrm{M}_0 A).$$

Proof. We have

$$(\mathrm{M}_n A)\mathrm{d}_0\mathrm{d}_k = (\mathrm{M}_n A)\mathrm{d}_{k+1}\mathrm{d}_0 = 0 \text{ for all } k \in [0, n-1],$$

that is, $(\mathrm{M}_n A)\mathrm{d}_0 \preceq \mathrm{Ker}\,\mathrm{d}_k$ for all $k \in [0, n-1]$ and therefore $(\mathrm{M}_n A)\mathrm{d}_0 \preceq \bigcap_{k \in [1,n-1]} \mathrm{Ker}\,\mathrm{d}_k = \mathrm{M}_{n-1} A$. Altogether, the differentials

$$\mathrm{M}_n A \xrightarrow{\partial} \mathrm{M}_{n-1} A$$

are well-defined and fulfill $\partial\partial = 0$. □

(2.22) Definition (Moore complex)**.** We let A be a simplicial object in $\mathcal{A}$. The complex $\mathrm{M}A$ given as in remark (2.21) is called the *Moore complex* of A.

(2.23) Proposition. Given a simplicial object A in $\mathcal{A}$, the Moore complex $\mathrm{M}A$ is a subcomplex of the associated complex $\mathrm{C}A$.

Proof. Letting $\mathrm{M}_n A \xrightarrow{\iota_n} A_n$ denote the embedding of $\mathrm{M}_n A$ into A_n for all $n \in \mathbb{N}_0$, we have

$$\iota_n \partial^{\mathrm{C}A} = \iota_n \Big(\sum_{k \in [0,n]} (-1)^k \mathrm{d}_k \Big) = \sum_{k \in [0,n]} (-1)^k \iota_n \mathrm{d}_k = \iota_n \mathrm{d}_0 = \partial^{\mathrm{M}A} \iota_{n-1},$$

because ι_n factorises over each kernel of d_k for $k \in [0,n]$, $n \in \mathbb{N}$. Hence we have a commutative diagram

$$\begin{array}{ccc} \mathrm{M}_n A & \xrightarrow{\partial^{\mathrm{M}A}} & \mathrm{M}_{n-1} A \\ {\scriptstyle \iota_n}\downarrow & \searrow^{\iota_n \mathrm{d}_0} & \downarrow{\scriptstyle \iota_{n-1}} \\ A_n & \xrightarrow{\partial^{\mathrm{C}A}} & A_{n-1} \end{array}$$

for each $n \in \mathbb{N}$, that is, the ι_n for $n \in \mathbb{N}_0$ yield a complex monomorphism

$$\mathrm{M}A \xrightarrow{\iota} \mathrm{C}A.$$

□

(2.24) Proposition.

(a) Given simplicial objects A, B in $\mathcal{A}$ and a simplicial morphism $A \xrightarrow{\varphi} B$, there exists an induced complex morphism

$$\mathrm{M}A \xrightarrow{\mathrm{M}\varphi} \mathrm{M}B,$$

where $\mathrm{M}_n \varphi := (\mathrm{M}\varphi)_n$ is induced by φ_n for all $n \in \mathbb{N}_0$.

(b) The construction in (a) yields a functor

$$\mathbf{s}\mathcal{A} \xrightarrow{\mathrm{M}} \mathbf{C}(\mathcal{A}).$$

Proof.

(a) The morphisms $A_n \xrightarrow{\varphi_n} B_n$ induce morphisms $\mathrm{M}_n A \xrightarrow{\mathrm{M}_n \varphi} \mathrm{M}_n B$ for $n \in \mathbb{N}_0$. Denoting the embeddings from $\mathrm{M}_n A$ into A_n resp. from $\mathrm{M}_n B$ into B_n by

$$\mathrm{M}_n A \xrightarrow{\iota_n^A} A_n \text{ resp. } \mathrm{M}_n B \xrightarrow{\iota_n^B} B_n,$$

we compute

$$\partial^{\mathrm{M}A} (\mathrm{M}_{n-1} \varphi) \iota_{n-1}^B = \partial^{\mathrm{M}A} \iota_{n-1}^A \varphi_{n-1} = \iota_n^A \partial^{\mathrm{C}A} \varphi_{n-1} = \iota_n^A \varphi_n \partial^{\mathrm{C}B} = \varphi_n \iota_n^B \partial^{\mathrm{C}B} = \varphi_n \partial^{\mathrm{M}B} \iota_{n-1}^B$$

and since ι_{n-1}^B is a monomorphism, this implies $\partial (\mathrm{M}_{n-1} \varphi) = \varphi_n \partial$ for all $n \in \mathbb{N}_0$. Hence the morphisms $\mathrm{M}_n \varphi$ for $n \in \mathbb{N}_0$ yield a complex morphism $\mathrm{M}A \xrightarrow{\mathrm{M}\varphi} \mathrm{M}B$.

(b) We suppose given simplicial objects A, B, C in $\mathcal{A}$ and simplicial morphisms $A \xrightarrow{\varphi} B$ and $B \xrightarrow{\varphi} C$. The embeddings are denoted as in (a) by

$$\mathrm{M}_n A \xrightarrow{\iota_n^A} A_n \text{ resp. } \mathrm{M}_n B \xrightarrow{\iota_n^B} B_n \text{ resp. } \mathrm{M}_n C \xrightarrow{\iota_n^C} C_n$$

for all $n \in \mathbb{N}_0$. Then we have

$$(\mathrm{M}_n \varphi)(\mathrm{M}_n \psi) \iota_n^C = (\mathrm{M}_n \varphi) \iota_n^B \psi_n = \iota_n^A \varphi_n \psi_n = \iota_n^A (\varphi \psi)_n = \mathrm{M}_n (\varphi \psi) \iota_n^C$$

and

$$(\mathrm{M}_n \mathrm{id}_A)\iota_n^A = \iota_n^A(\mathrm{id}_A)_n = \iota_n^A \mathrm{id}_{A_n} = \mathrm{id}_{\mathrm{M}_n A}\iota_n^A = (\mathrm{id}_{\mathrm{M}A})_n \iota_n^A$$

for all $n \in \mathbb{N}_0$. Since ι_n^C and ι_n^A are monomorphisms, we conclude that $(\mathrm{M}_n\varphi)(\mathrm{M}_n\psi) = \mathrm{M}_n(\varphi\psi)$ and $\mathrm{M}_n\mathrm{id}_A = (\mathrm{id}_{\mathrm{M}A})_n$ for all $n \in \mathbb{N}_0$. Thus we have $(\mathrm{M}\varphi)(\mathrm{M}\psi) = \mathrm{M}(\varphi\psi)$ and $\mathrm{Mid}_A = \mathrm{id}_{\mathrm{M}A}$ and hence, we have a functor

$$\mathbf{s}\mathcal{A} \xrightarrow{\mathrm{M}} \mathbf{C}(\mathcal{A}). \qquad \square$$

Our next aim is to ameliorate this proposition in the sense that we intend to show that the Moore complex is even a direct summand of the associated complex. A complement can be specified, namely the degenerate subcomplex. This will be introduced now.

(2.25) Remark. There exists a subcomplex $\mathrm{D}A \preceq \mathrm{C}A$ for every simplicial object A in $\mathcal{A}$ with $\mathrm{D}_n A := \sum_{k\in[0,n-1]} \mathrm{Im}\, \mathrm{s}_j$ for each $n \in \mathbb{N}_0$.

Proof. We have

$$\begin{aligned}(\mathrm{Im}\, \mathrm{s}_k)\partial &= \mathrm{Im}(\mathrm{s}_k\partial) = \mathrm{Im}(\mathrm{s}_k \sum_{l\in[0,n]} (-1)^l \mathrm{d}_l) = \mathrm{Im}(\sum_{l\in[0,k-1]} (-1)^l \mathrm{d}_l \mathrm{s}_{k-1} + \sum_{l\in[k+2,n]} (-1)^l \mathrm{d}_{l-1}\mathrm{s}_k) \\ &\preceq \sum_{l\in[0,k-1]} \mathrm{Im}(\mathrm{d}_l \mathrm{s}_{k-1}) + \sum_{l\in[k+2,n]} \mathrm{Im}(\mathrm{d}_{l-1}\mathrm{s}_k) \preceq \sum_{l\in[0,k-1]} \mathrm{Im}\, \mathrm{s}_{k-1} + \sum_{l\in[k+2,n]} \mathrm{Im}\, \mathrm{s}_k \preceq \sum_{m\in[0,n-2]} \mathrm{Im}\, \mathrm{s}_m \\ &= \mathrm{D}_{n-1}A\end{aligned}$$

and hence

$$(\mathrm{D}_n A)\partial \preceq \mathrm{D}_{n-1}A$$

for all $n \in \mathbb{N}$, that is, the subobjects $\mathrm{D}_n A \preceq A_n$ for $n \in \mathbb{N}_0$ yield a subcomplex $\mathrm{D}A \preceq \mathrm{C}A$. $\square$

(2.26) Definition (degenerate complex)**.** We let $A \in \mathrm{Ob}\, \mathbf{s}\mathcal{A}$ be a simplicial object in $\mathcal{A}$. The subcomplex $\mathrm{D}A \preceq \mathrm{C}A$ given as in remark (2.25) with $\mathrm{D}_n A = \sum_{k\in[0,n-1]} \mathrm{Im}\, \mathrm{s}_k$ for all $n \in \mathbb{N}_0$ is called the *degenerate complex* of A.

(2.27) Proposition. For $k \in [0, \alpha]$, $\alpha \in \mathbb{N}$, we let $F(-k) \in \mathbf{C}(\mathcal{A})$ be complexes with entries in $\mathcal{A}$. Further, we let

$$F(-k+1) \xrightarrow{\varphi(-k)} F(-k) \text{ and } F(-k) \xrightarrow{\psi(-k)} F(-k+1) \text{ for } k \in [1, \alpha]$$

be complex morphisms and we let $(h(-k)_n \mid n \in \mathbb{Z})$ be a complex homotopy from $\mathrm{id}_{F(-k+1)}$ to $\varphi(-k)\psi(-k)$, $k \in [1, \alpha]$. Note that $h(-k)_n$ is a morphism from $F(-k+1)_n$ to $F(-k+1)_{n+1}$. Then we have a complex homotopy $(H(-\alpha)_n \mid n \in \mathbb{Z})$ from $\mathrm{id}_{F(0)}$ to $\varphi(-1)\dots\varphi(-\alpha)\psi(-\alpha)\dots\psi(-1)$ given by

$$H(-\alpha)_n = \sum_{k\in[1,\alpha]} \varphi(-1)_n \dots \varphi(-k+1)_n h(-k)_n \psi(-k+1)_{n+1} \dots \psi(-1)_{n+1} \text{ for all } n \in \mathbb{Z}.$$

Proof. We proceed by induction on $\alpha \in \mathbb{N}$. For $\alpha = 1$, we have $H(-1)_n = h(-1)_n$ for all $n \in \mathbb{Z}$ and there is nothing to show. Now we assume that $\alpha > 1$ and that the assertion holds for all smaller natural numbers. Then we compute

$$\begin{aligned}&\mathrm{id}_{F(0)_n} - \varphi(-1)_n \dots \varphi(-\alpha)_n \psi(-\alpha)_n \dots \psi(-1)_n \\ &= \mathrm{id}_{F(0)_n} - \varphi(-1)_n\psi(-1)_n + \varphi(-1)_n\psi(-1)_n - \varphi(-1)_n \dots \varphi(-\alpha)_n\psi(-\alpha)_n \dots \psi(-1)_n \\ &= \mathrm{id}_{F(0)_n} - \varphi(-1)_n\psi(-1)_n + \varphi(-1)_n(\mathrm{id}_{F(-1)_n} - \varphi(-2)_n \dots \varphi(-\alpha)_n\psi(-\alpha)_n \dots \psi(-2)_n)\psi(-1)_n \\ &= h(-1)_n\partial + \partial h(-1)_{n-1} + \varphi(-1)_n\Big((\sum_{k\in[2,\alpha]} \varphi(-2)_n \dots \varphi(-k+1)_n h(-k)_n \psi(-k+1)_{n+1} \dots \psi(-2)_{n+1})\partial \\ &\qquad + \partial(\sum_{k\in[2,\alpha]} \varphi(-2)_{n-1} \dots \varphi(-k+1)_{n-1} h(-k)_{n-1}\psi(-k+1)_n \dots \psi(-2)_n)\Big)\psi(-1)_n\end{aligned}$$

$$= h(-1)_n\partial + \partial h(-1)_{n-1} + (\sum_{k\in[2,\alpha]} \varphi(-1)_n \dots \varphi(-k+1)_n h(-k)_n \psi(-k+1)_{n+1} \dots \psi(-2)_{n+1})\partial\psi(-1)_n$$
$$+ \varphi(-1)_n\partial(\sum_{k\in[2,\alpha]} \varphi(-2)_{n-1} \dots \varphi(-k+1)_{n-1} h(-k)_{n-1} \psi(-k+1)_n \dots \psi(-1)_n)$$
$$= h(-1)_n\partial + \partial h(-1)_{n-1} + (\sum_{k\in[2,\alpha]} \varphi(-1)_n \dots \varphi(-k+1)_n h(-k)_n \psi(-k+1)_{n+1} \dots \psi(-2)_{n+1})\psi(-1)_{n+1}\partial$$
$$+ \partial\varphi(-1)_{n-1}(\sum_{k\in[2,\alpha]} \varphi(-2)_{n-1} \dots \varphi(-k+1)_{n-1} h(-k)_{n-1} \psi(-k+1)_n \dots \psi(-1)_n)$$
$$= h(-1)_n\partial + \partial h(-1)_{n-1} + (\sum_{k\in[2,\alpha]} \varphi(-1)_n \dots \varphi(-k+1)_n h(-k)_n \psi(-k+1)_{n+1} \dots \psi(-1)_{n+1})\partial$$
$$+ \partial(\sum_{k\in[2,\alpha]} \varphi(-1)_{n-1} \dots \varphi(-k+1)_{n-1} h(-k)_{n-1} \psi(-k+1)_n \dots \psi(-1)_n)$$
$$= (\sum_{k\in[1,\alpha]} \varphi(-1)_n \dots \varphi(-k+1)_n h(-k)_n \psi(-k+1)_{n+1} \dots \psi(-1)_{n+1})\partial$$
$$+ \partial(\sum_{k\in[1,\alpha]} \varphi(-1)_{n-1} \dots \varphi(-k+1)_{n-1} h(-k)_{n-1} \psi(-k+1)_n \dots \psi(-1)_n)$$
$$= H(-\alpha)_n\partial + \partial H(-\alpha)_{n-1}$$

for all $n \in \mathbb{Z}$ and so $(H(-\alpha)_n \mid n \in \mathbb{Z})$ is a homotopy form $\mathrm{id}_{F(0)}$ to $\varphi(-1)\dots\varphi(-\alpha)\psi(-\alpha)\dots\psi(-1)$. □

(2.28) Theorem (normalisation theorem)**.** We have

$$\mathrm{C}A \cong \mathrm{D}A \oplus \mathrm{M}A \text{ and } \mathrm{C}A \simeq \mathrm{M}A$$

for each simplicial object $A \in \mathrm{Ob}\,\mathbf{s}\mathcal{A}$ in $\mathcal{A}$.

Proof. In the first step, we construct a pointwise finite pointwise split filtration from $\mathrm{M}A$ to $\mathrm{C}A$. Thereto, we let

$$F(-\alpha)_n := \bigcap_{k\in[\max(1,n-\alpha+1),n]} \mathrm{Ker}\,\mathrm{d}_k \text{ for all } \alpha, n \in \mathbb{N}_0,$$

and we denote by $F(-\alpha)_n \xrightarrow{\mu(\alpha)_n} A_n$ the embedding from $F(-\alpha)_n$ into A_n. We let a non-negative number $\alpha \in \mathbb{N}_0$ be given. Since

$$(F(-\alpha)_n)\mathrm{d}_k\mathrm{d}_l = (F(-\alpha)_n)\mathrm{d}_{l+1}\mathrm{d}_k = 0 \text{ for all } k \in [0, \max(0, n-\alpha)], l \in [\max(0, n-\alpha), n-1],$$

we have $F(-\alpha)_n\mathrm{d}_k \preceq \mathrm{Ker}\,\mathrm{d}_l$ for all $k \in [0, \max(0, n-\alpha)]$, $l \in [\max(0, n-\alpha), n-1]$, and therefore

$$F(-\alpha)_n\partial = (F(-\alpha)_n)(\sum_{k\in[0,n]} (-1)^k\mathrm{d}_k) = (F(-\alpha)_n)(\sum_{k\in[0,\max(0,n-\alpha)]} (-1)^k\mathrm{d}_k) \preceq \bigcap_{l\in[\max(1,n-\alpha),n-1]} \mathrm{Ker}\,\mathrm{d}_l$$
$$= F(-\alpha)_{n-1}.$$

Hence we have induced morphisms

$$F(-\alpha)_n \xrightarrow{\partial} F(-\alpha)_{n-1}$$

given by the commutative diagram

$$\begin{array}{ccc} F(-\alpha)_n & \xrightarrow{\partial} & F(-\alpha)_{n-1} \\ \downarrow{\scriptstyle \mu(-\alpha)_n} & & \downarrow{\scriptstyle \mu(-\alpha)_{n-1}} \\ A_n & \xrightarrow{\partial} & A_{n-1} \end{array}$$

where the vertical morphisms are the canonical embeddings. Since

$$\partial\partial\mu(-\alpha)_{n-2} = \partial\mu(-\alpha)_{n-1}\partial = \mu(-\alpha)_n\partial\partial = 0$$

and since $\mu(-\alpha_{n-2})$ is a monomorphism, we have $\partial\partial = 0$ as morphisms $F(-\alpha)_n \longrightarrow F(-\alpha)_{n-2}$ for all $n \in \mathbb{N}$, $n \geq 2$, that is, the objects $F(-\alpha)_n$ for $n \in \mathbb{N}_0$ together with the morphisms ∂ yield a complex $F(-\alpha) \preceq \mathrm{C}A$. We have embeddings

$$F(-\alpha)_n \xrightarrow{\iota(-\alpha)_n} F(-\alpha+1)_n \text{ for each } \alpha \in \mathbb{N}, n \in \mathbb{N}_0,$$

and we have

$$\mu(-\alpha)_n = \iota(-\alpha)_n\iota(-\alpha+1)_n \ldots \iota(-1)_n.$$

These embeddings yield complex morphisms

$$F(-\alpha) \xrightarrow{\iota(-\alpha)} F(-\alpha+1) \text{ for all } \alpha \in \mathbb{N},$$

because we have commutative diagrams

$$\begin{array}{ccc}
F(-\alpha)_n & \xrightarrow{\partial} & F(-\alpha)_{n-1} \\
{\scriptstyle \iota(-\alpha)_n}\downarrow & & \downarrow{\scriptstyle \iota(-\alpha)_{n-1}} \\
F(-\alpha+1)_n & \xrightarrow{\partial} & F(-\alpha+1)_{n-1} \\
{\scriptstyle \mu(-\alpha+1)_n}\downarrow & & \downarrow{\scriptstyle \mu(-\alpha+1)_{n-1}} \\
\mathrm{C}_nA & \xrightarrow{\partial} & \mathrm{C}_{n-1}A
\end{array}$$

for all $\alpha, n \in \mathbb{N}$ (the upper square commutes since $\mu(-\alpha+1)_{n-1}$ is a monomorphism). Since $F(-\alpha)_n = \mathrm{M}_nA$ for all $\alpha \in \mathbb{N}_0$, $\alpha \geq n$, the complexes $F(-\alpha)$ for $\alpha \in \mathbb{N}_0$ yield a pointwise finite filtration from $\mathrm{M}A$ to $\mathrm{C}A$.
To show that this filtration is pointwise split, we consider the morphisms

$$A_n \xrightarrow{\mathrm{id}-\mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}} A_n$$

for $\alpha \in [1, n]$, $n \in \mathbb{N}_0$. For $k \geq n - \alpha + 1$, we compute

$$\begin{aligned}
\mu(-\alpha+1)_n(\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha})\mathrm{d}_k &= \mu(-\alpha+1)_n(\mathrm{d}_k - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}\mathrm{d}_k) \\
&= \left\{\begin{array}{ll} \mu(-\alpha+1)_n(\mathrm{d}_k - \mathrm{d}_k) & \text{if } k = n-\alpha+1, \\ \mu(-\alpha+1)_n\mathrm{d}_k(\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}) & \text{if } k \geq n-\alpha+2 \end{array}\right\} \\
&= 0,
\end{aligned}$$

that is, we have an induced morphism $F(-\alpha+1)_n \xrightarrow{\pi(-\alpha)_n} F(-\alpha)_n$ such that the diagram

$$\begin{array}{ccc}
F(-\alpha+1)_n & \xrightarrow{\pi(-\alpha)_n} & F(-\alpha)_n \\
{\scriptstyle \mu(-\alpha+1)_n}\downarrow & & \downarrow{\scriptstyle \mu(-\alpha)_n} \\
A_n & \xrightarrow{\mathrm{id}-\mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}} & A_n
\end{array}$$

commutes. Additionally, we define $F(-\alpha+1)_n \xrightarrow{\pi(-\alpha)_n} F(-\alpha)_n$ for $\alpha > n$ by

$$\pi(-\alpha)_n := \mathrm{id}_{\mathrm{M}_nA}.$$

Then we have

$$(\mathrm{id} - \mathrm{d}_1\mathrm{s}_0)\mathrm{d}_0 = \mathrm{d}_0 - \mathrm{d}_1\mathrm{s}_0\mathrm{d}_0 = \mathrm{d}_0 - \mathrm{d}_1.$$

and

$$\begin{aligned}
&(\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}) \sum_{k\in[0,n-\alpha]} (-1)^k \mathrm{d}_k = \sum_{k\in[0,n-\alpha]} (-1)^k (\mathrm{d}_k - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}\mathrm{d}_k) \\
&= \sum_{k\in[0,n-\alpha-1]} (-1)^k (\mathrm{d}_k - \mathrm{d}_k\mathrm{d}_{n-\alpha}\mathrm{s}_{n-\alpha-1}) + (-1)^{n-\alpha}(\mathrm{d}_{n-\alpha} - \mathrm{d}_{n-\alpha+1}) \\
&= \sum_{k\in[0,n-\alpha-1]} (-1)^k \mathrm{d}_k(\mathrm{id} - \mathrm{d}_{n-\alpha}\mathrm{s}_{n-1-\alpha}) + (-1)^{n-\alpha}\mathrm{d}_{n-\alpha} + (-1)^{n-\alpha+1}\mathrm{d}_{n-\alpha+1} \\
&= \sum_{k\in[0,n-\alpha+1]} (-1)^k \mathrm{d}_k(\mathrm{id} - \mathrm{d}_{n-\alpha}\mathrm{s}_{n-1-\alpha})
\end{aligned}$$

for all $\alpha \in [1, n-1]$, $n \in \mathbb{N}$. This yields

$$\begin{aligned}
\pi(-n)_n\partial\mu(-n)_{n-1} &= \pi(-n)_n\mu(-n)_n\partial = \pi(-n)_n\mu(-n)_n\mathrm{d}_0 = \mu(-n+1)_n(\mathrm{id} - \mathrm{d}_1\mathrm{s}_0)\mathrm{d}_0 \\
&= \mu(-n+1)_n(\mathrm{d}_0 - \mathrm{d}_1) = \mu(-n+1)_n\partial = \partial\mu(-n+1)_{n-1} = \partial\pi(-n)_{n-1}\mu(-n)_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
\pi(-\alpha)_n\partial\mu(-\alpha)_{n-1} &= \pi(-\alpha)_n\mu(-\alpha)_n\partial = \pi(-\alpha)_n\mu(-\alpha)_n(\sum_{k\in[0,n-\alpha]} (-1)^k\mathrm{d}_k) \\
&= \mu(-\alpha+1)_n(\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha})(\sum_{k\in[0,n-\alpha]} (-1)^k\mathrm{d}_k) \\
&= \mu(-\alpha+1)_n(\sum_{k\in[0,n-\alpha+1]} (-1)^k\mathrm{d}_k)(\mathrm{id} - \mathrm{d}_{n-\alpha}\mathrm{s}_{n-1-\alpha}) \\
&= \mu(-\alpha+1)_n\partial(\mathrm{id} - \mathrm{d}_{n-\alpha}\mathrm{s}_{n-1-\alpha}) \\
&= \partial\mu(-\alpha+1)_{n-1}(\mathrm{id} - \mathrm{d}_{n-\alpha}\mathrm{s}_{n-1-\alpha}) = \partial\pi(-\alpha)_{n-1}\mu(-\alpha)_{n-1}
\end{aligned}$$

whence $\pi(-\alpha)_n\partial = \partial\pi(-\alpha)_{n-1}$ for all $\alpha \in [1, n]$, $n \in \mathbb{N}$. Since additionally $\pi(-\alpha)_n\partial = \mathrm{id}\partial = \partial\mathrm{id} = \partial\pi(-\alpha)_{n-1}$ for all $\alpha > n$, $n \in \mathbb{N}$, we have proven that the morphisms $\pi(-\alpha)_n$ yield complex morphisms

$$F(-\alpha+1) \xrightarrow{\pi(-\alpha)} F(-\alpha).$$

We get

$$\begin{aligned}
\iota(-\alpha)_n\pi(-\alpha)_n\mu(-\alpha)_n &= \left\{\begin{matrix} \iota(-\alpha)_n\mu(-\alpha+1)_n(\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}) & \text{for } \alpha \le n, \\ \iota(-\alpha)_n\mu(-\alpha+1)_n & \text{for } \alpha > n \end{matrix}\right\} \\
&= \left\{\begin{matrix} \mu(-\alpha)_n(\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}) & \text{for } \alpha \le n, \\ \mu(-\alpha)_n & \text{for } \alpha > n \end{matrix}\right\} = \mu(-\alpha)_n
\end{aligned}$$

for all $n \in \mathbb{N}_0$ and therefore $\iota(-\alpha)\pi(-\alpha) = \mathrm{id}_{F(-\alpha)}$ for each $\alpha \in \mathbb{N}$.
Now we recall resp. define

$$\mathrm{M}_nA \xrightarrow{\iota_n} \mathrm{C}_nA \text{ and } \mathrm{C}_nA \xrightarrow{\pi_n} \mathrm{M}_nA$$

by

$$\iota_n = \mu(-n)_n = \iota(-n)_n\iota(-n+1)_n \ldots \iota(-1)_n \text{ and } \pi_n := \pi(-1)_n \ldots \pi(-n+1)_n\pi(-n)_n \text{ for all } n \in \mathbb{N}_0.$$

The morphisms ι_n for $n \in \mathbb{N}_0$ yield a morphism of complexes by proposition (2.23). Additionally, the morphisms π_n for $n \in \mathbb{N}_0$ yield a complex morphism

$$\mathrm{C}A \xrightarrow{\pi} \mathrm{M}A$$

since we have

$$\pi_n\partial^{\mathrm{M}A} = \pi(-1)_n \ldots \pi(-n+1)_n\pi(-n)_n\mathrm{d}_0 = \pi(-1)_n \ldots \pi(-n+1)_n\pi(-n)_n\partial^{F(-n)}$$

$$= \pi(-1)_n \dots \pi(-n+1)_n \partial^{F(-n+1)} \pi(-n)_{n-1} = \partial^{F(0)} \pi(-1)_{n-1} \dots \pi(-n+1)_{n-1} \mathrm{id} = \partial^{\mathrm{C}A} \pi_{n-1}$$

for all $n \in \mathbb{N}$. Because of

$$\iota_n \pi_n = \iota(-n) \dots \iota(-1) \pi(-1) \dots \pi(-n) = \mathrm{id}_{\mathrm{M}_n A} \text{ for all } n \in \mathbb{N}_0$$

we obtain a split exact sequence

$$\operatorname{Ker} \pi \longrightarrow \mathrm{C}A \xrightarrow{\pi} \mathrm{M}A,$$

and thus $\mathrm{C}A \cong (\operatorname{Ker}\pi) \oplus \mathrm{M}A$. To show the desired decomposition of $\mathrm{C}A$, it remains to prove that $\operatorname{Ker}\pi \cong \mathrm{D}A$. We let $n \in \mathbb{N}_0$ be a non-negative integer. First, we have

$$\begin{aligned} \pi_n \iota_n &= \pi(-1) \dots \pi(-n) \mu(-n) = \pi(-1) \dots \pi(-n+1) \mu(-n+1)(\mathrm{id} - \mathrm{d}_1 \mathrm{s}_0) \\ &= \pi(-1) \dots \pi(-n+2) \mu(-n+2)(\mathrm{id} - \mathrm{d}_2 \mathrm{s}_1)(\mathrm{id} - \mathrm{d}_1 \mathrm{s}_0) = \dots \\ &= (\mathrm{id} - \mathrm{d}_n \mathrm{s}_{n-1})(\mathrm{id} - \mathrm{d}_{n-1} \mathrm{s}_{n-2}) \dots (\mathrm{id} - \mathrm{d}_2 \mathrm{s}_1)(\mathrm{id} - \mathrm{d}_1 \mathrm{s}_0), \end{aligned}$$

so $\pi_n \iota_n$ has the form $\pi_n \iota_n = \mathrm{id} - \varphi$, where φ is a signed sum of morphisms that can be written as composites of faces with at least one degeneracy. Note that if $\operatorname{Ker}(\mathrm{id} - \varphi) \xrightarrow{\kappa} \mathrm{C}_n A$ denotes a kernel of $\mathrm{id} - \varphi$, then $\kappa = \kappa\varphi$. Hence

$$\operatorname{Ker} \pi_n = \operatorname{Ker}(\pi_n \iota_n) = \operatorname{Ker}(\mathrm{id} - \varphi) \preceq \operatorname{Im} \varphi \preceq \mathrm{D}_n A.$$

Conversely, we have

$$\begin{aligned} \mathrm{s}_k \pi_n \iota_n &= \mathrm{s}_k (\mathrm{id} - \mathrm{d}_n \mathrm{s}_{n-1}) \dots (\mathrm{id} - \mathrm{d}_{k+2} \mathrm{s}_{k+1})(\mathrm{id} - \mathrm{d}_{k+1} \mathrm{s}_k)(\mathrm{id} - \mathrm{d}_k \mathrm{s}_{k-1}) \dots (\mathrm{id} - \mathrm{d}_1 \mathrm{s}_0) \\ &= (\mathrm{id} - \mathrm{d}_{n-1} \mathrm{s}_{n-2}) \dots (\mathrm{id} - \mathrm{d}_{k+1} \mathrm{s}_k) \mathrm{s}_k (\mathrm{id} - \mathrm{d}_{k+1} \mathrm{s}_k)(\mathrm{id} - \mathrm{d}_k \mathrm{s}_{k-1}) \dots (\mathrm{id} - \mathrm{d}_1 \mathrm{s}_0) \\ &= (\mathrm{id} - \mathrm{d}_{n-1} \mathrm{s}_{n-2}) \dots (\mathrm{id} - \mathrm{d}_{k+1} \mathrm{s}_k)(\mathrm{s}_k - \mathrm{s}_k)(\mathrm{id} - \mathrm{d}_k \mathrm{s}_{k-1}) \dots (\mathrm{id} - \mathrm{d}_1 \mathrm{s}_0) = 0, \end{aligned}$$

whence $\operatorname{Im} \mathrm{s}_k \preceq \operatorname{Ker}(\pi_n \iota_n) = \operatorname{Ker} \pi_n$ for all $k \in [0, n-1]$ and thus $\mathrm{D}_n A \preceq \operatorname{Ker} \pi_n$. Altogether, this implies $\operatorname{Ker} \pi = \mathrm{D}A$ (since both are subcomplexes of $\mathrm{C}A$, the pointwise proof is sufficient).
Finally, we want to show that $\mathrm{C}A \simeq \mathrm{M}A$. Thereto, we show that each embedding

$$F(-\alpha) \xrightarrow{\iota(-\alpha)} F(-\alpha+1)$$

for $\alpha \in \mathbb{N}$ is a homotopy equivalence. More specifically, since $\iota(-\alpha)\pi(-\alpha) = \mathrm{id}_{F(-\alpha)}$, we show that

$$\pi(-\alpha)\iota(-\alpha) \sim \mathrm{id}_{F(-\alpha+1)} \text{ for all } \alpha \in \mathbb{N}.$$

We suppose given $\alpha \in \mathbb{N}$ and $n \in \mathbb{N}_0$ such that $n \geq \alpha - 1$. Then

$$\mu(-\alpha+1)_n \mathrm{s}_{n-\alpha+1} \mathrm{d}_k = \mu(-\alpha+1)_n \mathrm{d}_{k-1} \mathrm{s}_{n-\alpha+1} = 0 \text{ for all } k \in [n-\alpha+3, n+1].$$

Hence we have an induced morphism $h(-\alpha)_n$ given by the commutative diagram

$$\begin{array}{ccc} F(-\alpha+1)_n & \xrightarrow{h(-\alpha)_n} & F(-\alpha+1)_{n+1} \\ \downarrow{\scriptstyle \mu(-\alpha+1)_n} & & \downarrow{\scriptstyle \mu(-\alpha+1)_{n+1}} \\ A_n & \xrightarrow{(-1)^{n-\alpha} \mathrm{s}_{n-\alpha+1}} & A_{n+1} \end{array}$$

Additionally, we set $h(-\alpha)_n := 0$ for $n \in \mathbb{N}$, $n \leq \alpha - 2$. Then we obtain

$$(-1)^{n-\alpha} \mathrm{s}_0 (\mathrm{d}_0 - \mathrm{d}_1) = 0 = \mathrm{id} - \mathrm{id}$$

for $n = \alpha - 1$. For $n \geq \alpha$, we get

$$(-1)^{n-\alpha} \mathrm{s}_{n-\alpha+1} \Big(\sum_{k \in [0, n-\alpha+2]} (-1)^k \mathrm{d}_k \Big) + \Big(\sum_{k \in [0, n-\alpha+1]} (-1)^k \mathrm{d}_k \Big) (-1)^{n-1-\alpha} \mathrm{s}_{n-\alpha}$$

$$= \sum_{k\in[0,n-\alpha]} (-1)^{n-\alpha+k} \mathrm{d}_k \mathrm{s}_{n-\alpha} + \sum_{k\in[0,n-\alpha+1]} (-1)^{n-1-\alpha+k} \mathrm{d}_k \mathrm{s}_{n-\alpha} = \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}$$
$$= \mathrm{id} - (\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha}).$$

This implies

$$\begin{aligned}(h(-n-1)_n\partial + \partial h(-n-1)_{n-1})\mu(-n)_n &= h(-n-1)_n\partial\mu(-n)_n = \mu(-n)_n(-\mathrm{s}_0(\mathrm{d}_0 - \mathrm{d}_1)) \\ &= \mu(-n)_n(\mathrm{id} - \mathrm{id}) = (\mathrm{id} - \pi(-n-1)_n\iota(-n-1)_n)\mu(-n)_n\end{aligned}$$

as well as

$$\begin{aligned}&(h(-\alpha)_n\partial + \partial h(-\alpha)_{n-1})\mu(-\alpha+1)_n \\ &= \mu(-\alpha+1)_n\Big((-1)^{n-\alpha}\mathrm{s}_{n-\alpha+1}(\sum_{k\in[0,n-\alpha+2]}(-1)^k\mathrm{d}_k) + (\sum_{k\in[0,n-\alpha+1]}(-1)^k\mathrm{d}_k)(-1)^{n-1-\alpha}\mathrm{s}_{n-\alpha}\Big) \\ &= \mu(-\alpha+1)_n(\mathrm{id} - (\mathrm{id} - \mathrm{d}_{n-\alpha+1}\mathrm{s}_{n-\alpha})) = (\mathrm{id} - \pi(-\alpha)_n\iota(-\alpha)_n)\mu(-\alpha+1)_n\end{aligned}$$

for $n \geq \alpha$. Altogether,

$$h(-\alpha)_n\partial + \partial h(-\alpha)_{n-1} = \mathrm{id} - \pi(-\alpha)_n\iota(-\alpha)_n \text{ for all } n \in \mathbb{N} \text{ with } n \geq \alpha - 1.$$

Since this equation also holds for $n < \alpha - 1$ (because of the trivial definition of h_n), we see that $(h(-\alpha)_n \mid n \in \mathbb{N}_0)$ is a complex homotopy from $\mathrm{id}_{F(-\alpha+1)}$ to $\pi(-\alpha)\iota(-\alpha)$. Thus

$$F(-\alpha) \xrightarrow{\iota(-\alpha)} F(-\alpha+1)$$

is a homotopy equivalence for all $\alpha \in \mathbb{N}$.
Still it remains to construct a homotopy from $\mathrm{id}_{\mathrm{C}A}$ to $\pi\iota$. Since $(h(-\alpha)_n \mid n \in \mathbb{N}_0)$ is a homotopy from $\mathrm{id}_{F(-\alpha+1)}$ to $\pi(-\alpha)\iota(-\alpha)$ for every $\alpha \in \mathbb{N}$, by proposition (2.27) we have a complex homotopy $(H(-\alpha)_n \mid n \in \mathbb{N}_0)$ from $\mathrm{id}_{\mathrm{C}A}$ to $\pi(-1)\ldots\pi(-\alpha)\iota(-\alpha)\ldots\iota(-1)$ given by

$$H(-\alpha)_n = \sum_{k\in[1,\alpha]} \pi(-1)_n \ldots \pi(-k+1)_n h(-k)_n \iota(-k+1)_{n+1} \ldots \iota(-1)_{n+1}$$

for every $n \in \mathbb{N}_0$, $\alpha \in \mathbb{N}$. But since $h(-k)_n = 0$ for all $k \in [n+2, \alpha]$, we get

$$H(-\alpha)_n = \sum_{k\in[1,\min(n+1,\alpha)]} \pi(-1)_n \ldots \pi(-k+1)_n h(-k)_n \iota(-k+1)_{n+1} \ldots \iota(-1)_{n+1}$$

for all $n \in \mathbb{N}_0$, $\alpha \in \mathbb{N}$, and hence

$$H(-\alpha)_n = H(-n-1)_n \text{ for all } \alpha \in \mathbb{N} \text{ with } \alpha \geq n+1, n \in \mathbb{N}_0.$$

With

$$H_n := H(-n-1)_n \text{ for all } n \in \mathbb{N}_0, \text{ going from } \mathrm{C}_nA \text{ to } \mathrm{C}_{n+1}A,$$

we compute

$$\begin{aligned}H_n\partial + \partial H_{n-1} &= H(-n-1)_n\partial + \partial H(-n)_{n-1} = H(-n-1)_n\partial + \partial H(-n-1)_{n-1} \\ &= \mathrm{id} - \pi(-1)_n \ldots \pi(-n)_n\pi(-n-1)_n\iota(-n-1)_n\iota(-n)_n \ldots \iota(-1)_1 \\ &= \mathrm{id} - \pi(-1)_n \ldots \pi(-n)_n\iota(-n)_n \ldots \iota(-1)_1 = \mathrm{id} - \pi_n\iota_n\end{aligned}$$

since $\pi(-n-1)_n\iota(-n-1)_n = \mathrm{id}$. Thus $(H_n \mid n \in \mathbb{N})$ is a complex homotopy from $\mathrm{id}_{\mathrm{C}A}$ to $\pi\iota$ and we have $\mathrm{M}A \simeq \mathrm{C}A$. □

§4 Path simplicial objects

We want to introduce the path simplicial object of a given simplicial object. Thereto, we have to study first a certain endofunctor of the category of simplex types $\mathbf{\Delta}$.

(2.29) Proposition.

(a) We may define a functor

$$\mathbf{\Delta} \xrightarrow{\mathrm{Sh}} \mathbf{\Delta}$$

by $\mathrm{Sh}[n] := [n+1]$ and

$$i(\mathrm{Sh}\theta) := \begin{cases} i\theta & i \in [0, m], \\ n+1 & i = m+1 \end{cases}$$

for every morphism $[m] \xrightarrow{\theta} [n]$ in the category of simplex types $\mathbf{\Delta}$, $m, n \in \mathbb{N}_0$.

(b) We have

$$\mathrm{Sh}\delta^k = \delta^k \text{ for all } k \in [0, n], n \in \mathbb{N}, \text{ as morphisms } [n] \longrightarrow [n+1]$$

and

$$\mathrm{Sh}\sigma^k = \sigma^k \text{ for all } k \in [0, n], n \in \mathbb{N}_0, \text{ as morphisms } [n+2] \longrightarrow [n+1].$$

(c) The cofaces $[n] \xrightarrow{\delta^{n+1}} [n+1]$ yield a natural transformation

$$\mathrm{id}_{\mathbf{\Delta}} \xrightarrow{\delta^{\bullet+1}} \mathrm{Sh}.$$

(d) We have $(\mathrm{Sh}\theta)\sigma^n = \sigma^m\theta$ for all morphisms $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$ such that $m\theta = n$.

Proof.

(a) We let $m, n, p \in \mathbb{N}_0$ be non-negative integers and $[m] \xrightarrow{\theta} [n]$, $[n] \xrightarrow{\rho} [p]$ be morphisms in $\mathbf{\Delta}$. Then we have

$$i(\mathrm{Sh}(\theta\rho)) = \left.\begin{cases} i\theta\rho & i \in [0, m], \\ p+1 & i = m+1 \end{cases}\right\} = \left.\begin{cases} (i\theta)(\mathrm{Sh}\rho) & i \in [0, m], \\ (n+1)(\mathrm{Sh}\rho) & i = m+1 \end{cases}\right\} = i(\mathrm{Sh}\theta)(\mathrm{Sh}\rho)$$

and

$$i(\mathrm{Shid}_{[m]}) = \left.\begin{cases} i\mathrm{id}_{[m]} & i \in [0, m], \\ m+1 & i = m+1 \end{cases}\right\} = \left.\begin{cases} i & i \in [0, m], \\ m+1 & i = m+1, \end{cases}\right\} = i$$

for all $i \in [0, m+1]$, that is, $\mathrm{Sh}(\theta\rho) = (\mathrm{Sh}\theta)(\mathrm{Sh}\rho)$ and $\mathrm{Shid}_{[m]} = \mathrm{id}_{\mathrm{Sh}[m]}$. Hence

$$\mathbf{\Delta} \xrightarrow{\mathrm{Sh}} \mathbf{\Delta}$$

is a functor.

(b) We have

$$i(\mathrm{Sh}\delta^k) = \left.\begin{cases} i\delta^k & i \in [0, n-1], \\ n+1 & i = n \end{cases}\right\} = \left.\begin{cases} i & i \in [0, k-1], \\ i+1 & i \in [k, n-1], \\ n+1 & i = n \end{cases}\right\} = \left.\begin{cases} i & i \in [0, k-1], \\ i+1 & i \in [k, n] \end{cases}\right\} = i\delta^k$$

for all $i \in [0, n]$ and

$$i(\mathrm{Sh}\sigma^k) = \left.\begin{cases} i\sigma^k & i \in [0, n+1], \\ n+1 & i = n+2 \end{cases}\right\} = \left.\begin{cases} i & i \in [0, k], \\ i-1 & i \in [k+1, n+1], \\ n+1 & i = n+2 \end{cases}\right\} = \left.\begin{cases} i & i \in [0, k], \\ i-1 & i \in [k+1, n+2] \end{cases}\right\}$$
$$= i\sigma^k$$

for all $i \in [0, n+2]$. Hence we have $\mathrm{Sh}\delta^k = \delta^k$ for all $k \in [0, n]$, $n \in \mathbb{N}$, and $\mathrm{Sh}\sigma^k = \sigma^k$ for all $k \in [0, n]$, $n \in \mathbb{N}_0$.

(c) We let $[m] \xrightarrow{\theta} [n]$ be a morphism in $\mathbf{\Delta}$. Then we have

$$i\delta^{m+1}(\mathrm{Sh}\theta) = i(\mathrm{Sh}\theta) = i\theta = i\theta\delta^{n+1}$$

for all $i \in [0, m]$, that is, $\delta^{m+1}(\mathrm{Sh}\theta) = \theta\delta^{n+1}$. Hence the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\delta^{m+1}} & \mathrm{Sh}[m] \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\mathrm{Sh}\theta} \\ [n] & \xrightarrow{\delta^{n+1}} & \mathrm{Sh}[n] \end{array}$$

commutes and therefore we have a natural transformation

$$\mathrm{id}_{\mathbf{\Delta}} \xrightarrow{\delta^{\bullet+1}} \mathrm{Sh}.$$

(d) For every morphism $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$ we get

$$i(\mathrm{Sh}\theta)\sigma^n = i\theta\sigma^n = i\theta = i\sigma^m\theta$$

for all $i \in [0, m]$. Since $m\theta = n$, we have moreover

$$(m+1)(\mathrm{Sh}\theta)\sigma^n = (n+1)\sigma^n = n = m\theta = (m+1)\sigma^m\theta.$$

Altogether, $(\mathrm{Sh}\theta)\sigma^n = \sigma^m\theta$. □

(2.30) Definition (path simplicial object)**.**

(a) The endofunctor $\mathbf{\Delta} \xrightarrow{\mathrm{Sh}} \mathbf{\Delta}$ given as in proposition (2.29) by $\mathrm{Sh}[n] = [n+1]$ and

$$i(\mathrm{Sh}\theta) = \begin{cases} i\theta & \text{for } i \in [0, m], \\ n+1 & \text{for } i = m+1 \end{cases}$$

for all $i \in [0, m+1]$, morphisms $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$, $m, n \in \mathbb{N}_0$, is called *shift functor* of $\mathbf{\Delta}$.

(b) For a given category $\mathcal{C}$, we define the functor

$$\mathbf{s}\mathcal{C} \xrightarrow{\mathrm{P}} \mathbf{s}\mathcal{C}$$

by $\mathrm{P} := ((\mathrm{Sh})^{\mathrm{op}}, \mathcal{C})$ (remember $\mathbf{s}\mathcal{C} = (\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$). Given a simplicial object $X \in \mathrm{Ob}\,\mathbf{s}\mathcal{C}$ in $\mathcal{C}$, the functor P assigns to X the simplicial object $\mathrm{P}X = X \circ (\mathrm{Sh})^{\mathrm{op}}$. It is called the *path simplicial object* of X.

(2.31) Proposition (classical description of the path simplicial object)**.** We let X be a simplicial object in a category $\mathcal{C}$.

(a) We have $\mathrm{P}_n X = X_{n+1}$ for every $n \in \mathbb{N}_0$.

(b) The faces and degeneracies in the path simplicial object $\mathrm{P}X$ are given by $\mathrm{d}_k^{\mathrm{P}X} = \mathrm{d}_k^X$ for every $k \in [0, n]$, $n \in \mathbb{N}$, and $\mathrm{s}_k^{\mathrm{P}X} = \mathrm{s}_k^X$ for every $k \in [0, n]$, $n \in \mathbb{N}_0$.

(c) The morphisms $X_{n+1} \xrightarrow{\mathrm{d}_{n+1}} X_n$ for $n \in \mathbb{N}_0$ form a simplicial morphism $\mathrm{P}X \xrightarrow{\mathrm{d}_{\bullet+1}} X$.

Proof. This is a consequence of proposition (2.29). □

The following notion is similar to that of a *frontal morphism* in [12], [13].

(2.32) Definition (backal morphisms)**.** We let X be a simplicial object in a category $\mathcal{C}$. A morphism $X_n \xrightarrow{f} X_m$ for $m, n \in \mathbb{N}_0$ is called *backal*, if there exists a morphism $[m] \xrightarrow{\theta} [n]$ in the category of simplex types $\mathbf{\Delta}$ such that $f = X_\theta$ and $m\theta = n$.

(2.33) Proposition. We let X be a simplicial object in a category $\mathcal{C}$.

(a) If X_θ is backal, where $[m] \xrightarrow{\theta} [n]$ is a morphism in $\mathbf{\Delta}$, then we have

$$\mathrm{s}_n \mathrm{P}_\theta X = X_\theta \mathrm{s}_m.$$

(b) A morphism $X_n \xrightarrow{f} X_m$ for $m, n \in \mathbb{N}_0$ is backal if and only if there exists a representation

$$f = \mathrm{d}_{n_{\lfloor u,1 \rfloor}} \mathrm{s}_{m_{\lceil 1,t \rceil}}$$

with $0 \leq m_1 < \cdots < m_t < m$ and $0 \leq n_1 < \cdots < n_u < n$, $t, ut \in \mathbb{N}_0$.

(c) If there exists a morphism $[m-1] \xrightarrow{\rho} [n-1]$ in $\mathbf{\Delta}$ such that $f = \mathrm{P}_\rho X$, then f is backal.

Proof.

(a) This is also a consequence of proposition (2.29).

(b) That follows from theorem (1.7) and its proof. Therein, the assertion $n_t \neq n$ is equivalent to $n \in [0, m]\theta$ and since θ is a monotonically increasing map, this is equivalent to $m\theta = n$.

(c) By our assumption, we have $f = \mathrm{P}_\rho X = X_{\mathrm{Sh}\rho}$ and $m(\mathrm{Sh}\rho) = n$ by the definition of the shift functor Sh.

□

The next theorem describes the significance of path simplicial objects in homotopy theory.

(2.34) Theorem. The path simplicial object $\mathrm{P}X$ of a simplicial object X in a category $\mathcal{C}$ is simplicially homotopy equivalent to the constant simplicial object $\mathrm{Const}\, X_0$.

Proof. Since $\mathrm{d}_k \mathrm{d}_{\lfloor n-1,0 \rfloor} = \mathrm{d}_{\lfloor n,0 \rfloor}$ for all $k \in [0, n]$, $n \in \mathbb{N}$, and since $\mathrm{s}_k \mathrm{d}_{\lfloor n+1,0 \rfloor} = \mathrm{d}_{\lfloor n,0 \rfloor}$ for all $k \in [0, n]$, $n \in \mathbb{N}_0$, we have a simplicial morphism

$$\mathrm{P}X \xrightarrow{D} \mathrm{Const}\, X_0$$

given by $D_n = \mathrm{d}_{\lfloor n,0 \rfloor}$ for all $n \in \mathbb{N}_0$. Further, we have $\mathrm{s}_{\lceil 0,n \rceil} \mathrm{d}_k = \mathrm{s}_{\lceil 0,n-1 \rceil}$ for all $k \in [0, n]$, $n \in \mathbb{N}$, and $\mathrm{s}_{\lceil 0,n \rceil} \mathrm{s}_k = \mathrm{s}_{\lceil 0,n+1 \rceil}$ for all $k \in [0, n]$, $n \in \mathbb{N}_0$, so that there exists a simplicial morphism

$$\mathrm{Const}\, X_0 \xrightarrow{S} \mathrm{P}X$$

given by $S_n = \mathrm{s}_{\lceil 0,n \rceil}$ for all $n \in \mathbb{N}_0$. We have $S_n D_n = \mathrm{s}_{\lceil 0,n \rceil} \mathrm{d}_{\lfloor n,0 \rfloor} = \mathrm{id}_{X_0}$ for all $n \in \mathbb{N}_0$. Hence D is a retraction with coretraction S.

For each $k \in [0, n]$, $n \in \mathbb{N}_0$, we define the morphism $\mathrm{P}_n X \xrightarrow{h_k} \mathrm{P}_{n+1} X$ by $h_k := \mathrm{d}_{\lfloor n,k+1 \rfloor} \mathrm{s}_{\lceil k+1,n+1 \rceil}$. Then we have $h_n \mathrm{d}_{n+1} = \mathrm{s}_{n+1} \mathrm{d}_{n+1} = \mathrm{id}_{\mathrm{P}_n X}$ and $h_0 \mathrm{d}_0 = \mathrm{d}_{\lfloor n,1 \rfloor} \mathrm{s}_{\lceil 1,n+1 \rceil} \mathrm{d}_0 = \mathrm{d}_{\lfloor n,0 \rfloor} \mathrm{s}_{\lceil 0,n \rceil} = D_n S_n$ for all $n \in \mathbb{N}_0$. Further, we get

$$h_l \mathrm{d}_k = \mathrm{d}_{\lfloor n,l+1 \rfloor} \mathrm{s}_{\lceil l+1,n+1 \rceil} \mathrm{d}_k = \left\{ \begin{matrix} \mathrm{d}_{\lfloor n,l+1 \rfloor} \mathrm{d}_k \mathrm{s}_{\lceil l,n \rceil} & \text{for } k < l, \\ \mathrm{d}_{\lfloor n,k+1 \rfloor} \mathrm{d}_k \mathrm{s}_{\lceil k,n \rceil} & \text{for } k = l, k \neq 0 \\ \mathrm{d}_{\lfloor n,l+1 \rfloor} \mathrm{s}_{\lceil l+1,n \rceil} & \text{for } k > l+1 \end{matrix} \right\}$$

$$= \begin{cases} \mathrm{d}_k \mathrm{d}_{\lfloor n-1,l \rfloor} \mathrm{s}_{\lceil l,n \rceil} & \text{for } k < l, \\ \mathrm{d}_{\lfloor n,k \rfloor} \mathrm{s}_{\lceil k,n+1 \rceil} \mathrm{d}_k & \text{for } k = l, k \neq 0 \\ \mathrm{d}_{k-1} \mathrm{d}_{\lfloor n-1,l+1 \rfloor} \mathrm{s}_{\lceil l+1,n \rceil} & \text{for } k > l+1 \end{cases} = \begin{cases} \mathrm{d}_k h_{l-1} & \text{for } k < l, \\ h_{k-1} \mathrm{d}_k & \text{for } k = l, k \neq 0, \\ \mathrm{d}_{k-1} h_l & \text{for } k > l+1 \end{cases}$$

and

$$h_l \mathrm{s}_k = \mathrm{d}_{\lfloor n,l+1 \rfloor} \mathrm{s}_{\lceil l+1,n+1 \rceil} \mathrm{s}_k = \begin{cases} \mathrm{d}_{\lfloor n,l+1 \rfloor} \mathrm{s}_k \mathrm{s}_{\lceil l+2,n+2 \rceil} & \text{for } k \leq l, \\ \mathrm{d}_{\lfloor n,l+1 \rfloor} \mathrm{s}_{\lceil l+1,n+2 \rceil} & \text{for } k > l \end{cases}$$

$$= \begin{cases} \mathrm{s}_k \mathrm{d}_{\lfloor n+1,l+2 \rfloor} \mathrm{s}_{\lceil l+2,n+2 \rceil} & \text{for } k \leq l, \\ \mathrm{s}_{k-1} \mathrm{d}_{\lfloor n+1,l+1 \rfloor} \mathrm{s}_{\lceil l+1,n+2 \rceil} & \text{for } k > l \end{cases} = \begin{cases} \mathrm{s}_k h_{l+1} & \text{for } k \leq l, \\ \mathrm{s}_{k-1} h_l & \text{for } k > l \end{cases}$$

for all $k \in [0, n+1]$, $l \in [0, n]$, $n \in \mathbb{N}_0$. Hence $(h_k \in {}_{\mathcal{C}}(\mathrm{P}_n X, \mathrm{P}_{n+1} X) \mid k \in [0, n], n \in \mathbb{N}_0)$ is a simplicial homotopy from $\mathrm{id}_{\mathrm{P}X}$ to DS and we have proven that D is a simplicial homotopy equivalence, whence $\mathrm{P}X \simeq \mathrm{Const}\, X_0$. □

§5 The classifying simplicial set of a group

As an example for the notions introduced in this chapter, the homology of a group via its classifying simplicial set is studied now.
Throughout this section, we suppose given a group G.

(2.35) Definition (classifying simplicial set of a group)**.** The nerve $\mathrm{N}G$ of G is also called the *classifying simplicial set* of G; cf. definition (1.35). In this context, we write $\mathrm{B}G := \mathrm{N}G$.

Recall that $\mathrm{B}_n G = G^{\times n}$ and that the faces and degeneracies in the simplicial set $\mathrm{B}G$ are given by

$$(g_j)_{j \in \lfloor n-1,0 \rfloor} \mathrm{d}_k = \begin{cases} (g_j)_{j \in \lfloor n-1,1 \rfloor} & \text{for } k = 0, \\ (g_j)_{j \in \lfloor n-1,k+1 \rfloor} \cup (g_k g_{k-1}) \cup (g_j)_{j \in \lfloor k-2,0 \rfloor} & \text{for } k \in [1, n-1], \\ (g_j)_{j \in \lfloor n-2,0 \rfloor} & \text{for } k = n \end{cases}$$

for all $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \mathrm{B}_n G$, $k \in [0, n]$, $n \in \mathbb{N}$, and

$$(g_j)_{j \in \lfloor n-1,0 \rfloor} \mathrm{s}_k = (g_j)_{j \in \lfloor n-1,k \rfloor} \cup (1) \cup (g_j)_{j \in \lfloor k-1,0 \rfloor}$$

for all $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \mathrm{B}_n G$, $k \in [0, n]$, $n \in \mathbb{N}_0$.

(2.36) Definition (resolving simplicial set)**.** The path simplicial set $\mathrm{E}G := \mathrm{PB}G$ of the classifying simplicial set of G is called *resolving simplicial set* of G.

(2.37) Proposition (classical description of the resolving simplicial set)**.** The set of n-simplices of the resolving simplicial set of G are given by $\mathrm{E}_n G = G^{\times (n+1)}$ for $n \in \mathbb{N}_0$. The faces $\mathrm{E}_n G \xrightarrow{\mathrm{d}_k} \mathrm{E}_{n-1} G$ and degeneracies $\mathrm{E}_n G \xrightarrow{\mathrm{s}_k} \mathrm{E}_{n+1} G$ are given by

$$(g_j)_{j \in \lfloor n,0 \rfloor} \mathrm{d}_k = \begin{cases} (g_j)_{j \in \lfloor n,1 \rfloor} & \text{for } k = 0, \\ (g_j)_{j \in \lfloor n,k+1 \rfloor} \cup (g_k g_{k-1}) \cup (g_j)_{j \in \lfloor k-2,0 \rfloor} & \text{for } k \in [1, n] \end{cases}$$

for all $(g_j)_{j \in \lfloor n,0 \rfloor} \in \mathrm{E}_n G$, $k \in [0, n]$, $n \in \mathbb{N}$, and

$$(g_j)_{j \in \lfloor n,0 \rfloor} \mathrm{s}_k = (g_j)_{j \in \lfloor n,k \rfloor} \cup (1) \cup (g_j)_{j \in \lfloor k-1,0 \rfloor}$$

for all $(g_j)_{j \in \lfloor n,0 \rfloor} \in \mathrm{E}_n G$, $k \in [0, n]$, $n \in \mathbb{N}_0$.

Proof. Follows from proposition (2.31) and definition (2.35). □

From a homotopical point of view, the total simplicial set $\mathrm{E}G$ is trivial.

(2.38) Corollary. The resolving simplicial set $\mathrm{E}G$ of G is contractible.

Proof. Follows from theorem (2.34) and proposition (2.37). □

With the help of EG, we are able to connect the homology of G and the homology of BG, as we will see now.

(2.39) Remark. For the following facts, we refer the reader for example to [21, section 2.3].
Recall that the *bar resolution* of G with entries in a commutative ring R is defined by $(\mathrm{Bar}_{G;R})_n = RG^{\otimes_R (n+1)}$ for all $n \in \mathbb{N}_0$. It is a free resolution of the trivial RG-module R. A typical element of $RG^{\otimes_R (n+1)}$ is written by

$$(g_i)^{\otimes}_{i \in \lfloor n,0 \rfloor} = g_n \otimes g_{n-1} \otimes \cdots \otimes g_0.$$

Note that $RG^{\otimes_R (n+1)}$ is free as an RG-module over the basis

$$\{(\prod_{i \in \lfloor n-1,k \rfloor} g_i)^{\otimes}_{k \in \lfloor n,0 \rfloor} \mid g_i \in G \text{ for } i \in \lfloor n-1, 0 \rfloor\}$$

and free as an R-module over the basis

$$\{(\prod_{i \in \lfloor n,k \rfloor} g_i)^{\otimes}_{k \in \lfloor n,0 \rfloor} \mid g_i \in G \text{ for } i \in \lfloor n, 0 \rfloor\}.$$

The differentials

$$(\mathrm{Bar}_{G;R})_n \xrightarrow{\partial} (\mathrm{Bar}_{G;R})_{n-1}$$

in the bar resolution are given by

$$(g_i)^{\otimes}_{i \in \lfloor n,0 \rfloor} \partial = \sum_{k \in [0,n]} (-1)^k (g_i)^{\otimes}_{i \in \lfloor n,0 \rfloor \wedge k}.$$

(2.40) Proposition. We let R be a commutative ring. Then we have an isomorphism

$$\mathrm{C}(\mathrm{E}G; R) \longrightarrow \mathrm{Bar}_{G;R}$$

given by

$$\mathrm{C}_n(\mathrm{E}G; R) \to (\mathrm{Bar}_{G;R})_n, (g_j)_{j \in \lfloor n,0 \rfloor} \mapsto (\prod_{i \in \lfloor n,j \rfloor} g_i)^{\otimes}_{j \in \lfloor n,0 \rfloor} \text{ for all } n \in \mathbb{N}_0.$$

In particular, C(EG; R) inherits the structure of a complex of RG-modules, where the multiplication with an element $g \in G$ is given by

$$g(g_j)_{j \in \lfloor n,0 \rfloor} := (gg_n) \cup (g_j)_{j \in \lfloor n,0 \rfloor}$$

for all $(g_j)_{j \in \lfloor n,0 \rfloor} \in \mathrm{E}_n G$, $n \in \mathbb{N}_0$.

Proof. We let

$$\varphi_n \colon \mathrm{C}_n(\mathrm{E}G; R) \to (\mathrm{Bar}_{G;R})_n, (g_j)_{j \in \lfloor n,0 \rfloor} \mapsto (\prod_{i \in \lfloor n,j \rfloor} g_i)^{\otimes}_{j \in \lfloor n,0 \rfloor} \text{ for all } n \in \mathbb{N}_0.$$

Since these maps are bijective and linear over R, it remains to prove the compatibility with the differentials. Indeed, we have

$$\begin{aligned}(g_j)_{j \in \lfloor n,0 \rfloor} \varphi_n \partial &= (\prod_{i \in \lfloor n,j \rfloor} g_i)^{\otimes}_{j \in \lfloor n,0 \rfloor} \partial = \sum_{k \in [0,n]} (-1)^k (\prod_{i \in \lfloor n,j \rfloor} g_i)^{\otimes}_{j \in \lfloor n,0 \rfloor \wedge k} = \sum_{k \in [0,n]} (-1)^k (g_j)_{j \in \lfloor n,0 \rfloor} \mathrm{d}_k \varphi_{n-1} \\ &= (g_j)_{j \in \lfloor n,0 \rfloor} (\sum_{k \in [0,n]} (-1)^k \mathrm{d}_k) \varphi_{n-1} = (g_j)_{j \in \lfloor n,0 \rfloor} \partial \varphi_{n-1}\end{aligned}$$

for all $(g_j)_{j \in \lfloor n,0 \rfloor} \in \mathrm{E}_n G$, $n \in \mathbb{N}$, and since $\mathrm{E}_n G$ is an R-linear basis of $\mathrm{C}_n(\mathrm{E}G; R)$, this means $\varphi_n \partial = \partial \varphi_{n-1}$ for all $n \in \mathbb{N}$. Hence the morphisms φ_n for $n \in \mathbb{N}_0$ yield an isomorphism of complexes

$$\mathrm{C}(\mathrm{E}G; R) \xrightarrow{\varphi} \mathrm{Bar}_{G;R}.$$

Via transfer of structure, the modules $\mathrm{C}_n(\mathrm{E}G;R)$ become RG-modules and the morphisms φ_n become isomorphisms of RG-modules with

$$\begin{aligned} g(g_j)_{j\in\lfloor n,0\rfloor} &= (g((g_j)_{j\in\lfloor n,0\rfloor}\varphi_n))\varphi_n^{-1} = (g(\prod_{i\in\lfloor n,j\rfloor} g_i)^{\otimes}_{j\in\lfloor n,0\rfloor})\varphi_n^{-1} = (g\prod_{i\in\lfloor n,j\rfloor} g_i)^{\otimes}_{j\in\lfloor n,0\rfloor}\varphi_n^{-1} \\ &= (gg_n)\cup(g_j)_{j\in\lfloor n-1,0\rfloor} \end{aligned}$$

for all $g\in G$, $(g_j)_{j\in\lfloor n,0\rfloor}\in \mathrm{E}_nG$, $n\in\mathbb{N}_0$. As the differentials of $\mathrm{Bar}_{G;R}$ are RG-linear, the differentials of $\mathrm{C}(\mathrm{E}G;R)$ become RG-linear as well and hence $\mathrm{C}(\mathrm{E}G;R)$ becomes a complex of RG-modules. □

We relate the complex of the classifying simplicial set with the bar resolution.

(2.41) Proposition. We let R be a commutative ring. Then we have

$$\mathrm{C}(\mathrm{B}G;R)\cong R\otimes_{RG}\mathrm{C}(\mathrm{E}G;R).$$

Proof. Since

$$1\otimes(g_j)_{j\in\lfloor n,0\rfloor} = 1\otimes((g_n)\cup(g_j)_{j\in\lfloor n-1,0\rfloor}) = 1\otimes g_n((1)\cup(g_j)_{j\in\lfloor n-1,0\rfloor}) = 1\otimes((1)\cup(g_j)_{j\in\lfloor n-1,0\rfloor})$$

for all $(g_j)_{j\in\lfloor n,0\rfloor}\in\mathrm{E}_nG$, we obtain well-defined R-linear maps

$$\varphi_n\colon R\otimes_{RG}\mathrm{C}_n(\mathrm{E}G;R)\to\mathrm{C}_n(\mathrm{B}G;R),$$

which are given by

$$(1\otimes(g_j)_{j\in\lfloor n,0\rfloor})\varphi_n := (g_j)_{j\in\lfloor n-1,0\rfloor} \text{ for all } (g_j)_{j\in\lfloor n,0\rfloor}\in\mathrm{E}_nG, n\in\mathbb{N}_0.$$

An inverse R-linear map to φ_n is given by

$$\mathrm{C}_n(\mathrm{B}G;R)\to R\otimes_{RG}\mathrm{C}_n(\mathrm{E}G;R), (g_j)_{j\in\lfloor n-1,0\rfloor}\mapsto 1\otimes((1)\cup(g_j)_{j\in\lfloor n-1,0\rfloor}).$$

Hence φ_n is an isomorphism for all $n\in\mathbb{N}_0$. So again, it remains to verify the compatibility with the differentials: We have

$$\begin{aligned} (1\otimes(g_j)_{j\in\lfloor n,0\rfloor})(R\otimes_{RG}\partial)\varphi_{n-1} &= (1\otimes((g_j)_{j\in\lfloor n,0\rfloor}\partial))\varphi_{n-1} = (1\otimes(\sum_{k\in[0,n]}(-1)^k(g_j)_{j\in\lfloor n,0\rfloor}\mathrm{d}_k))\varphi_{n-1} \\ &= \sum_{k\in[0,n]}(-1)^k(1\otimes((g_j)_{j\in\lfloor n,0\rfloor}\mathrm{d}_k))\varphi_{n-1} = \sum_{k\in[0,n]}(-1)^k(g_j)_{j\in\lfloor n-1,0\rfloor}\mathrm{d}_k \\ &= (g_j)_{j\in\lfloor n-1,0\rfloor}\partial = (1\otimes(g_j)_{j\in\lfloor n,0\rfloor})\varphi_n\partial \end{aligned}$$

for all $(g_j)_{j\in\lfloor n,0\rfloor}\in\mathrm{E}_nG$. Thus we have $(R\otimes_{RG}\partial)\varphi_{n-1}=\varphi_n\partial$ for all $n\in\mathbb{N}$.

$$\begin{array}{ccc} R\otimes_{RG}\mathrm{C}_n(\mathrm{E}G;R) & \xrightarrow{R\otimes_{RG}\partial} & R\otimes_{RG}\mathrm{C}_{n-1}(\mathrm{E}G;R) \\ \downarrow{\scriptstyle\varphi_n} & & \downarrow{\scriptstyle\varphi_{n-1}} \\ \mathrm{C}_n(\mathrm{B}G;R) & \xrightarrow{\partial} & \mathrm{C}_{n-1}(\mathrm{B}G;R) \end{array}$$

Hence the maps φ_n for $n\in\mathbb{N}_0$ yield an isomorphism of complexes

$$R\otimes_{RG}\mathrm{C}(\mathrm{E}G;R)\xrightarrow{\varphi}\mathrm{C}(\mathrm{B}G;R).$$ □

Now we are able to relate the (co)homology of the group G defined via Ext and Tor with the (co)homology of its classifying simplicial set.

(2.42) Theorem (simplicial definition of homology and cohomology of groups)**.** We let R be a commutative ring and M be an R-module. Then we have

$$\mathrm{H}_n(\mathrm{B}G,M;R)\cong\mathrm{H}_n(G,M;R) \text{ and } \mathrm{H}^n(\mathrm{B}G,M;R)\cong\mathrm{H}^n(G,M;R) \text{ for all } n\in\mathbb{N}_0.$$

Proof. By the propositions (2.40) and (2.41) we have

$$\begin{aligned}\mathrm{H}_n(\mathrm{B}G, M; R) &= \mathrm{H}_n(\mathrm{C}(\mathrm{B}G; R) \otimes_R M) \cong \mathrm{H}_n(R \otimes_{RG} \mathrm{C}(\mathrm{E}G; R) \otimes_R M) \cong \mathrm{H}_n(\mathrm{C}(\mathrm{E}G; R) \otimes_{RG} M)\\ &\cong \mathrm{H}_n(\mathrm{Bar}_{G;R} \otimes_{RG} M) \cong \mathrm{Tor}_n^{RG}(R, M) = \mathrm{H}_n(G, M; R)\end{aligned}$$

and

$$\begin{aligned}\mathrm{H}^n(\mathrm{B}G, M; R) &= \mathrm{H}^n{}_R(\mathrm{C}(\mathrm{B}G; R), M) \cong \mathrm{H}^n{}_R(R \otimes_{RG} \mathrm{C}(\mathrm{E}G; R), M) \cong \mathrm{H}^n{}_{RG}(\mathrm{C}(\mathrm{E}G; R), {}_R(R, M))\\ &\cong \mathrm{H}^n{}_{RG}(\mathrm{Bar}_{G;R}, M) \cong \mathrm{Ext}^n_{RG}(R, M) = \mathrm{H}^n(G, M; R)\end{aligned}$$

for all $n \in \mathbb{N}_0$. □

Chapter III

Bisimplicial objects

In chapter II, §5, we associated to every group G its classifying simplicial set $\mathrm{B}G$. As we will see in chapter IV, this procedure can be applied dimensionwise to simplicial groups, and we obtain a simplicial object in the category of simplicial sets. Such an object can be described more easily as a bisimplicial set, a notion we will introduce now.
Furthermore, we have seen that one can associate a complex to every simplicial object in an abelian category. This notion can be generalised to bisimplicial objects, in this case we obtain a double complex. A comparison between the associated complex of the diagonal simplicial object and the total complex of this double complex is given in the generalised Eilenberg-Zilber theorem in the last section of this chapter.
For further information, we refer the reader (again) to [8], [9], [17], [23], [26], [29, §8].

§1 From bisimplicial objects to simplicial objects

(3.1) Definition (bisimplicial objects and their morphisms)**.** We let $\mathcal{C}$ be a category. The *category of bisimplicial objects* in $\mathcal{C}$ is defined to be the functor category

$$\mathbf{s}^2\mathcal{C} := (\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}).$$

An object in $\mathbf{s}^2\mathcal{C}$ is called a *bisimplicial object* in $\mathcal{C}$, a morphism in $\mathbf{s}^2\mathcal{C}$ is called a *morphism of bisimplicial objects* in $\mathcal{C}$.

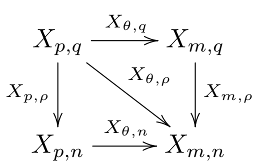
The categories $\mathbf{s}^2\mathcal{C}$ and $\mathbf{ss}\mathcal{C}$ for a given category $\mathcal{C}$ are equivalent in an obvious way. Most time, we will not distinguish both concepts and, whenever necessary or helpful, we change our point of view without any comment.

(3.2) Definition (objects and morphisms of a bisimplicial object)**.** We suppose given a bisimplicial object X in a category $\mathcal{C}$ and morphisms $[m] \xrightarrow{\theta} [p]$, $[n] \xrightarrow{\rho} [q]$ in $\mathbf{\Delta}$, where $m, n, p, q \in \mathbb{N}_0$. In this situation, we set $X_{m,n} := X_{([m],[n])}$ and $X_{\theta,\rho} := X_{(\theta,\rho)}$. Moreover, we abbreviate $X_{\theta,q} := X_{\theta,\mathrm{id}_{[q]}}$ and $X_{p,\rho} := X_{\mathrm{id}_{[p]},\rho}$. Likewise for morphisms of bisimplicial objects.

(3.3) Remark. Given a bisimplicial object X and morphisms $[m] \xrightarrow{\theta} [p]$, $[n] \xrightarrow{\rho} [q]$ in $\mathbf{\Delta}$, where $m, n, p, q \in \mathbb{N}_0$, we have

$$X_{\theta,q} X_{m,\rho} = X_{\theta,\rho} = X_{p,\rho} X_{\theta,n}.$$

That is, the diagram



commutes.

Proof. We have

$$X_{\theta,q}X_{m,\rho} = X_{(\theta,\mathrm{id}_{[q]})}X_{(\mathrm{id}_{[m]},\rho)} = X_{(\mathrm{id}_{[m]},\rho)(\theta,\mathrm{id}_{[q]})} = X_{(\mathrm{id}_{[m]}\theta,\rho\mathrm{id}_{[q]})} = X_{\theta,\rho}$$

and

$$X_{p,\rho}X_{\theta,n} = X_{(\mathrm{id}_{[p]},\rho)}X_{(\theta,\mathrm{id}_{[n]})} = X_{(\theta,\mathrm{id}_{[n]})(\mathrm{id}_{[p]},\rho)} = X_{(\theta\mathrm{id}_{[p]},\mathrm{id}_{[n]}\rho)} = X_{\theta,\rho}. \qquad \square$$

(3.4) Definition (horizontal and vertical faces and degeneracies)**.** For every bisimplicial object X in a category $\mathcal{C}$, we define the *horizontal* and *vertical faces*

$$X_{p,q} \xrightarrow{\mathrm{d}^{\mathrm{h}}_k} X_{p-1,q} \text{ resp. } X_{p,q} \xrightarrow{\mathrm{d}^{\mathrm{v}}_k} X_{p,q-1}$$

by $\mathrm{d}^{\mathrm{h}}_k := X_{\delta^k,q}$ for $k \in [0,p]$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and $\mathrm{d}^{\mathrm{v}}_k := X_{p,\delta^k}$ for $k \in [0,q]$, $q \in \mathbb{N}$, $p \in \mathbb{N}_0$. Analogously, the *horizontal* and *vertical degeneracies*

$$X_{p,q} \xrightarrow{\mathrm{s}^{\mathrm{h}}_k} X_{p+1,q} \text{ resp. } X_{p,q} \xrightarrow{\mathrm{s}^{\mathrm{v}}_k} X_{p,q+1}$$

are defined to be $\mathrm{s}^{\mathrm{h}}_k := X_{\sigma^k,q}$ for $k \in [0,p]$, $p \in \mathbb{N}$, $q \in \mathbb{N}_0$, and $\mathrm{s}^{\mathrm{v}}_k := X_{p,\sigma^k}$ for $k \in [0,q]$, $q \in \mathbb{N}$, $p \in \mathbb{N}_0$.

Our next aim is to present two different ways of obtaining a simplicial set from a given bisimplicial set.

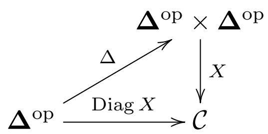
(3.5) Definition (diagonal simplicial object)**.** We let $\mathcal{C}$ be a category. The functor

$$\mathbf{s}^2\mathcal{C} \xrightarrow{\mathrm{Diag}} \mathbf{s}\mathcal{C}$$

is defined to be the induced functor $\mathrm{Diag} := {}_{\mathbf{Cat}}(\Delta,\mathcal{C})$ that we obtain by applying the hom-functor ${}_{\mathbf{Cat}}(-,\mathcal{C})$ to the diagonal morphism

$$\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{\Delta} \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}$$

in **Cat**. For every bisimplicial object X in $\mathcal{C}$, the simplicial object $\mathrm{Diag}\,X$ is called its *diagonal simplicial object*.



(3.6) Proposition. The faces and degeneracies of $\mathrm{Diag}\,X$ for a bisimplicial object X in a category $\mathcal{C}$ are given by $\mathrm{d}_k = \mathrm{d}^{\mathrm{h}}_k\mathrm{d}^{\mathrm{v}}_k = \mathrm{d}^{\mathrm{v}}_k\mathrm{d}^{\mathrm{h}}_k$ for every $k \in [0,n]$, $n \in \mathbb{N}$, resp. $\mathrm{s}_k = \mathrm{s}^{\mathrm{h}}_k\mathrm{s}^{\mathrm{v}}_k = \mathrm{s}^{\mathrm{v}}_k\mathrm{s}^{\mathrm{h}}_k$ for every $k \in [0,n]$, $n \in \mathbb{N}_0$.

Proof. We have

$$\mathrm{d}^{\mathrm{Diag}\,X}_k = (\mathrm{Diag}\,X)_{\delta^k} = X_{\Delta(\delta^k)} = X_{\delta^k,\delta^k} = X_{\delta^k,n}X_{n-1,\delta^k} = \mathrm{d}^{\mathrm{h}}_k\mathrm{d}^{\mathrm{v}}_k$$

for all $k \in [0,n]$, $n \in \mathbb{N}$, where $\delta^k \in {}_{\mathbf{\Delta}}([n-1],[n])$, and

$$\mathrm{s}^{\mathrm{Diag}\,X}_k = (\mathrm{Diag}\,X)_{\sigma^k} = X_{\Delta(\sigma^k)} = X_{\sigma^k,\sigma^k} = X_{\sigma^k,n}X_{n+1,\sigma^k} = \mathrm{s}^{\mathrm{h}}_k\mathrm{s}^{\mathrm{v}}_k$$

for all $k \in [0,n]$, $n \in \mathbb{N}_0$, where $\sigma^k \in {}_{\mathbf{\Delta}}([n+1],[n])$. $\square$

(3.7) Proposition. We let X and Y be bisimplicial objects in a category $\mathcal{C}$ and we let $X \xrightarrow{f} Y$ be a morphism of bisimplicial objects in $\mathcal{C}$. The induced morphism

$$\mathrm{Diag}\,X \xrightarrow{\mathrm{Diag}\,f} \mathrm{Diag}\,Y$$

is given by $\mathrm{Diag}_n f = f_{n,n}$ for all $n \in \mathbb{N}_0$.

Proof. For all $n \in \mathbb{N}_0$, we have

$$\mathrm{Diag}_n f = \mathrm{Diag}_{[n]} f = f_{\Delta([n])} = f_{([n],[n])} = f_{n,n}. \qquad \square$$

(3.8) Definition (splitting)**.** For a morphism $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$ we define its *splitting at* $p \in [0, m]$ by $\operatorname{Spl}_p(\theta) := (\operatorname{Spl}_{\leq p}(\theta), \operatorname{Spl}_{\geq p}(\theta)) \in \operatorname{Mor}(\mathbf{\Delta} \times \mathbf{\Delta})$, where

$$[p] \xrightarrow{\operatorname{Spl}_{\leq p}(\theta)} [p\theta] \text{ and } [m-p] \xrightarrow{\operatorname{Spl}_{\geq p}(\theta)} [n - p\theta]$$

are given by $i\operatorname{Spl}_{\leq p}(\theta) := i\theta$ for $i \in [0, p]$ and $i\operatorname{Spl}_{\geq p}(\theta) := (i+p)\theta - p\theta$ for $i \in [0, m-p]$.

(3.9) Lemma. We have

$$\operatorname{Spl}_p(\theta\rho) = \operatorname{Spl}_p(\theta)\operatorname{Spl}_{p\theta}(\rho)$$

for all morphisms $[m] \xrightarrow{\theta} [n]$ and $[n] \xrightarrow{\rho} [l]$ in $\mathbf{\Delta}$ and all $p \in [0, m]$.

Proof. We have

$$i\operatorname{Spl}_{\leq p}(\theta\rho) = i(\theta\rho) = (i\theta)\rho = (i\theta)\operatorname{Spl}_{\leq p\theta}(\rho) = i\operatorname{Spl}_{\leq p}(\theta)\operatorname{Spl}_{\leq p\theta}(\rho)$$

for all $i \in [0, p]$ as well as

$$\begin{aligned} i\operatorname{Spl}_{\geq p}(\theta\rho) &= (i+p)(\theta\rho) - p(\theta\rho) = ((i+p)\theta - p\theta + p\theta)\rho - (p\theta)\rho = ((i+p)\theta - p\theta)\operatorname{Spl}_{\geq p\theta}(\rho) \\ &= i\operatorname{Spl}_{\geq p}(\theta)\operatorname{Spl}_{\geq p\theta}(\rho) \end{aligned}$$

for all $i \in [0, m-p]$, that is,

$$\begin{aligned} \operatorname{Spl}_p(\theta\rho) &= (\operatorname{Spl}_{\leq p}(\theta\rho), \operatorname{Spl}_{\geq p}(\theta\rho)) = (\operatorname{Spl}_{\leq p}(\theta)\operatorname{Spl}_{\leq p\theta}(\rho), \operatorname{Spl}_{\geq p}(\theta)\operatorname{Spl}_{\geq p\theta}(\rho)) \\ &= (\operatorname{Spl}_{\leq p}(\theta), \operatorname{Spl}_{\geq p}(\theta))(\operatorname{Spl}_{\leq p\theta}(\rho), \operatorname{Spl}_{\geq p\theta}(\rho)) = \operatorname{Spl}_p(\theta)\operatorname{Spl}_{p\theta}(\rho). \end{aligned}$$ □

(3.10) Remark (cf. Artin and Mazur [1])**.** We suppose given a bisimplicial set X. There is a simplicial set $\operatorname{Tot} X$ given by

$$\operatorname{Tot}_n X := \{(x_q)_{q \in \lfloor n, 0 \rfloor} \in \bigtimes_{q \in \lfloor n, 0 \rfloor} X_{q, n-q} \mid x_q \mathrm{d}^{\mathrm{h}}_q = x_{q-1}\mathrm{d}^{\mathrm{v}}_0 \text{ for all } q \in \lfloor n, 1 \rfloor\} \text{ for all } n \in \mathbb{N}_0$$

and by $(x_q)_{q \in \lfloor n, 0 \rfloor} \operatorname{Tot}_\theta X = (x_{p\theta} X_{\operatorname{Spl}_p(\theta)})_{p \in \lfloor m, 0 \rfloor}$ for all $(x_q)_{q \in \lfloor n, 0 \rfloor} \in \operatorname{Tot}_n X$ and for all morphisms $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$.

Proof. For a morphism $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$, we have

$$\begin{aligned} x_{p\theta} X_{\operatorname{Spl}_p(\theta)} \mathrm{d}^{\mathrm{h}}_p &= x_{p\theta} X_{\operatorname{Spl}_{\leq p}(\theta), \operatorname{Spl}_{\geq p}(\theta)} X_{\delta^p, m-p} = x_{p\theta} X_{\delta^p \operatorname{Spl}_{\leq p}(\theta), \operatorname{Spl}_{\geq p}(\theta)} \\ &= x_{p\theta} X_{\operatorname{Spl}_{\leq p-1}(\theta)\delta^{\lceil (p-1)\theta+1, p\theta \rceil}, \operatorname{Spl}_{\geq p}(\theta)} = x_{p\theta} X_{\delta^{\lceil (p-1)\theta+1, p\theta \rceil}, n-p\theta} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p}(\theta)} \\ &= x_{p\theta} \mathrm{d}^{\mathrm{h}}_{\lfloor p\theta, (p-1)\theta+1 \rfloor} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p}(\theta)} \end{aligned}$$

for all $p \in \lfloor m, 1 \rfloor$. Using $x_q \mathrm{d}^{\mathrm{h}}_q = x_{q-1}\mathrm{d}^{\mathrm{v}}_0$ for $q \in \lfloor p\theta, (p-1)\theta + 1 \rfloor$, we obtain

$$x_{p\theta} \mathrm{d}^{\mathrm{h}}_{\lfloor p\theta, (p-1)\theta+1 \rfloor} = x_{(p-1)\theta} \mathrm{d}^{\mathrm{v}}_{\lfloor p\theta - (p-1)\theta - 1, 0 \rfloor}.$$

Since

$$\begin{aligned} x_{(p-1)\theta} \mathrm{d}^{\mathrm{v}}_{\lfloor p\theta - (p-1)\theta - 1, 0 \rfloor} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p}(\theta)} &= x_{(p-1)\theta} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p}(\theta)\delta^{\lceil p\theta - (p-1)\theta - 1, 0 \rceil}} \\ &= x_{(p-1)\theta} X_{\operatorname{Spl}_{\leq p-1}(\theta), \delta^0 \operatorname{Spl}_{\geq p-1}(\theta)} \\ &= x_{(p-1)\theta} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p-1}(\theta)} X_{p-1, \delta^0} = x_{(p-1)\theta} X_{\operatorname{Spl}_{p-1}(\theta)} \mathrm{d}^{\mathrm{v}}_0, \end{aligned}$$

we finally have

$$\begin{aligned} x_{p\theta} X_{\operatorname{Spl}_p(\theta)} \mathrm{d}^{\mathrm{h}}_p &= x_{p\theta} \mathrm{d}^{\mathrm{h}}_{\lfloor p\theta, (p-1)\theta+1 \rfloor} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p}(\theta)} = x_{(p-1)\theta} \mathrm{d}^{\mathrm{v}}_{\lfloor p\theta - (p-1)\theta - 1, 0 \rfloor} X_{\operatorname{Spl}_{\leq p-1}(\theta), \operatorname{Spl}_{\geq p}(\theta)} \\ &= x_{(p-1)\theta} X_{\operatorname{Spl}_{p-1}(\theta)} \mathrm{d}^{\mathrm{v}}_0 \end{aligned}$$

for all $p \in \lfloor m, 1 \rfloor$, that is, $\operatorname{Tot}_\theta X \colon \operatorname{Tot}_n X \to \operatorname{Tot}_m X, (x_q)_{q \in \lfloor n,0 \rfloor} \mapsto (x_{p\theta} X_{\operatorname{Spl}_p(\theta)})_{p \in \lfloor m,0 \rfloor}$ is a well-defined map. By lemma (3.9),

$$\begin{aligned}(x_r)_{r \in \lfloor l,0 \rfloor}(\operatorname{Tot}_{\theta\rho} X) &= (x_{p\theta\rho} X_{\operatorname{Spl}_p(\theta\rho)})_{p \in \lfloor m,0 \rfloor} = (x_{p\theta\rho} X_{\operatorname{Spl}_p(\theta)\operatorname{Spl}_{p\theta}(\rho)})_{p \in \lfloor m,0 \rfloor} \\ &= (x_{p\theta\rho} X_{\operatorname{Spl}_{p\theta}(\rho)} X_{\operatorname{Spl}_p(\theta)})_{p \in \lfloor m,0 \rfloor} = (x_{q\rho} X_{\operatorname{Spl}_q(\rho)})_{q \in \lfloor n,0 \rfloor}(\operatorname{Tot}_\theta X) \\ &= (x_r)_{r \in \lfloor l,0 \rfloor}(\operatorname{Tot}_\rho X)(\operatorname{Tot}_\theta X)\end{aligned}$$

for all $(x_r)_{r \in \lfloor l,0 \rfloor} \in \operatorname{Tot}_r X$ and all morphisms $[m] \xrightarrow{\theta} [n]$ and $[n] \xrightarrow{\rho} [l]$ in $\mathbf{\Delta}$ and since

$$(x_q)_{q \in \lfloor n,0 \rfloor}(\operatorname{Tot}_{\operatorname{id}_{[n]}} X) = (x_{q\operatorname{id}_{[n]}} X_{\operatorname{Spl}_q(\operatorname{id}_{[n]})})_{q \in \lfloor n,0 \rfloor} = (x_q X_{\operatorname{id}_{[q]},\operatorname{id}_{[n-q]}})_{q \in \lfloor n,0 \rfloor} = (x_q)_{q \in \lfloor n,0 \rfloor}$$

for all $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$, $n \in \mathbb{N}_0$, we have in fact a simplicial set. □

(3.11) Definition (total simplicial set)**.** For every bisimplicial set X, the simplicial set $\operatorname{Tot} X$ given as in remark (3.10) by

$$\operatorname{Tot}_n X = \{(x_q)_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} X_{q,n-q} \mid x_q \mathrm{d}^{\mathrm{h}}_q = x_{q-1} \mathrm{d}^{\mathrm{v}}_0 \text{ for all } q \in \lfloor n, 1 \rfloor\} \text{ for all } n \in \mathbb{N}_0$$

and by $(x_q)_{q \in \lfloor n,0 \rfloor}(\operatorname{Tot}_\theta X) = (x_{p\theta} X_{\operatorname{Spl}_p(\theta)})_{p \in \lfloor m,0 \rfloor}$ for all $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$ and for all morphisms $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$, is called the *total simplicial set* of X.

(3.12) Proposition. We let X be a bisimplicial set. The faces in its total simplicial set $\operatorname{Tot} X$ are given by

$$(x_q)_{q \in \lfloor n,0 \rfloor} \mathrm{d}_k = (x_q \mathrm{d}^{\mathrm{h}}_k)_{q \in \lfloor n,k+1 \rfloor} \cup (x_q \mathrm{d}^{\mathrm{v}}_{k-q})_{q \in \lfloor k-1,0 \rfloor}$$

for $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$, $k \in [0, n]$, $n \in \mathbb{N}$, and the degeneracies are given by

$$(x_q)_{q \in \lfloor n,0 \rfloor} \mathrm{s}_k = (x_q \mathrm{s}^{\mathrm{h}}_k)_{q \in \lfloor n,k \rfloor} \cup (x_q \mathrm{s}^{\mathrm{v}}_{k-q})_{q \in \lfloor k,0 \rfloor}$$

for $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$, $k \in [0, n]$, $n \in \mathbb{N}_0$.

Proof. We compute

$$\begin{aligned}(x_q)_{q \in \lfloor n,0 \rfloor} \mathrm{d}_k &= (x_q)_{q \in \lfloor n,0 \rfloor}(\operatorname{Tot}_{\delta^k} X) = (x_{p\delta^k} X_{\operatorname{Spl}_p(\delta^k)})_{p \in \lfloor n-1,0 \rfloor} \\ &= (x_{p+1} X_{\delta^k,\operatorname{id}_{[n-1-p]}})_{p \in \lfloor n-1,k \rfloor} \cup (x_p X_{\operatorname{id}_{[p]},\delta^{k-p}})_{p \in \lfloor k-1,0 \rfloor} \\ &= (x_p \mathrm{d}^{\mathrm{h}}_k)_{p \in \lfloor n,k+1 \rfloor} \cup (x_p \mathrm{d}^{\mathrm{v}}_{k-p})_{p \in \lfloor k-1,0 \rfloor}\end{aligned}$$

for $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$, $k \in [0, n]$, $n \in \mathbb{N}$, and

$$\begin{aligned}(x_q)_{q \in \lfloor n,0 \rfloor} \mathrm{s}_k &= (x_q)_{q \in \lfloor n,0 \rfloor}(\operatorname{Tot}_{\sigma^k} X) = (x_{p\sigma^k} X_{\operatorname{Spl}_p(\sigma^k)})_{p \in \lfloor n-1,0 \rfloor} \\ &= (x_{p-1} X_{\sigma^k,\operatorname{id}_{[n+1-p]}})_{p \in \lfloor n+1,k+1 \rfloor} \cup (x_p X_{\operatorname{id}_{[p]},\sigma^{k-p}})_{p \in \lfloor k,0 \rfloor} = (x_p \mathrm{s}^{\mathrm{h}}_k)_{p \in \lfloor n,k \rfloor} \cup (x_p \mathrm{s}^{\mathrm{v}}_{k-p})_{p \in \lfloor k,0 \rfloor}\end{aligned}$$

for $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$, $k \in [0, n]$, $n \in \mathbb{N}_0$. □

(3.13) Proposition.

(a) We let $X \xrightarrow{f} Y$ be a bisimplicial map between bisimplicial sets X and Y. There exists an induced simplicial map $\operatorname{Tot} f \colon \operatorname{Tot} X \to \operatorname{Tot} Y$ between the corresponding total simplicial sets $\operatorname{Tot} X$ and $\operatorname{Tot} Y$, given by

$$(x_q)_{q \in \lfloor n,0 \rfloor}(\operatorname{Tot}_n f) = (x_q f_{q,n-q})_{q \in \lfloor n,0 \rfloor} \text{ for } (x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X, n \in \mathbb{N}_0.$$

(b) The construction in (a) yields a functor

$$\mathbf{s}^2\mathbf{Set} \xrightarrow{\operatorname{Tot}} \mathbf{sSet}.$$

Proof.

(a) For a given $(x_q)_{q\in\lfloor n,0\rfloor} \in \operatorname{Tot}_n X$, we have

$$x_q f_{q,n-q} \mathrm{d}^{\mathrm{h}}_q = x_q \mathrm{d}^{\mathrm{h}}_q f_{q-1,n-q} = x_{q-1} \mathrm{d}^{\mathrm{v}}_0 f_{q-1,n-q} = x_{q-1} f_{q-1,n-q+1} \mathrm{d}^{\mathrm{v}}_0.$$

Thus the map

$$\operatorname{Tot}_n f\colon \operatorname{Tot}_n X \to \operatorname{Tot}_n Y, (x_q)_{q\in\lfloor n,0\rfloor} \mapsto (x_q f_{q,n-q})_{q\in\lfloor n,0\rfloor}$$

is well-defined for every $n \in \mathbb{N}_0$. We let $[m] \xrightarrow{\theta} [n]$ be a morphism in $\mathbf{\Delta}$. For $(x_q)_{q\in\lfloor n,0\rfloor} \in \operatorname{Tot}_n X$, we compute

$$\begin{aligned}(x_q)_{q\in\lfloor n,0\rfloor}(\operatorname{Tot}_\theta X)(\operatorname{Tot}_m f) &= (x_{p\theta} X_{\operatorname{Spl}_p(\theta)})_{p\in\lfloor m,0\rfloor}(\operatorname{Tot}_m f) = (x_{p\theta} X_{\operatorname{Spl}_p(\theta)} f_{p,m-p})_{p\in\lfloor m,0\rfloor} \\ &= (x_{p\theta} f_{p\theta,n-p\theta} Y_{\operatorname{Spl}_p(\theta)})_{p\in\lfloor m,0\rfloor} = (x_q f_{q,n-q})_{q\in\lfloor n,0\rfloor}(\operatorname{Tot}_\theta Y) \\ &= (x_q)_{q\in\lfloor n,0\rfloor}(\operatorname{Tot}_n f)(\operatorname{Tot}_\theta Y).\end{aligned}$$

Hence we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Tot}_n X & \xrightarrow{\operatorname{Tot}_\theta X} & \operatorname{Tot}_m X \\ \downarrow{\scriptstyle \operatorname{Tot}_n f} & & \downarrow{\scriptstyle \operatorname{Tot}_m f} \\ \operatorname{Tot}_n(Y) & \xrightarrow{\operatorname{Tot}_\theta Y} & \operatorname{Tot}_m Y \end{array}$$

that is, the maps $\operatorname{Tot}_n f$ for $n \in \mathbb{N}$ yield a simplicial map

$$\operatorname{Tot} X \xrightarrow{\operatorname{Tot} f} \operatorname{Tot} Y.$$

(b) We let X, Y and Z be bisimplicial sets and we let $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ be bisimplicial maps. Then we have

$$\begin{aligned}(x_q)_{q\in\lfloor n,0\rfloor} \operatorname{Tot}_n(fg) &= (x_q (fg)_{q,n-q})_{q\in\lfloor n,0\rfloor} = (x_q f_{q,n-q} g_{q,n-q})_{q\in\lfloor n,0\rfloor} = (x_q f_{q,n-q})_{q\in\lfloor n,0\rfloor}(\operatorname{Tot}_n g) \\ &= (x_q)_{q\in\lfloor n,0\rfloor}(\operatorname{Tot}_n f)(\operatorname{Tot}_n g)\end{aligned}$$

and

$$(x_q)_{q\in\lfloor n,0\rfloor}(\operatorname{Tot}_n \mathrm{id}_X) = (x_q (\mathrm{id}_X)_{q,n-q})_{q\in\lfloor n,0\rfloor} = (x_q \mathrm{id}_{X_{q,n-q}})_{q\in\lfloor n,0\rfloor} = (x_q)_{q\in\lfloor n,0\rfloor}$$

for all $(x_q)_{q\in\lfloor n,0\rfloor} \in \operatorname{Tot}_n X$, $n \in \mathbb{N}_0$. Hence we have a functor

$$\mathbf{s^2Set} \xrightarrow{\operatorname{Tot}} \mathbf{sSet}.$$ □

(3.14) Lemma. We have

$$\delta^{\lceil p+1,m\rceil}\theta = \operatorname{Spl}_{\leq p}(\theta)\delta^{\lceil p\theta+1,n\rceil} \text{ and } \delta^{\lceil 0,p-1\rceil}\theta = \operatorname{Spl}_{\geq p}(\theta)\delta^{\lceil 0,p\theta-1\rceil}$$

for all $\theta \in \mathbf{\Delta}([m],[n])$, where $m, n, p \in \mathbb{N}_0$ and $p \leq m$.

Proof. We compute

$$i\delta^{\lceil p+1,m\rceil}\theta = i\theta = i\operatorname{Spl}_{\leq p}(\theta) = i\operatorname{Spl}_{\leq p}(\theta)\delta^{\lceil p\theta+1,n\rceil}$$

and

$$i\delta^{\lceil 0,p-1\rceil}\theta = (i+p)\theta = (i+p)\theta - p\theta + p\theta = i\operatorname{Spl}_{\geq p}(\theta) + p\theta = i\operatorname{Spl}_{\geq p}(\theta)\delta^{\lceil 0,p\theta-1\rceil}$$

for all $i \in [0,p]$. □

The following proposition states a connection between the diagonal simplicial set and the total simplicial set construction.

(3.15) Proposition. We have a natural transformation

$$\mathrm{Diag} \xrightarrow{\phi} \mathrm{Tot}$$

between the functors

$$\mathbf{s}^2\mathbf{Set} \xrightarrow{\mathrm{Diag}} \mathbf{sSet} \text{ and } \mathbf{s}^2\mathbf{Set} \xrightarrow{\mathrm{Tot}} \mathbf{sSet},$$

where the simplicial map

$$\mathrm{Diag}\, X \xrightarrow{\phi_X} \mathrm{Tot}\, X$$

at $X \in \mathrm{Ob}\, \mathbf{sSet}$ is given by $x_n(\phi_X)_n = (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor}$ for all $x_n \in \mathrm{Diag}_n X$, $n \in \mathbb{N}_0$.

Proof. We let X be a bisimplicial set. First of all,

$$x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor} \mathrm{d}^{\mathrm{h}}_q = x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor} = x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-2,0 \rfloor} \mathrm{d}^{\mathrm{v}}_0$$

for $x_n \in \mathrm{Diag}_n X$, $q \in \lfloor n,1 \rfloor$, that is, $(\phi_X)_n \colon \mathrm{Diag}_n X \to \mathrm{Tot}_n X, x_n \mapsto (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor}$ is a well-defined map for all $n \in \mathbb{N}_0$. Given a morphism $\theta \in \mathbf{\Delta}([m],[n])$, lemma (3.14) yields

$$\begin{aligned}
x_n(\mathrm{Diag}_\theta X)(\phi_X)_m &= (x_n X_{\theta,\theta})(\phi_X)_m = (x_n X_{\theta,\theta} \mathrm{d}^{\mathrm{h}}_{\lfloor m,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor})_{p \in \lfloor m,0 \rfloor} \\
&= (x_n X_{\theta,\theta} X_{\delta^{\lceil p+1,m \rceil}, \delta^{\lceil 0,p-1 \rceil}})_{p \in \lfloor m,0 \rfloor} = (x_n X_{\delta^{\lceil p+1,m \rceil}\theta, \delta^{\lceil 0,p-1 \rceil}\theta})_{p \in \lfloor m,0 \rfloor} \\
&= (x_n X_{\mathrm{Spl}_{\leq p}(\theta) \delta^{\lceil p\theta+1,n \rceil}, \mathrm{Spl}_{\geq p}(\theta) \delta^{\lceil 0,p\theta-1 \rceil}})_{p \in \lfloor m,0 \rfloor} \\
&= (x_n X_{\delta^{\lceil p\theta+1,n \rceil}, \delta^{\lceil p\theta-1,0 \rceil}} X_{\mathrm{Spl}_{\leq p}(\theta), \mathrm{Spl}_{\geq p}(\theta)})_{p \in \lfloor m,0 \rfloor} \\
&= (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,p\theta+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p\theta-1,0 \rfloor} X_{\mathrm{Spl}_p(\theta)})_{p \in \lfloor m,0 \rfloor} = (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} (\mathrm{Tot}_\theta X) \\
&= x_n(\phi_X)_n(\mathrm{Tot}_\theta X)
\end{aligned}$$

for all $x_n \in \mathrm{Diag}_n X$. Hence the diagram

$$\begin{array}{ccc}
\mathrm{Diag}_n X & \xrightarrow{\mathrm{Diag}_\theta X} & \mathrm{Diag}_m X \\
\big\downarrow{\scriptstyle (\phi_X)_n} & & \big\downarrow{\scriptstyle (\phi_X)_m} \\
\mathrm{Tot}_n X & \xrightarrow{\mathrm{Tot}_\theta X} & \mathrm{Tot}_m X
\end{array}$$

commutes and we get a simplicial map

$$\mathrm{Diag}\, X \xrightarrow{\phi_X} \mathrm{Tot}\, X.$$

To show naturality, we let X and Y be bisimplicial sets and $X \xrightarrow{f} Y$ be a bisimplicial map. We compute

$$\begin{aligned}
x_n(\phi_X)_n(\mathrm{Tot}_n f) &= (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} (\mathrm{Tot}_n f) = (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor} f_{q,n-q})_{q \in \lfloor n,0 \rfloor} \\
&= (x_n f_{n,n} \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} = x_n f_{n,n} (\phi_Y)_n = x_n (\mathrm{Diag}_n f)(\phi_Y)_n.
\end{aligned}$$

for all $x_n \in \mathrm{Diag}_n X$, $n \in \mathbb{N}_0$, which shows the commutativity of

$$\begin{array}{ccc}
\mathrm{Diag}\, X & \xrightarrow{\phi_X} & \mathrm{Tot}\, X \\
\big\downarrow{\scriptstyle \mathrm{Diag}\, f} & & \big\downarrow{\scriptstyle \mathrm{Tot}\, f} \\
\mathrm{Diag}\, Y & \xrightarrow{\phi_Y} & \mathrm{Tot}\, Y
\end{array}$$

Hence we have indeed a natural transformation

$$\mathrm{Diag} \xrightarrow{\phi} \mathrm{Tot}\,.$$ □

Cegarra and Remedios have proven in [7] that ϕ_X is a so-called *weak homotopy equivalence* for each bisimplicial object X (cf. theorem (4.32)).

§2 Homotopy of double complexes

Recall the normalisation theorem (2.28), which states that the associated complex and the Moore complex of a given simplicial object in an abelian category are homotopy equivalent. Since the corresponding notions also exist for bisimplicial objects (cf. §3), a natural question is to ask for the analogous theorem in this case. Indeed, there is a normalisation theorem for bisimplicial objects, cf. (3.24). In this section, the necessary notion of a double complex homotopy is introduced (cf. [22, section 1.1.5]).
In this section, we suppose given an additive category $\mathcal{A}$.

(3.16) Definition (double complex homotopy)**.** We let $C, D \in \mathbf{C}^2(\mathcal{A})$ be double complexes in $\mathcal{A}$ and we let

$$C \xrightarrow{\varphi} D \text{ and } C \xrightarrow{\psi} D$$

be morphisms of double complexes. The morphisms f and g are said to be *homotopic*, if for all $p, q \in \mathbb{Z}$ there are complex morphisms

$$C_{p,-} \xrightarrow{h^{\mathrm{h}}_{p,-}} D_{p+1,-} \text{ and } C_{-,q} \xrightarrow{h^{\mathrm{v}}_{-,q}} D_{-,q+1}$$

such that

$$h^{\mathrm{h}}_{p,q}\partial^{\mathrm{h}} + \partial^{\mathrm{h}} h^{\mathrm{h}}_{p-1,q} + h^{\mathrm{v}}_{p,q}\partial^{\mathrm{v}} + \partial^{\mathrm{v}} h^{\mathrm{v}}_{p,q-1} = \varphi_{p,q} - \psi_{p,q} \text{ for all } p, q \in \mathbb{Z}.$$

In this case, we write $\varphi \sim \psi$ and we call

$$(h^{\mathrm{h}}_{p,q} \in {}_{\mathcal{A}}(C_{p,q}, D_{p+1,q}) \mid p, q \in \mathbb{Z}) \cup (h^{\mathrm{v}}_{p,q} \in {}_{\mathcal{A}}(C_{p,q}, D_{p,q+1}) \mid p, q \in \mathbb{Z})$$

a *double complex homotopy* from φ to ψ. If $h^{\mathrm{v}}_{-,q} = 0$ for all $q \in \mathbb{Z}$, we say that f and g are *horizontally homotopic* and we call

$$(h^{\mathrm{h}}_{p,q} \in {}_{\mathcal{A}}(C_{p,q}, D_{p+1,q}) \mid p, q \in \mathbb{Z})$$

a *horizontal double complex homotopy* from φ to ψ. Similarly, if $h^{\mathrm{h}}_{p,-} = 0$ for all $p \in \mathbb{Z}$, then f and g are said to be *vertically homotopic* and

$$(h^{\mathrm{v}}_{p,q} \in {}_{\mathcal{A}}(C_{p,q}, D_{p,q+1}) \mid p, q \in \mathbb{Z})$$

is called a *vertical double complex homotopy* from φ to ψ.

Given a double complex $C \in \operatorname{Ob} \mathbf{C}^2_{\urcorner}(\mathcal{A})$, recall that its total complex $\operatorname{Tot} C$ has entries $\operatorname{Tot}_n C = \bigoplus_{p \in \lfloor n,0 \rfloor} C_{p,n-p}$ for $n \in \mathbb{N}_0$ and differentials $\operatorname{Tot}_n C \xrightarrow{\partial} \operatorname{Tot}_{n-1} C$ given by

$$\partial_{p,q} = \begin{cases} \partial^{\mathrm{h}} & \text{if } q = p - 1, \\ (-1)^p \partial^{\mathrm{v}} & \text{if } q = p, \\ 0 & \text{else} \end{cases}$$

for $p \in \lfloor n, 0 \rfloor$, $q \in \lfloor n-1, 0 \rfloor$, $n \in \mathbb{N}$. Moreover, given a morphism φ in $\mathbf{C}^2_{\urcorner}(\mathcal{A})$, we have $\operatorname{Tot}_n \varphi = \bigoplus_{p \in \lfloor n,0 \rfloor} \varphi_{p,n-p}$ for $n \in \mathbb{N}_0$.
The total simplicial set $\operatorname{Tot} X$ of a bisimplicial set $X \in \operatorname{Ob} \mathbf{sSet}$ is not to be confused with the total complex $\operatorname{Tot} C$ of a double complex $C \in \operatorname{Ob} \mathbf{C}^2_{\urcorner}(\mathcal{A})$.

(3.17) Proposition. We let $C, D \in \operatorname{Ob} \mathbf{C}^2_{\urcorner}(\mathcal{A})$ be double complexes in $\mathcal{A}$ and we let

$$C \xrightarrow{\varphi} D \text{ and } C \xrightarrow{\psi} D$$

be double complex morphisms. If φ is homotopic to ψ, then $\operatorname{Tot} \varphi$ is homotopic to $\operatorname{Tot} \psi$.

Proof. We suppose that $\varphi \sim \psi$ via a given double complex homotopy $(h^{\mathrm{h}}_{p,q} \in {}_{\mathcal{A}}(C_{p,q}, D_{p+1,q}) \mid p, q \in \mathbb{Z}) \cup (h^{\mathrm{v}}_{p,q} \in {}_{\mathcal{A}}(C_{p,q}, D_{p,q+1}) \mid p, q \in \mathbb{Z})$. Then we can define morphisms

$$\bigoplus_{p \in \lfloor n,0 \rfloor} C_{p,n-p} \xrightarrow{h_n} \bigoplus_{q \in \lfloor n+1,0 \rfloor} D_{q,n+1-q}$$

by setting

$$(h_n)_{p,q} = \begin{cases} (-1)^p h^{\mathrm{v}}_{p,n-p} & \text{if } q = p, \\ h^{\mathrm{h}}_{p,n-p} & \text{if } q = p+1, \\ 0 & \text{else,} \end{cases}$$

for all $p \in \lfloor n, 0 \rfloor$, $q \in \lfloor n+1, 0 \rfloor$, that is

$$h_n = \begin{pmatrix} h^{\mathrm{h}}_{n,0} & (-1)^n h^{\mathrm{v}}_{n,0} & & & 0 \\ & h^{\mathrm{h}}_{n-1} & \ddots & & \\ & & \ddots & -h^{\mathrm{v}}_{1,n-1} & \\ 0 & & & h^{\mathrm{h}}_{0,n} & h^{\mathrm{v}}_{0,n} \end{pmatrix}.$$

We get

$$\begin{aligned}(h_n \partial + \partial h_{n-1})_{p,r} &= (h_n \partial)_{p,r} + (\partial h_{n-1})_{p,r} = \sum_{q \in \lfloor n+1,0 \rfloor} (h_n)_{p,q} \partial_{q,r} + \sum_{q \in \lfloor n-1,0 \rfloor} \partial_{p,q} (h_{n-1})_{q,r} \\ &= (-1)^p h^{\mathrm{v}}_{p,n-p} \partial_{p,r} + h^{\mathrm{h}}_{p,n-p} \partial_{p+1,r} + \partial_{p,r-1} h^{\mathrm{h}}_{r-1,n-r} + (-1)^r \partial_{p,r} h^{\mathrm{v}}_{r,n-1-r} \\ &= \left\{ \begin{array}{ll} (-1)^p h^{\mathrm{v}}_{p,n-p} \partial^{\mathrm{h}} + (-1)^{p-1} \partial^{\mathrm{h}} h^{\mathrm{v}}_{p-1,n-p} & \text{if } r = p-1, \\ h^{\mathrm{v}}_{p,n-p} \partial^{\mathrm{v}} + h^{\mathrm{h}}_{p,n-p} \partial^{\mathrm{h}} + \partial^{\mathrm{h}} h^{\mathrm{h}}_{p-1,n-p} + \partial^{\mathrm{v}} h^{\mathrm{v}}_{p,n-1-p} & \text{if } r = p, \\ (-1)^{p+1} h^{\mathrm{h}}_{p,n-p} \partial^{\mathrm{v}} + (-1)^p \partial^{\mathrm{v}} h^{\mathrm{h}}_{p,n-1-p} & \text{if } r = p+1, \\ 0 & \text{else} \end{array} \right\} \\ &= \left\{ \begin{array}{ll} \varphi_{p,n-p} - \psi_{p,n-p} & \text{if } r = p, \\ 0 & \text{else} \end{array} \right\} = (\operatorname{Tot}_n \varphi - \operatorname{Tot}_n \psi)_{p,r}\end{aligned}$$

for all $p, r \in \lfloor n, 0 \rfloor$, $n \in \mathbb{N}_0$, that is, $(h_n \in {}_{\mathcal{A}}(\operatorname{Tot}_n C, \operatorname{Tot}_{n+1} D) \mid n \in \mathbb{N}_0)$ is a complex homotopy from $\operatorname{Tot} \varphi$ to $\operatorname{Tot} \psi$. □

The preceeding proposition shows that Tot is a functor from the so-called homotopy category of bounded double complexes to the so-called homotopy category of bounded complexes. We will not introduce these categories here, because we do not need them; however, we shall make use of one consequence of Tot being a functor between these homotopy categories.

(3.18) Definition (double complex homotopy equivalence)**.** Double complexes $C, D \in \mathbf{C}^2(\mathcal{A})$ in $\mathcal{A}$ are said to be (*double complex*) *homotopy equivalent*, if there are morphisms

$$C \xrightarrow{\varphi} D \text{ and } D \xrightarrow{\psi} C$$

such that $\varphi\psi \sim \mathrm{id}_C$ and $\psi\varphi \sim \mathrm{id}_D$. In this case, we call φ and ψ mutually inverse (*double complex*) *homotopy equivalences*.

(3.19) Proposition. If $C, D \in \operatorname{Ob} \mathbf{C}^2_{\urcorner}(\mathcal{A})$ are homotopy equivalent double complexes in $\mathcal{A}$, then their total complexes $\operatorname{Tot} C$ and $\operatorname{Tot} D$ are homotopy equivalent, too.

Proof. Suppose there is a double complex homotopy equivalence $C \xrightarrow{\varphi} D$ with inverse $D \xrightarrow{\psi} C$, that is, $\varphi\psi \sim \mathrm{id}_C$ and $\psi\varphi \sim \mathrm{id}_D$. According to proposition (3.17), it follows that

$$(\operatorname{Tot} \varphi)(\operatorname{Tot} \psi) = \operatorname{Tot}(\varphi\psi) \sim \operatorname{Tot}(\mathrm{id}_C) = \mathrm{id}_{\operatorname{Tot} C} \text{ and } (\operatorname{Tot} \psi)(\operatorname{Tot} \varphi) = \operatorname{Tot}(\psi\varphi) \sim \operatorname{Tot}(\mathrm{id}_D) = \mathrm{id}_{\operatorname{Tot} D}.$$

Thus $\operatorname{Tot} \varphi$ is a complex homotopy equivalence from $\operatorname{Tot} C$ to $\operatorname{Tot} D$ with inverse $\operatorname{Tot} \psi$. □

§3 Homology of bisimplicial objects

In this section, double complexes attached to a bisimplicial object are introduced and a normalisation theorem is proven, cf. theorem (2.28).
We note that the categories $\mathbf{C}(\mathbf{s}\mathcal{A})$ and $\mathbf{sC}(\mathcal{A})$ for an additive category $\mathcal{A}$ are equivalent, and we identify them.

(3.20) Definition (associated double complex)**.** We let $A \in \operatorname{Ob} \mathbf{s}^2\mathcal{A}$ be a bisimplicial object in an additive category $\mathcal{A}$. The *associated double complex* of A is defined by $\mathrm{C}^{(2)}A := \mathrm{CC}A$. Here, $\mathrm{C}_p\mathrm{C}_q A = A_{p,q}$ for all $p, q \in \mathbb{N}_0$.

(3.21) Definition (associated double complex to a bisimplicial set)**.** We let R be a commutative ring. For a bisimplicial set X, the double complex $\mathrm{C}^{(2)}(X; R) := \mathrm{C}^{(2)}RX$ is called the *double complex associated to* X *over* R. If $R = \mathbb{Z}$, we will just speak of the *double complex associated to* X and we write $\mathrm{C}^{(2)}(X) := \mathrm{C}^{(2)}(X; \mathbb{Z})$.

(3.22) Remark. There exists a double complex $\mathrm{D}^{(2)}A \preceq \mathrm{C}^{(2)}A$ for every bisimplicial object $A \in \operatorname{Ob} \mathbf{s}^2\mathcal{A}$ in an abelian category $\mathcal{A}$ with $\mathrm{D}^{(2)}_{p,q}A := \sum_{i \in [0,p-1]} \operatorname{Im} \mathrm{s}^{\mathrm{h}}_i + \sum_{j \in [0,q-1]} \operatorname{Im} \mathrm{s}^{\mathrm{v}}_j$ for all $p, q \in \mathbb{N}_0$.

Proof. Analogous to remark (2.25), using remark (3.3). □

(3.23) Definition (Moore double complex, degenerate double complex)**.** We let $A \in \operatorname{Ob} \mathbf{s}^2\mathcal{A}$ be a bisimplicial object in an abelian category $\mathcal{A}$.

(a) The *Moore double complex* of A is defined by $\mathrm{M}^{(2)}A := \mathrm{MM}A$.

(b) The *degenerate double complex* of A is the sub-double complex $\mathrm{D}^{(2)}A \preceq \mathrm{C}^{(2)}A$ given as in remark (3.22) by $\mathrm{D}^{(2)}_{p,q}A := \sum_{i \in [0,p-1]} \operatorname{Im} \mathrm{s}^{\mathrm{h}}_i + \sum_{j \in [0,q-1]} \operatorname{Im} \mathrm{s}^{\mathrm{v}}_j$ for all $p, q \in \mathbb{N}_0$.

(3.24) Theorem (normalisation theorem for bisimplicial objects)**.** We have

$$\mathrm{C}^{(2)}A \cong \mathrm{D}^{(2)}A \oplus \mathrm{M}^{(2)}A \text{ and } \mathrm{C}^{(2)}A \simeq \mathrm{M}^{(2)}A$$

for each bisimplicial object $A \in \operatorname{Ob} \mathbf{s}^2\mathcal{A}$ in an abelian category $\mathcal{A}$.

Proof. We let $A \in \operatorname{Ob} \mathbf{s}^2\mathcal{A}$ be a bisimplicial object in $\mathcal{A}$. Then we have

$$\mathrm{C}^{(2)}A = \mathrm{CC}A \simeq \mathrm{MC}A \simeq \mathrm{MM}A = \mathrm{M}^{(2)}A,$$

where the first homotopy equivalence is an application of the normalisation theorem (2.28) to the category $\mathbf{sC}(\mathcal{A})$ and the second homotopy equivalence results from the fact that the additive functor M maps homotopy equivalent complexes over $\mathbf{s}\mathcal{A}$ to homotopy equivalent complexes over $\mathbf{C}(\mathcal{A})$. Furthermore, the normalisation theorem (2.28) yields

$$\mathrm{C}^{(2)}A = \mathrm{CC}A \cong \mathrm{DC}A \oplus \mathrm{MC}A \cong \mathrm{DC}A \oplus \mathrm{MD}A \oplus \mathrm{MM}A.$$

Since $\mathrm{M}^{(2)}A = \mathrm{MM}A$, it remains to show that $\mathrm{D}^{(2)}A = \mathrm{DC}A \oplus \mathrm{MD}A$. On the one hand, we have

$$\mathrm{D}_p\mathrm{C}_qA \oplus \mathrm{M}_p\mathrm{D}_qA \preceq \mathrm{D}_p\mathrm{C}_qA + \mathrm{C}_p\mathrm{D}_qA = \sum_{i \in [0,p-1]} \operatorname{Im} \mathrm{s}^{\mathrm{h}}_i + \sum_{j \in [0,q-1]} \operatorname{Im} \mathrm{s}^{\mathrm{v}}_j = \mathrm{D}^{(2)}_{p,q}A$$

for all $p, q \in \mathbb{N}_0$. Conversely, we have

$$\operatorname{Im} \mathrm{s}^{\mathrm{h}}_i = (A_{p-1,q})\mathrm{s}^{\mathrm{h}}_i \cong (\mathrm{M}_qA_{p-1,-} \oplus \mathrm{D}_qA_{p-1,-})\mathrm{s}^{\mathrm{h}}_i \preceq (\mathrm{M}_qA_{p-1,-})\mathrm{s}^{\mathrm{h}}_i + (\mathrm{D}_qA_{p-1,-})\mathrm{s}^{\mathrm{h}}_i \preceq \mathrm{D}_p\mathrm{M}_qA + \mathrm{D}_p\mathrm{D}_qA$$

for all $i \in [0, p-1]$ and analogously

$$\operatorname{Im} \mathrm{s}^{\mathrm{v}}_j \preceq \mathrm{M}_p\mathrm{D}_qA + \mathrm{D}_p\mathrm{D}_qA$$

for all $j \in [0, q-1]$. Hence

$$\mathrm{D}^{(2)}_{p,q}A = \sum_{i \in [0,p-1]} \operatorname{Im} \mathrm{s}^{\mathrm{h}}_i + \sum_{j \in [0,q-1]} \operatorname{Im} \mathrm{s}^{\mathrm{v}}_j \preceq \mathrm{D}_p\mathrm{M}_qA + \mathrm{D}_p\mathrm{D}_qA + \mathrm{M}_p\mathrm{D}_qA + \mathrm{D}_p\mathrm{D}_qA = \mathrm{D}_p\mathrm{C}_qA \oplus \mathrm{M}_p\mathrm{D}_qA$$

for all $p, q \in \mathbb{N}_0$. □

(3.25) Definition (path bisimplicial object)**.** For a given category $\mathcal{C}$, we define the functor

$$\mathrm{P}^{(2)} := \big((\mathrm{Sh})^{\mathrm{op}} \times (\mathrm{Sh})^{\mathrm{op}}, \mathcal{C}\big).$$

Given a bisimplicial object $X \in \operatorname{Ob} \mathbf{s}\mathcal{C}$ in $\mathcal{C}$, the functor $\mathrm{P}^{(2)}$ assigns to X the bisimplicial object $\mathrm{P}^{(2)}X = X \circ ((\mathrm{Sh})^{\mathrm{op}} \times (\mathrm{Sh})^{\mathrm{op}})$. It is called the *path bisimplicial object* of X. In particular, we have $\mathrm{P}^{(2)}_{p,q}X = X_{p+1,q+1}$ for $p, q \in \mathbb{N}_0$.

§4 The generalised Eilenberg-Zilber theorem

We give a proof of the generalised Eilenberg-Zilber theorem of Dold, Puppe and Cartier [9]. The arguments used here are adapted from the articles [12], [13] of Eilenberg and Mac Lane.
Throughout this section, we suppose given a bisimplicial object A in an abelian category $\mathcal{A}$.

(3.26) Definition (shuffle)**.** We let $n \in \mathbb{N}_0$ and $p \in \lfloor n, 0\rfloor$. A $(p, n-p)$*-shuffle* is a permutation $\mu \in \mathrm{S}_{[0,n-1]}$ such that $\mu|_{[0,p-1]}$ and $\mu|_{[p,n-1]}$ are strictly monotonically increasing maps, where we write $\mathrm{S}_{[0,n-1]}$ for the symmetric group on $[0, n-1]$. The set of all $(p, n-p)$-shuffles is denoted by $\mathrm{Sh}_{p,n-p}$.

(3.27) Definition. We let $n \in \mathbb{N}_0$, $p \in \lfloor n, 0\rfloor$.

(a) The morphism

$$A_{n,n} \xrightarrow{\mathsf{AW}^A_{p,n-p}} A_{p,n-p}$$

is defined by

$$\mathsf{AW}_{p,n-p} := \mathsf{AW}^A_{p,n-p} := \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1\rfloor}\mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0\rfloor}.$$

(b) Further, we let

$$A_{p,n-p} \xrightarrow{\nabla_{p,n-p}} A_{n,n}$$

be given by

$$\nabla_{p,n-p} := \nabla^A_{p,n-p} := \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn}\mu)\mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1\rceil\mu}\mathrm{s}^{\mathrm{h}}_{\lceil p,n-1\rceil\mu}.$$

Our first aim is to show that these morphisms yield complex morphisms between $\mathrm{C}\operatorname{Diag} A$ and $\operatorname{Tot}\mathrm{C}^{(2)}A$.

(3.28) Remark. Defining

$$\mathrm{C}_n \operatorname{Diag} A \xrightarrow{\mathsf{AW}_n} \operatorname{Tot}_n \mathrm{C}^{(2)}A$$

by $\mathsf{AW}_n\mathrm{pr}_{p,n-p} := \mathsf{AW}_{p,n-p}$ for all $p \in \lfloor n, 0\rfloor$, $n \in \mathbb{N}_0$, we obtain a complex morphism

$$\mathrm{C}\operatorname{Diag} A \xrightarrow{\mathsf{AW}} \operatorname{Tot}\mathrm{C}^{(2)}A.$$

Proof. We have

$$\begin{aligned}
\mathsf{AW}_n\partial\mathrm{pr}_{p,n-1-p} &= \mathsf{AW}_{p+1,n}\partial^{\mathrm{h}} + (-1)^p\mathsf{AW}_{p,n-p}\partial^{\mathrm{v}} \\
&= \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+2\rfloor}\mathrm{d}^{\mathrm{v}}_{\lfloor p,0\rfloor}\Big(\sum_{i\in[0,p+1]}(-1)^i\mathrm{d}^{\mathrm{h}}_i\Big) + (-1)^p\mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1\rfloor}\mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0\rfloor}\Big(\sum_{j\in[0,n-p]}(-1)^j\mathrm{d}^{\mathrm{v}}_j\Big) \\
&= \sum_{i\in[0,p+1]}(-1)^i\mathrm{d}^{\mathrm{h}}_{\lfloor n,p+2\rfloor}\mathrm{d}^{\mathrm{h}}_i\mathrm{d}^{\mathrm{v}}_{\lfloor p,0\rfloor} + \sum_{j\in[0,n-p]}(-1)^{j+p}\mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1\rfloor}\mathrm{d}^{\mathrm{v}}_{j+p}\mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0\rfloor} \\
&= \sum_{i\in[0,p+1]}(-1)^i\mathrm{d}^{\mathrm{h}}_{\lfloor n,p+2\rfloor}\mathrm{d}^{\mathrm{h}}_i\mathrm{d}^{\mathrm{v}}_{\lfloor p,0\rfloor} + \sum_{j\in[p,n]}(-1)^j\mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1\rfloor}\mathrm{d}^{\mathrm{v}}_j\mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0\rfloor}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i \in [0,p]} (-1)^i \mathrm{d}^{\mathrm{h}}_i \mathrm{d}^{\mathrm{v}}_i \mathrm{d}^{\mathrm{h}}_{\lfloor n-1,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor} + \sum_{j \in [p+1,n]} (-1)^j \mathrm{d}^{\mathrm{h}}_j \mathrm{d}^{\mathrm{v}}_j \mathrm{d}^{\mathrm{h}}_{\lfloor n-1,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor} \\
&= (\sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{h}}_k \mathrm{d}^{\mathrm{v}}_k) \mathrm{d}^{\mathrm{h}}_{\lfloor n-1,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor} = (\sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{Diag}\,A}_k) \mathrm{d}^{\mathrm{h}}_{\lfloor n-1,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor} \\
&= \partial \mathsf{AW}_{p,n-1-p} = \partial \mathsf{AW}_{n-1} \mathrm{pr}_{p,n-1-p}
\end{aligned}
$$

for all $p \in \lfloor n-1, 0 \rfloor$, $n \in \mathbb{N}$, that is, we get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{C}_n \operatorname{Diag} A & \xrightarrow{\partial} & \mathrm{C}_{n-1} \operatorname{Diag} A \\
{\scriptstyle \mathsf{AW}_n} \big\downarrow & & \big\downarrow {\scriptstyle \mathsf{AW}_{n-1}} \\
\operatorname{Tot}_n \mathrm{C}^{(2)} A & \xrightarrow{\partial} & \operatorname{Tot}_{n-1} \mathrm{C}^{(2)} A
\end{array}
$$

for every $n \in \mathbb{N}$. Thus the morphisms AW_n for $n \in \mathbb{N}_0$ yield a complex morphism

$$\mathrm{C} \operatorname{Diag} A \xrightarrow{\mathsf{AW}} \operatorname{Tot} \mathrm{C}^{(2)} A.$$ □

(3.29) Definition (Alexander-Whitney morphism)**.** The complex morphism

$$\mathrm{C} \operatorname{Diag} A \xrightarrow{\mathsf{AW}} \operatorname{Tot} \mathrm{C}^{(2)} A$$

given as in remark (3.28) by $\mathsf{AW}_n \mathrm{pr}_{p,n-p} = \mathsf{AW}_{p,n-p}$ for all $p \in \lfloor n, 0 \rfloor$, that is,

$$\mathsf{AW}_n = \begin{pmatrix} \mathsf{AW}_{n,0} & \dots & \mathsf{AW}_{0,n} \end{pmatrix}$$

as a morphism from $\mathrm{C}_n \operatorname{Diag} A = A_{n,n}$ to $\operatorname{Tot}_n \mathrm{C}^{(2)} A = \bigoplus_{p \in \lfloor n,0 \rfloor} A_{p,n-p}$ for all $n \in \mathbb{N}_0$, is called *Alexander-Whitney morphism.*

(3.30) Proposition (recursive characterisation of the shuffle morphism via path simplicial objects)**.** We have $\nabla^A_{0,0} = \mathrm{id}_{A_{0,0}}$ and

$$
\nabla^A_{p,n-p} = \begin{cases}
\mathrm{s}^{\mathrm{v}}_0 \nabla^{\mathrm{P}^{(2)} A}_{n-1,0} & \text{if } p = n, \\
(-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \nabla^{\mathrm{P}^{(2)} A}_{p-1,n-p} + \mathrm{s}^{\mathrm{h}}_p \nabla^{\mathrm{P}^{(2)} A}_{p,n-1-p} & \text{if } p \in \lfloor n-1, 1 \rfloor, \\
\mathrm{s}^{\mathrm{h}}_0 \nabla^{\mathrm{P}^{(2)} A}_{0,n-1} & \text{if } p = 0
\end{cases}
$$

for all $p \in \lfloor n, 0 \rfloor$, $n \in \mathbb{N}$.

Proof. We let $n \in \mathbb{N}$. For $p \in \lfloor n-1, 1 \rfloor$ we have

$$
\begin{aligned}
\nabla^A_{p,n-p} &= \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil} \\
&= \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (p-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} + \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (n-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \\
&= \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (p-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-2 \rceil \mu} \mathrm{s}^{\mathrm{v}}_{n-1} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} + \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (n-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-2 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{n-1} \\
&= \mathrm{s}^{\mathrm{v}}_{n-p} \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (p-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-2 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} + \mathrm{s}^{\mathrm{h}}_p \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (n-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-2 \rceil \mu}.
\end{aligned}
$$

Since there are bijections

$$\{\mu \in \mathrm{Sh}_{p,n-p} \mid (p-1)\mu = n-1\} \to \{\mu \in \mathrm{Sh}_{p-1,n-p+1} \mid (n-1)\mu = n-1\}, \mu \mapsto (p-1, p, \dots, n-2, n-1)\mu$$

and

$$\{\mu \in \mathrm{Sh}_{p,n-p} \mid (n-1)\mu = n-1\} \to \mathrm{Sh}_{p,n-1-p}, \mu \mapsto \mu|_{\lceil 0,n-2 \rceil}$$

we can conclude

$$\begin{aligned}\nabla^A_{p,n-p} &= \mathrm{s}^{\mathrm{v}}_{n-p} \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (p-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-2 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} + \mathrm{s}^{\mathrm{h}}_p \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (n-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-2 \rceil \mu} \\
&= (-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \sum_{\substack{\mu \in \mathrm{Sh}_{p-1,n-p+1} \\ (n-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-2 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p-1,n-2 \rceil \mu} + \mathrm{s}^{\mathrm{h}}_p \sum_{\substack{\mu \in \mathrm{Sh}_{p,n-p} \\ (n-1)\mu = n-1}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-2 \rceil \mu} \\
&= (-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \sum_{\mu \in \mathrm{Sh}_{p-1,n-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-2 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p-1,n-2 \rceil \mu} + \mathrm{s}^{\mathrm{h}}_p \sum_{\mu \in \mathrm{Sh}_{p,n-1-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-2 \rceil \mu} \\
&= (-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \nabla^{\mathrm{P}^{(2)} A}_{p-1,n-p} + \mathrm{s}^{\mathrm{h}}_p \nabla^{\mathrm{P}^{(2)} A}_{p,n-1-p}\end{aligned}$$

The proof for $p = n$ or $p = 0$ is easier since in this case the sum in the shuffle morphism does not split into two sums and since the only $(n, 0)$-shuffle resp. $(0, n)$-shuffle is the identity: We have

$$\nabla^A_{n,0} = \mathrm{s}^{\mathrm{v}}_{\lceil 0,n-1 \rceil} = \mathrm{s}^{\mathrm{v}}_0 \mathrm{s}^{\mathrm{v}}_{\lceil 0,n-2 \rceil} = \mathrm{s}^{\mathrm{v}}_0 \nabla^{\mathrm{P}^{(2)} A}_{n-1,0}$$

and

$$\nabla^A_{0,n} = \mathrm{s}^{\mathrm{h}}_{\lceil 0,n-1 \rceil} = \mathrm{s}^{\mathrm{h}}_0 \mathrm{s}^{\mathrm{h}}_{\lceil 0,n-2 \rceil} = \mathrm{s}^{\mathrm{h}}_0 \nabla^{\mathrm{P}^{(2)} A}_{0,n-1}.$$ □

(3.31) Lemma. We have

$$\nabla^A_{p,n-p} \mathrm{s}^{\operatorname{Diag} A}_n = \mathrm{s}^{\mathrm{v}}_{n-p} \mathrm{s}^{\mathrm{h}}_p \nabla^{\mathrm{P}^{(2)} A}_{p,n-p}$$

and

$$\nabla^{\mathrm{P}^{(2)} A}_{p,n-p} \mathrm{d}^{\operatorname{Diag} A}_{n+1} = \mathrm{d}^{\mathrm{v}}_{n+1-p} \mathrm{d}^{\mathrm{h}}_{p+1} \nabla^A_{p,n-p}$$

for all $p \in \lfloor n, 0 \rfloor$, $n \in \mathbb{N}_0$.

Proof. We compute

$$\begin{aligned}\nabla^A_{p,n-p} \mathrm{s}^{\operatorname{Diag} A}_n &= \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \mathrm{s}^{\mathrm{v}}_n \mathrm{s}^{\mathrm{h}}_n = \mathrm{s}^{\mathrm{v}}_{n-p} \mathrm{s}^{\mathrm{h}}_p \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \\
&= \mathrm{s}^{\mathrm{v}}_{n-p} \mathrm{s}^{\mathrm{h}}_p \nabla^{\mathrm{P}^{(2)} A}_{p,n-p}\end{aligned}$$

and

$$\begin{aligned}\nabla^{\mathrm{P}^{(2)} A}_{p,n-p} \mathrm{d}^{\operatorname{Diag} A}_{n+1} &= \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \mathrm{d}^{\mathrm{v}}_{n+1} \mathrm{d}^{\mathrm{h}}_{n+1} \\
&= \mathrm{d}^{\mathrm{v}}_{n+1-p} \mathrm{d}^{\mathrm{h}}_{p+1} \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu) \mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} = \mathrm{d}^{\mathrm{v}}_{n+1-p} \mathrm{d}^{\mathrm{h}}_{p+1} \nabla^A_{p,n-p}\end{aligned}$$

for all $p \in \lfloor n, 0 \rfloor$, $n \in \mathbb{N}_0$. □

(3.32) Remark. We have

$$\nabla_{p,n-p} \mathrm{d}^{\operatorname{Diag} A}_n = \begin{cases} \mathrm{d}^{\mathrm{h}}_n \nabla_{n-1,0} & \text{if } p = n, \\ (-1)^{n-p} \mathrm{d}^{\mathrm{h}}_p \nabla_{p-1,n-p} + \mathrm{d}^{\mathrm{v}}_{n-p} \nabla_{p,n-1-p} & \text{if } p \in \lfloor n-1, 1 \rfloor, \\ \mathrm{d}^{\mathrm{v}}_n \nabla_{0,n-1} & \text{if } p = 0 \end{cases}$$

for all $p \in \lfloor n, 0 \rfloor$, $n \in \mathbb{N}$.

Proof. According to proposition (3.30) and lemma (3.31), we have

$$\begin{aligned}\nabla^A_{p,n-p} \mathrm{d}^{\operatorname{Diag} A}_n &= ((-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \nabla^{\mathrm{P}^{(2)} A}_{p-1,n-p} + \mathrm{s}^{\mathrm{h}}_p \nabla^{\mathrm{P}^{(2)} A}_{p,n-1-p}) \mathrm{d}^{\operatorname{Diag} A}_n \\
&= (-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \nabla^{\mathrm{P}^{(2)} A}_{p-1,n-p} \mathrm{d}^{\operatorname{Diag} A}_n + \mathrm{s}^{\mathrm{h}}_p \nabla^{\mathrm{P}^{(2)} A}_{p,n-1-p} \mathrm{d}^{\operatorname{Diag} A}_n\end{aligned}$$

$$= (-1)^{n-p} \mathrm{s}^{\mathrm{v}}_{n-p} \mathrm{d}^{\mathrm{v}}_{n+1-p} \mathrm{d}^{\mathrm{h}}_{p} \nabla^{A}_{p-1,n-p} + \mathrm{s}^{\mathrm{h}}_{p} \mathrm{d}^{\mathrm{v}}_{n-p} \mathrm{d}^{\mathrm{h}}_{p+1} \nabla^{A}_{p,n-1-p}$$
$$= (-1)^{n-p} \mathrm{d}^{\mathrm{h}}_{p} \nabla^{A}_{p-1,n-p} + \mathrm{d}^{\mathrm{v}}_{n-p} \nabla^{A}_{p,n-1-p}$$

for $p \in \lfloor n-1, 1\rfloor$. The computation for $p = n$ and $p = 0$ is analogous: We have

$$\nabla^{A}_{n,0} \mathrm{d}^{\mathrm{Diag}\, A}_{n} = (\mathrm{s}^{\mathrm{v}}_{0} \nabla^{\mathrm{P}^{(2)} A}_{n-1,0}) \mathrm{d}^{\mathrm{Diag}\, A}_{n} = \mathrm{s}^{\mathrm{v}}_{0} \mathrm{d}^{\mathrm{v}}_{1} \mathrm{d}^{\mathrm{h}}_{n} \nabla^{A}_{n-1,0} = \mathrm{d}^{\mathrm{h}}_{n} \nabla^{A}_{n-1,0}$$

and

$$\nabla^{A}_{0,n} \mathrm{d}^{\mathrm{Diag}\, A}_{n} = (\mathrm{s}^{\mathrm{h}}_{0} \nabla^{\mathrm{P}^{(2)} A}_{0,n-1}) \mathrm{d}^{\mathrm{Diag}\, A}_{n} = \mathrm{s}^{\mathrm{h}}_{0} \mathrm{d}^{\mathrm{v}}_{n} \mathrm{d}^{\mathrm{h}}_{1} \nabla^{A}_{0,n-1} = \mathrm{d}^{\mathrm{v}}_{n} \nabla^{A}_{0,n-1}. \qquad \square$$

(3.33) Remark. Defining

$$\mathrm{Tot}_n\, \mathrm{C}^{(2)} A \xrightarrow{\nabla_n} \mathrm{C}_n\, \mathrm{Diag}\, A$$

by $\mathrm{emb}_{p,n-p} \nabla_n := \nabla_{p,n-p}$ for all $p \in \lfloor n, 0\rfloor$, $n \in \mathbb{N}_0$, we obtain a complex morphism

$$\mathrm{Tot}\, \mathrm{C}^{(2)} A \xrightarrow{\nabla} \mathrm{C}\, \mathrm{Diag}\, A.$$

Proof. By the definition of the differential morphisms in $\mathrm{Tot}\, \mathrm{C}^{(2)} A$, we have to show that

$$\nabla^{A}_{p,n-p} \partial^{\mathrm{C}\, \mathrm{Diag}\, A} = \begin{cases} \partial^{\mathrm{C}^{(2)} A, \mathrm{h}} \nabla^{A}_{n-1,0} & \text{if } p = n, \\ \partial^{\mathrm{C}^{(2)} A, \mathrm{h}} \nabla^{A}_{p-1,n-p} + (-1)^p \partial^{\mathrm{C}^{(2)} A, \mathrm{v}} \nabla^{A}_{p,n-1-p} & \text{if } p \in \lfloor n-1, 1\rfloor, \\ \partial^{\mathrm{C}^{(2)} A, \mathrm{v}} \nabla^{A}_{0,n-1} & \text{if } p = 0 \end{cases}$$

for all $p \in \lfloor n, 0\rfloor$, $n \in \mathbb{N}_0$. First, we consider the boundary cases: We have

$$\nabla^{A}_{n,0} \partial^{\mathrm{C}\, \mathrm{Diag}\, A} = \mathrm{s}^{\mathrm{v}}_{\lceil 0,n-1\rceil} \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{Diag}\, A}_{k} = \mathrm{s}^{\mathrm{v}}_{\lceil 0,n-1\rceil} \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{v}}_{k} \mathrm{d}^{\mathrm{h}}_{k}$$
$$= \sum_{k \in [0,n]} (-1)^k \mathrm{s}^{\mathrm{v}}_{\lceil 0,n-1\rceil} \mathrm{d}^{\mathrm{v}}_{k} \mathrm{d}^{\mathrm{h}}_{k} = \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{h}}_{k} \mathrm{s}^{\mathrm{v}}_{\lceil 0,n-2\rceil} = \partial^{\mathrm{C}^{(2)} A, \mathrm{h}} \nabla^{A}_{n-1,0}$$

and analogously

$$\nabla^{A}_{0,n} \partial^{\mathrm{C}\, \mathrm{Diag}\, A} = \mathrm{s}^{\mathrm{h}}_{\lceil 0,n-1\rceil} \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{Diag}\, A}_{k} = \mathrm{s}^{\mathrm{h}}_{\lceil 0,n-1\rceil} \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{v}}_{k} \mathrm{d}^{\mathrm{h}}_{k}$$
$$= \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{v}}_{k} \mathrm{s}^{\mathrm{h}}_{\lceil 0,n-1\rceil} \mathrm{d}^{\mathrm{h}}_{k} = \sum_{k \in [0,n]} (-1)^k \mathrm{d}^{\mathrm{v}}_{k} \mathrm{s}^{\mathrm{h}}_{\lceil 0,n-2\rceil} = \partial^{\mathrm{C}^{(2)} A, \mathrm{v}} \nabla^{A}_{0,n-1}$$

for all $n \in \mathbb{N}$.
It remains to prove

$$\nabla^{A}_{p,n-p} \partial^{\mathrm{C}\, \mathrm{Diag}\, A} = \partial^{\mathrm{C}^{(2)} A, \mathrm{h}} \nabla^{A}_{p-1,n-p} + (-1)^p \partial^{\mathrm{C}^{(2)} A, \mathrm{v}} \nabla^{A}_{p,n-1-p}$$

for $p \in \lfloor n-1, 1\rfloor$, $n \in \mathbb{N}$, $n \geq 2$. Thereto, it suffices to show that

$$\nabla^{A}_{p,n-p} \partial^{\mathrm{C}\, \mathrm{Diag}\, \mathrm{P}^{(2)} A} = \partial^{\mathrm{C}^{(2)} \mathrm{P}^{(2)} A, \mathrm{h}} \nabla^{A}_{p-1,n-p} + (-1)^p \partial^{\mathrm{C}^{(2)} \mathrm{P}^{(2)} A, \mathrm{v}} \nabla^{A}_{p,n-1-p}$$

because according to remark (3.32) this implies

$$\nabla^{A}_{p,n-p} \partial^{\mathrm{C}\, \mathrm{Diag}\, A} = \nabla^{A}_{p,n-p} (\partial^{\mathrm{C}\, \mathrm{Diag}\, \mathrm{P}^{(2)} A} + (-1)^n \mathrm{d}^{\mathrm{Diag}\, A}_{n}) = \nabla^{A}_{p,n-p} \partial^{\mathrm{C}\, \mathrm{Diag}\, \mathrm{P}^{(2)} A} + (-1)^n \nabla^{A}_{p,n-p} \mathrm{d}^{\mathrm{Diag}\, A}_{n}$$
$$= \partial^{\mathrm{C}^{(2)} \mathrm{P}^{(2)} A, \mathrm{h}} \nabla^{A}_{p-1,n-p} + (-1)^p \partial^{\mathrm{C}^{(2)} \mathrm{P}^{(2)} A, \mathrm{v}} \nabla^{A}_{p,n-1-p} + (-1)^n ((-1)^{n-p} \mathrm{d}^{\mathrm{h}}_{p} \nabla^{A}_{p-1,n-p} + \mathrm{d}^{\mathrm{v}}_{n-p} \nabla^{A}_{p,n-1-p})$$
$$= \partial^{\mathrm{C}^{(2)} \mathrm{P}^{(2)} A, \mathrm{h}} \nabla^{A}_{p-1,n-p} + (-1)^p \partial^{\mathrm{C}^{(2)} \mathrm{P}^{(2)} A, \mathrm{v}} \nabla^{A}_{p,n-1-p} + (-1)^p \mathrm{d}^{\mathrm{h}}_{p} \nabla^{A}_{p-1,n-p} + (-1)^n \mathrm{d}^{\mathrm{v}}_{n-p} \nabla^{A}_{p,n-1-p}$$
$$= \partial^{\mathrm{C}^{(2)} A, \mathrm{h}} \nabla^{A}_{p-1,n-p} + (-1)^p \partial^{\mathrm{C}^{(2)} A, \mathrm{v}} \nabla^{A}_{p,n-1-p}.$$

We proceed by induction on $n \in \mathbb{N}$, $n \geq 2$, to show the second identity involving the path bisimplicial object $\mathrm{P}^{(2)}A$ and use the recursive characterisation of proposition (3.30). However, by the induction hypothesis, we may also use the first identity involving A during our calculations since this is implied by the second as already shown.
First, for $n = 2$ and $p = 1$ we compute

$$\begin{aligned}
\nabla^{A}_{1,1}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} &= (\mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,0} - \mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{0,1})\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} = \mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,0}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} - \mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{0,1}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= \mathrm{s}^{\mathrm{h}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{0,0} - \mathrm{s}^{\mathrm{v}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{0,0} = \mathrm{s}^{\mathrm{h}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}} - \mathrm{s}^{\mathrm{v}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}} \\
&= \mathrm{s}^{\mathrm{h}}_1(\mathrm{d}^{\mathrm{h}}_0 - \mathrm{d}^{\mathrm{h}}_1) - \mathrm{s}^{\mathrm{v}}_1(\mathrm{d}^{\mathrm{v}}_0 - \mathrm{d}^{\mathrm{v}}_1) = \mathrm{d}^{\mathrm{h}}_0\mathrm{s}^{\mathrm{h}}_0 - \mathrm{id}_{A_{1,1}} - \mathrm{d}^{\mathrm{v}}_0\mathrm{s}^{\mathrm{v}}_0 + \mathrm{id}_{A_{1,1}} = \mathrm{d}^{\mathrm{h}}_0\mathrm{s}^{\mathrm{h}}_0 - \mathrm{d}^{\mathrm{v}}_0\mathrm{s}^{\mathrm{v}}_0 \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{A}_{0,1} - \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{A}_{1,0}.
\end{aligned}$$

Next, we show the asserted formula for $p = n - 1$, $n \in \mathbb{N}$, $n \geq 3$:

$$\begin{aligned}
\nabla^{A}_{n-1,1}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} &= (-\mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{n-2,1} + \mathrm{s}^{\mathrm{h}}_{n-1}\nabla^{\mathrm{P}^{(2)}A}_{n-1,0})\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= -\mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{n-2,1}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + \mathrm{s}^{\mathrm{h}}_{n-1}\nabla^{\mathrm{P}^{(2)}A}_{n-1,0}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= -\mathrm{s}^{\mathrm{v}}_1(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{n-3,1} + (-1)^{n-2}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{n-2,0}) + \mathrm{s}^{\mathrm{h}}_{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} \\
&= -\mathrm{s}^{\mathrm{v}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{n-3,1} + (-1)^{n-1}\mathrm{s}^{\mathrm{v}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} + \mathrm{s}^{\mathrm{h}}_{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} \\
&= -\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{n-3,1} + (-1)^{n-1}(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_0 - \mathrm{id}_{A_{n-1,1}})\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} \\
&\quad + (\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_{n-2} + (-1)^{n-1}\mathrm{id}_{A_{n-1,1}})\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} \\
&= -\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{n-3,1} + (-1)^{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_0\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} + \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_{n-2}\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}(\mathrm{s}^{\mathrm{h}}_{n-2}\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} - \mathrm{s}^{\mathrm{v}}_1\nabla^{\mathrm{P}^{(2)}A}_{n-3,1}) + (-1)^{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_0\nabla^{\mathrm{P}^{(2)}A}_{n-2,0} \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{A}_{n-2,1} + (-1)^{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{A}_{n-1,0}.
\end{aligned}$$

Analogously, for $p = 1$, $n \in \mathbb{N}$, $n \geq 3$, we have

$$\begin{aligned}
\nabla^{A}_{1,n-1}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} &= ((-1)^{n-1}\mathrm{s}^{\mathrm{v}}_{n-1}\nabla^{\mathrm{P}^{(2)}A}_{0,n-1} + \mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,n-2})\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= (-1)^{n-1}\mathrm{s}^{\mathrm{v}}_{n-1}\nabla^{\mathrm{P}^{(2)}A}_{0,n-1}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + \mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,n-2}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= (-1)^{n-1}\mathrm{s}^{\mathrm{v}}_{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} + \mathrm{s}^{\mathrm{h}}_1(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} - \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{1,n-3}) \\
&= (-1)^{n-1}\mathrm{s}^{\mathrm{v}}_{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} + \mathrm{s}^{\mathrm{h}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} - \mathrm{s}^{\mathrm{h}}_1\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{1,n-3} \\
&= (-1)^{n-1}(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_{n-2} + (-1)^{n-1}\mathrm{id}_{A_{1,n-1}})\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} \\
&\quad + (\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_0 - \mathrm{id}_{A_{1,n-1}})\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} - \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,n-3} \\
&= (-1)^{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_{n-2}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} + \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_0\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} - \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,n-3} \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_0\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} - ((-1)^{n-2}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_{n-2}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} + \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,n-3}) \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_0\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} - \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}((-1)^{n-2}\mathrm{s}^{\mathrm{v}}_{n-2}\nabla^{\mathrm{P}^{(2)}A}_{0,n-2} + \mathrm{s}^{\mathrm{h}}_1\nabla^{\mathrm{P}^{(2)}A}_{1,n-3}) \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{A}_{0,n-1} - \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{A}_{1,n-2}.
\end{aligned}$$

Finally, we let $p \in [n-2, 2]$, $n \in \mathbb{N}$, $n \geq 4$. Then we get

$$\begin{aligned}
\nabla^{A}_{p,n-p}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} &= ((-1)^{n-p}\mathrm{s}^{\mathrm{v}}_{n-p}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-p} + \mathrm{s}^{\mathrm{h}}_p\nabla^{\mathrm{P}^{(2)}A}_{p,n-1-p})\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= (-1)^{n-p}\mathrm{s}^{\mathrm{v}}_{n-p}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-p}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + \mathrm{s}^{\mathrm{h}}_p\nabla^{\mathrm{P}^{(2)}A}_{p,n-1-p}\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\
&= (-1)^{n-p}\mathrm{s}^{\mathrm{v}}_{n-p}(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{p-2,n-p} + (-1)^{p-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p}) \\
&\quad + \mathrm{s}^{\mathrm{h}}_p(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} + (-1)^p\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{p,n-2-p}) \\
&= (-1)^{n-p}\mathrm{s}^{\mathrm{v}}_{n-p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{p-2,n-p} + (-1)^{n-1}\mathrm{s}^{\mathrm{v}}_{n-p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} \\
&\quad + \mathrm{s}^{\mathrm{h}}_p\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} + (-1)^p\mathrm{s}^{\mathrm{h}}_p\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{\mathrm{P}^{(2)}A}_{p,n-2-p}
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{n-p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{v}}_{n-p}\nabla^{\mathrm{P}^{(2)}A}_{p-2,n-p} \\
&\quad + (-1)^{n-1}(\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_{n-1-p} + (-1)^{n-p}\mathrm{id}_{A_{p,n-p}})\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} \\
&\quad + (\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_{p-1} + (-1)^{p}\mathrm{id}_{A_{p,n-p}})\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} + (-1)^{p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{h}}_{p}\nabla^{\mathrm{P}^{(2)}A}_{p,n-2-p} \\
&= (-1)^{n-p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{v}}_{n-p}\nabla^{\mathrm{P}^{(2)}A}_{p-2,n-p} + (-1)^{n-1}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{v}}_{n-1-p}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} \\
&\quad + \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\mathrm{s}^{\mathrm{h}}_{p-1}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} + (-1)^{p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\mathrm{s}^{\mathrm{h}}_{p}\nabla^{\mathrm{P}^{(2)}A}_{p,n-2-p} \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}((-1)^{n-p}\mathrm{s}^{\mathrm{v}}_{n-p}\nabla^{\mathrm{P}^{(2)}A}_{p-2,n-p} + \mathrm{s}^{\mathrm{h}}_{p-1}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p}) \\
&\quad + (-1)^{p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}((-1)^{n-p-1}\mathrm{s}^{\mathrm{v}}_{n-1-p}\nabla^{\mathrm{P}^{(2)}A}_{p-1,n-1-p} + \mathrm{s}^{\mathrm{h}}_{p}\nabla^{\mathrm{P}^{(2)}A}_{p,n-2-p}) \\
&= \partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{h}}\nabla^{A}_{p-1,n-p} + (-1)^{p}\partial^{\mathrm{C}^{(2)}\mathrm{P}^{(2)}A,\mathrm{v}}\nabla^{A}_{p,n-1-p}
\end{aligned}
$$

By induction, we have shown that the morphisms ∇_n for $n \in \mathbb{N}_0$ yield a complex morphism

$$\operatorname{Tot} \mathrm{C}^{(2)} A \xrightarrow{\nabla} \mathrm{C} \operatorname{Diag} A.$$

□

(3.34) Definition (shuffle morphism)**.** The complex morphism

$$\operatorname{Tot} \mathrm{C}^{(2)} A \xrightarrow{\nabla} \mathrm{C} \operatorname{Diag} A$$

given as in remark (3.33) by $\mathrm{emb}_{p,n-p}\nabla_n = \nabla_{p,n-p}$ for all $p \in \lfloor n, 0 \rfloor$, that is,

$$\nabla_n = \begin{pmatrix} \nabla_{n,0} \\ \vdots \\ \nabla_{0,n} \end{pmatrix}$$

as a morphism from $\operatorname{Tot}_n \mathrm{C}^{(2)} A = \bigoplus_{p \in \lfloor n,0 \rfloor} A_{p,n-p}$ to $\mathrm{C}_n \operatorname{Diag} A = A_{n,n}$ for all $n \in \mathbb{N}_0$, is called (*Eilenberg-Mac Lane*) *shuffle morphism*.

At next, we will show that the Alexander-Whitney morphism and the Eilenberg-Mac Lane shuffle morphism restrict to well-defined morphisms on $\mathrm{M} \operatorname{Diag} A$ resp. $\operatorname{Tot} \mathrm{M}^{(2)} A$.

(3.35) Proposition.

(a) We have a morphism of split short exact sequences

$$
\begin{array}{ccccc}
\mathrm{D} \operatorname{Diag} A & \longrightarrow & \mathrm{C} \operatorname{Diag} A & \longrightarrow & \mathrm{M} \operatorname{Diag} A \\
\downarrow & & \downarrow {\scriptstyle \mathsf{AW}} & & \downarrow \\
\operatorname{Tot} \mathrm{D}^{(2)} A & \longrightarrow & \operatorname{Tot} \mathrm{C}^{(2)} A & \longrightarrow & \operatorname{Tot} \mathrm{M}^{(2)} A
\end{array}
$$

By abuse of notation, the induced morphism $\mathrm{M} \operatorname{Diag} A \longrightarrow \operatorname{Tot} \mathrm{M}^{(2)} A$ is also denoted by $\mathsf{AW} := \mathsf{AW}^A$.

(b) We have a morphism of split short exact sequences

$$
\begin{array}{ccccc}
\operatorname{Tot} \mathrm{D}^{(2)} A & \longrightarrow & \operatorname{Tot} \mathrm{C}^{(2)} A & \longrightarrow & \operatorname{Tot} \mathrm{M}^{(2)} A \\
\downarrow & & \downarrow {\scriptstyle \nabla} & & \downarrow \\
\mathrm{D} \operatorname{Diag} A & \longrightarrow & \mathrm{C} \operatorname{Diag} A & \longrightarrow & \mathrm{M} \operatorname{Diag} A
\end{array}
$$

By abuse of notation, the induced morphism $\operatorname{Tot} \mathrm{M}^{(2)} A \longrightarrow \mathrm{M} \operatorname{Diag} A$ is also denoted by $\nabla := \nabla^A$.

Proof.

(a) We have

$$\mathrm{s}^{\operatorname{Diag} A}_k \mathsf{AW}_{p,n-p} = \mathrm{s}^{\mathrm{h}}_k \mathrm{s}^{\mathrm{v}}_k \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor} = \mathrm{s}^{\mathrm{h}}_k \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor} \mathrm{s}^{\mathrm{v}}_k \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor} = \begin{cases} (\mathrm{s}^{\mathrm{h}}_k \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor}) \mathrm{s}^{\mathrm{v}}_{k-p} & \text{if } p \leq k, \\ (\mathrm{d}^{\mathrm{h}}_{\lfloor n-1,p \rfloor} \mathrm{s}^{\mathrm{v}}_k \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor}) \mathrm{s}^{\mathrm{h}}_k & \text{if } p > k, \end{cases}$$

for every $k \in [0, n-1]$, that is, $(\operatorname{Im} \mathrm{s}_k^{\operatorname{Diag} A})\mathsf{AW}_{p,n-p} = \operatorname{Im}(\mathrm{s}_k^{\operatorname{Diag} A}\mathsf{AW}_{p,n-p}) \preceq \mathrm{D}_{p,n-p}^{(2)}A$ for all $k \in [0, n-1]$ and therefore $(\mathrm{D}_n \operatorname{Diag} A)\mathsf{AW}_{p,n-p} \preceq \mathrm{D}_{p,n-p}^{(2)}A$ for all $p \in \lfloor n, 0\rfloor$. Hence we have an induced morphism

$$\mathrm{D} \operatorname{Diag} A \longrightarrow \operatorname{Tot} \mathrm{D}^{(2)}A.$$

Moreover, $\mathrm{M} \operatorname{Diag} A \cong \mathrm{C} \operatorname{Diag} A/\mathrm{D} \operatorname{Diag} A$ and $\operatorname{Tot} \mathrm{M}^{(2)}A \cong \operatorname{Tot}(\mathrm{C}^{(2)}A/\mathrm{D}^{(2)}A) \cong \operatorname{Tot} \mathrm{C}^{(2)}A/ \operatorname{Tot} \mathrm{D}^{(2)}A$ by the normalisation theorem (2.28). Hence we have an induced morphism on the cokernels

$$\mathrm{M} \operatorname{Diag} A \xrightarrow{\mathsf{AW}} \operatorname{Tot} \mathrm{M}^{(2)}A.$$

(b) We will show that

$$(\mathrm{D}_{p,n-p}^{(2)}A)\nabla_{p,n-p}^{A} \preceq \mathrm{D}_n \operatorname{Diag} A$$

for all $p \in \lfloor n, 0\rfloor$, $n \in \mathbb{N}_0$. Thereto, we proceed by induction on n, where for $n = 0$ the assertion is trivial since $\mathrm{D}_{0,0}^{(2)}A \cong 0$ and $\mathrm{D}_0 \operatorname{Diag} A \cong 0$. So we let a natural number $n \in \mathbb{N}$ with $n \geq 1$ and $p \in \lfloor n, 0\rfloor$ be given and we assume that the asserted inclusion holds for all bisimplicial sets up to dimension $n-1$. By proposition (3.30), we compute

$$\begin{aligned}
\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-p}^{A} &= \left.\begin{cases} \mathrm{s}_i^{\mathrm{h}}\mathrm{s}_0^{\mathrm{v}}\nabla_{n-1,0}^{\mathrm{P}^{(2)}A} & \text{if } p = n, \\ \mathrm{s}_i^{\mathrm{h}}((-1)^{n-p}\mathrm{s}_{n-p}^{\mathrm{v}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_p^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) & \text{if } p \in \lfloor n-1, 1\rfloor \end{cases}\right\} \\
&= \left.\begin{cases} \mathrm{s}_i^{\mathrm{h}}\mathrm{s}_0^{\mathrm{v}}\nabla_{n-1,0}^{\mathrm{P}^{(2)}A} & \text{if } p = n, \\ (-1)^{n-p}\mathrm{s}_i^{\mathrm{h}}\mathrm{s}_{n-p}^{\mathrm{v}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_i^{\mathrm{h}}\mathrm{s}_p^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A} & \text{if } p \in \lfloor n-1, 1\rfloor \end{cases}\right\} \\
&= \begin{cases} \mathrm{s}_0^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{n-1,0}^{\mathrm{P}^{(2)}A} & \text{if } p = n, \\ (-1)^{n-p}\mathrm{s}_{n-p}^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_{p-1}^{\mathrm{h}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A} & \text{if } p \in \lfloor n-1, 1\rfloor \end{cases}
\end{aligned}$$

for $i \in [0, p-1]$, $p \in \lfloor n, 1\rfloor$, and therefore

$$\operatorname{Im}(\mathrm{s}_i^{\mathrm{h}}\nabla_{n,0}^{A}) \preceq \operatorname{Im}(\mathrm{s}_0^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{n-1,0}^{\mathrm{P}^{(2)}A})$$

for $i \in [0, n-1]$ and

$$\begin{aligned}
\operatorname{Im}(\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-p}^{A}) &\preceq \operatorname{Im}((-1)^{n-p}\mathrm{s}_{n-p}^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_{p-1}^{\mathrm{h}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \\
&\preceq \operatorname{Im}((-1)^{n-p}\mathrm{s}_{n-p}^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) + \operatorname{Im}(\mathrm{s}_{p-1}^{\mathrm{h}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \\
&\preceq \operatorname{Im}(\mathrm{s}_{n-p}^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) + \operatorname{Im}(\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A})
\end{aligned}$$

for $i \in [0, p-1]$, $p \in \lfloor n-1, 1\rfloor$. Now by the induction hypothesis, we have

$$\operatorname{Im}(\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \preceq (\mathrm{D}_{p,n-1-p}^{(2)}\mathrm{P}^{(2)}A)\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A} \preceq \mathrm{D}_{n-1} \operatorname{Diag} \mathrm{P}^{(2)}A \preceq \mathrm{D}_n \operatorname{Diag} A$$

for $i \in [0, p-1]$, $p \in \lfloor n-1, 1\rfloor$, and

$$\begin{aligned}
\operatorname{Im}(\mathrm{s}_{n-p}^{\mathrm{v}}\mathrm{s}_i^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) &\preceq \operatorname{Im}(\mathrm{s}_i^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) \preceq (\mathrm{D}_{p-1,n-p}^{(2)}\mathrm{P}^{(2)}A)\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} \preceq \mathrm{D}_{n-1} \operatorname{Diag} \mathrm{P}^{(2)}A \\
&\preceq \mathrm{D}_n \operatorname{Diag} A
\end{aligned}$$

for $i \in [0, p-2]$, $p \in \lfloor n, 1\rfloor$. Since additionally, by lemma (3.31),

$$\operatorname{Im}(\mathrm{s}_{n-p}^{\mathrm{v}}\mathrm{s}_{p-1}^{\mathrm{h}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) = \operatorname{Im}(\nabla_{p-1,n-p}^{A}\mathrm{s}_{n-1}^{\operatorname{Diag} A}) \preceq \operatorname{Im}(\mathrm{s}_{n-1}^{\operatorname{Diag} A}) \preceq \mathrm{D}_n \operatorname{Diag} A$$

for $p \in \lfloor n, 1\rfloor$, we can conclude that $\operatorname{Im}(\mathrm{s}_i^{\mathrm{h}}\nabla_{p,n-p}^{A}) \preceq \mathrm{D}_n \operatorname{Diag} A$ for $i \in [0, p-1]$, $p \in \lfloor n, 1\rfloor$.

Analogously, we show $\operatorname{Im}(\mathrm{s}_j^{\mathrm{v}}\nabla_{p,n-p}^{A}) \preceq \mathrm{D}_n \operatorname{Diag} A$ for all $j \in [0, n-1-p]$, $p \in \lfloor n-1, 0\rfloor$. Indeed, by proposition (3.30), we have

$$\mathrm{s}_j^{\mathrm{v}}\nabla_{p,n-p}^{A} = \left.\begin{cases} \mathrm{s}_j^{\mathrm{v}}((-1)^{n-p}\mathrm{s}_{n-p}^{\mathrm{v}}\nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_p^{\mathrm{h}}\nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) & \text{if } p \in \lfloor n-1, 1\rfloor, \\ \mathrm{s}_j^{\mathrm{v}}\mathrm{s}_0^{\mathrm{h}}\nabla_{0,n-1}^{\mathrm{P}^{(2)}A} & \text{if } p = 0 \end{cases}\right\}$$

$$= \begin{cases} (-1)^{n-p} \mathrm{s}_j^{\mathrm{v}} \mathrm{s}_{n-p}^{\mathrm{v}} \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_j^{\mathrm{v}} \mathrm{s}_p^{\mathrm{h}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A} & \text{if } p \in \lfloor n-1, 1 \rfloor, \\ \mathrm{s}_j^{\mathrm{v}} \mathrm{s}_0^{\mathrm{h}} \nabla_{0,n-1}^{\mathrm{P}^{(2)}A} & \text{if } p = 0 \end{cases}$$

$$= \begin{cases} (-1)^{n-p} \mathrm{s}_{n-1-p}^{\mathrm{v}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_p^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A} & \text{if } p \in \lfloor n-1, 1 \rfloor, \\ \mathrm{s}_0^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{0,n-1}^{\mathrm{P}^{(2)}A} & \text{if } p = 0 \end{cases}$$

for $j \in [0, n-1-p]$, $p \in \lfloor n-1, 0 \rfloor$, and therefore

$$\begin{aligned} \operatorname{Im}(\mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-p}^{A}) &\preceq \operatorname{Im}((-1)^{n-p} \mathrm{s}_{n-1-p}^{\mathrm{v}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} + \mathrm{s}_p^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \\ &\preceq \operatorname{Im}((-1)^{n-p} \mathrm{s}_{n-1-p}^{\mathrm{v}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) + \operatorname{Im}(\mathrm{s}_p^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \\ &\preceq \operatorname{Im}(\mathrm{s}_j^{\mathrm{v}} \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) + \operatorname{Im}(\mathrm{s}_p^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \end{aligned}$$

for $j \in [0, n-1-p]$, $p \in \lfloor n-1, 1 \rfloor$, and

$$\operatorname{Im}(\mathrm{s}_j^{\mathrm{v}} \nabla_{0,n}^{A}) \preceq \operatorname{Im}(\mathrm{s}_0^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{0,n-1}^{\mathrm{P}^{(2)}A})$$

for $j \in [0, n-1]$. With the induction hypothesis, it follows that

$$\operatorname{Im}(\mathrm{s}_j^{\mathrm{v}} \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A}) \preceq (\mathrm{D}_{p-1,n-p}^{(2)} \mathrm{P}^{(2)}A) \nabla_{p-1,n-p}^{\mathrm{P}^{(2)}A} \preceq \mathrm{D}_{n-1} \operatorname{Diag} \mathrm{P}^{(2)}A \preceq \mathrm{D}_n \operatorname{Diag} A$$

for $j \in [0, n-1-p]$, $p \in \lfloor n-1, 1 \rfloor$ and

$$\operatorname{Im}(\mathrm{s}_p^{\mathrm{h}} \mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \preceq \operatorname{Im}(\mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) \preceq (\mathrm{D}_{p,n-1-p}^{(2)} \mathrm{P}^{(2)}A) \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A} \preceq \mathrm{D}_{n-1} \operatorname{Diag} \mathrm{P}^{(2)}A \preceq \mathrm{D}_n \operatorname{Diag} A$$

for $j \in [0, n-2-p]$, $p \in \lfloor n-1, 0 \rfloor$. Since additionally, by lemma (3.31),

$$\operatorname{Im}(\mathrm{s}_p^{\mathrm{h}} \mathrm{s}_{n-1-p}^{\mathrm{v}} \nabla_{p,n-1-p}^{\mathrm{P}^{(2)}A}) = \operatorname{Im}(\nabla_{p,n-1-p}^{A} \mathrm{s}_{n-1}^{\operatorname{Diag} A}) \preceq \operatorname{Im}(\mathrm{s}_{n-1}^{\operatorname{Diag} A}) \preceq \mathrm{D}_n \operatorname{Diag} A$$

for $p \in \lfloor n-1, 0 \rfloor$. Hence $\operatorname{Im}(\mathrm{s}_j^{\mathrm{v}} \nabla_{p,n-p}^{A}) \preceq \mathrm{D}_n \operatorname{Diag} A$ for $j \in [0, n-1-p]$, $p \in \lfloor n-1, 0 \rfloor$. Therefore

$$(\mathrm{D}_{p,n-p}^{(2)} A) \nabla_{p,n-p}^{A} \preceq \mathrm{D}_n \operatorname{Diag} A.$$

So we have induced morphisms

$$\operatorname{Tot} \mathrm{D}^{(2)} A \longrightarrow \mathrm{D} \operatorname{Diag} A$$

and

$$\operatorname{Tot} \mathrm{M}^{(2)} A \xrightarrow{\nabla^A} \mathrm{M} \operatorname{Diag} A. \qquad \square$$

(3.36) Theorem (generalised Eilenberg-Zilber theorem, normalised case)**.** The Alexander-Whitney morphism

$$\mathrm{CM}A \xrightarrow{\mathsf{AW}} \operatorname{Tot} \mathrm{M}^{(2)} A$$

and the Eilenberg-Mac Lane shuffle morphism

$$\operatorname{Tot} \mathrm{M}^{(2)} A \xrightarrow{\nabla} \mathrm{M} \operatorname{Diag} A$$

are mutually inverse homotopy equivalences. In particular,

$$\mathrm{M} \operatorname{Diag} A \simeq \operatorname{Tot} \mathrm{M}^{(2)} A.$$

Proof. First, we want to show that $\nabla^A \mathsf{AW}^A = \mathrm{id}_{\operatorname{Tot} \mathrm{M}^{(2)} A}$. We let $n \in \mathbb{N}_0$ be given. For each $p, q \in \lfloor n, 0 \rfloor$ we have

$$\nabla_{p,n-p}^{A} \mathsf{AW}_{q,n-q}^{A} = (\sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu) \mathrm{s}_{\lceil 0,p-1 \rceil \mu}^{\mathrm{v}} \mathrm{s}_{\lceil p,n-1 \rceil \mu}^{\mathrm{h}}) \mathrm{d}_{\lfloor n,q+1 \rfloor}^{\mathrm{h}} \mathrm{d}_{\lfloor q-1,0 \rfloor}^{\mathrm{v}}$$

$$= \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu)(\mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})(\mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor}).$$

By applying the simplicial identities, we recognise that each summand ends with a vertical degeneracy if $q < p$ resp. with a horizontal degeneracy if $q > p$. Since we are in the normalised case, this means that $\nabla^A_{p,n-p} \mathsf{AW}^A_{q,n-q} = 0$ for $p \neq q$. It remains to consider the case $q = p$. Then we have

$$\nabla^A_{p,n-p} \mathsf{AW}^A_{p,n-p} = \sum_{\mu \in \mathrm{Sh}_{p,n-p}} (\operatorname{sgn} \mu)(\mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil \mu} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor})(\mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor}).$$

Now if $\mu \neq \mathrm{id}$, then $p\mu \leq p-1$ and hence

$$\operatorname{Im}(\mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil \mu} \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor}) = \operatorname{Im}(\mathrm{s}^{\mathrm{h}}_{p\mu} \mathrm{s}^{\mathrm{h}}_{\lceil p+1,n-1 \rceil \mu} \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor}) \preceq \operatorname{Im}(\mathrm{s}^{\mathrm{h}}_{p\mu} \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor}) \preceq \operatorname{Im} \mathrm{s}^{\mathrm{h}}_{p\mu}.$$

Therefore

$$\nabla^A_{p,n-p} \mathsf{AW}^A_{p,n-p} = (\mathrm{s}^{\mathrm{v}}_{\lceil 0,p-1 \rceil} \mathrm{d}^{\mathrm{v}}_{\lfloor p-1,0 \rfloor})(\mathrm{s}^{\mathrm{h}}_{\lceil p,n-1 \rceil} \mathrm{d}^{\mathrm{h}}_{\lfloor n,p+1 \rfloor}) = \mathrm{id}_{\mathrm{M}^{(2)}_{p,n-p} A}$$

since the only summand that is not trivial because it ends with a degeneracy, is the one where $\mu = \mathrm{id}_{[0,n-1]}$. Thus

$$\nabla^A_n \mathsf{AW}^A_n = \begin{pmatrix} \nabla^A_{n,0} \\ \vdots \\ \nabla^A_{0,n} \end{pmatrix} \begin{pmatrix} \mathsf{AW}^A_{n,0} & \dots & \mathsf{AW}^A_{0,n} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{\mathrm{M}^{(2)}_{n,0} A} & & 0 \\ & \ddots & \\ 0 & & \mathrm{id}_{\mathrm{M}^{(2)}_{0,n} A} \end{pmatrix} = \mathrm{id}_{\operatorname{Tot}_n \mathrm{M}^{(2)} A}.$$

Now we consider the composition $f^A := \mathsf{AW}^A \nabla^A$ and we will show that $f^A \sim \mathrm{id}_{\mathrm{M} \operatorname{Diag} A}$. Thereto, we define recursively morphisms

$$\mathrm{C}_n \operatorname{Diag} A \xrightarrow{h^A_n} \mathrm{C}_{n+1} \operatorname{Diag} A$$

by $h_n := 0$ for $n < 0$ and

$$h^A_n := h^{\mathrm{P}^{(2)} A}_{n-1} + (-1)^n \mathrm{s}^{\operatorname{Diag} A}_n f^{\mathrm{P}^{(2)} A}_n \text{ for } n \in \mathbb{N}_0.$$

We have to show that these morphisms induce morphisms on the entries of the normalised complex $\mathrm{M} \operatorname{Diag} A$. Thereto, we prove that they restrict to morphisms

$$\mathrm{D}_n \operatorname{Diag} A \xrightarrow{h^A_n} \mathrm{D}_{n+1} \operatorname{Diag} A \text{ for } n \in \mathbb{N}_0.$$

For $n = 0$, this holds since $\mathrm{D}_0 \operatorname{Diag} A \cong 0$. We let $n \in \mathbb{N}$ be given. Then we get

$$\begin{aligned} \mathrm{s}^{\operatorname{Diag} A}_k h^A_n &= \mathrm{s}^{\operatorname{Diag} A}_k (h^{\mathrm{P}^{(2)} A}_{n-1} + (-1)^n \mathrm{s}^{\operatorname{Diag} A}_n f^{\mathrm{P}^{(2)} A}_n) = \mathrm{s}^{\operatorname{Diag} A}_k h^{\mathrm{P}^{(2)} A}_{n-1} + (-1)^n \mathrm{s}^{\operatorname{Diag} A}_k \mathrm{s}^{\operatorname{Diag} A}_n f^{\mathrm{P}^{(2)} A}_n \\ &= \mathrm{s}^{\operatorname{Diag} A}_k h^{\mathrm{P}^{(2)} A}_{n-1} + (-1)^n \mathrm{s}^{\operatorname{Diag} A}_{n-1} \mathrm{s}^{\operatorname{Diag} A}_k f^{\mathrm{P}^{(2)} A}_n \end{aligned}$$

for each $k \in [0, n-1]$. Since AW^A and ∇^A restrict to morphisms on $\mathrm{D} \operatorname{Diag} A$ and $\operatorname{Tot} \mathrm{D}^{(2)} A$ by proposition (3.35), we see that

$$\operatorname{Im}((-1)^n \mathrm{s}^{\operatorname{Diag} A}_{n-1} \mathrm{s}^{\operatorname{Diag} A}_k f^{\mathrm{P}^{(2)} A}_n) \preceq \mathrm{D}_n \operatorname{Diag} \mathrm{P}^{(2)} A \preceq \mathrm{D}_{n+1} \operatorname{Diag} A.$$

For the first summand, we get by induction that

$$\operatorname{Im}(\mathrm{s}^{\operatorname{Diag} A}_k h^{\mathrm{P}^{(2)} A}_{n-1}) \preceq \mathrm{D}_n \operatorname{Diag} \mathrm{P}^{(2)} A \preceq \mathrm{D}_{n+1} \operatorname{Diag} A$$

if $k \leq n-2$. We consider the case $k = n-1$. By definition and proposition (2.33)(c), h^A_n is a certain linear combination of morphisms, each one being a composite of a horizonal backal and a vertical backal morphism. Thus, by proposition (2.33)(a) applied vertically and horizontally, we have $\mathrm{s}^{\operatorname{Diag} A}_{n-1} h^{\mathrm{P}^{(2)} A}_{n-1} = h^A_{n-1} \mathrm{s}^{\operatorname{Diag} A}_n$ and hence

$$\operatorname{Im}(\mathrm{s}^{\operatorname{Diag} A}_{n-1} h^{\mathrm{P}^{(2)} A}_{n-1}) \preceq \mathrm{D}_{n+1} \operatorname{Diag} A.$$

Altogether,

$$\operatorname{Im}(\mathrm{s}_k^{\mathrm{Diag}\,A} h_n^A) \preceq \mathrm{D}_{n+1} \,\mathrm{Diag}\, A.$$

It remains to show that $(h_n \in {}_{\mathcal{A}}(\mathrm{M}_n \,\mathrm{Diag}\, A, \mathrm{M}_{n+1} \,\mathrm{Diag}\, A) \mid n \in \mathbb{N}_0)$ is a complex homotopy from $\mathrm{id}_{\mathrm{M}\,\mathrm{Diag}\,A}$ to f^A, that is,

$$h_n^A \partial^{\mathrm{C}\,\mathrm{Diag}\,A} + \partial^{\mathrm{C}\,\mathrm{Diag}\,A} h_{n-1}^A = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_n^A \text{ for all } n \in \mathbb{N}_0$$

up to sums of morphisms whose images are in the degenerate complex. We proceed by induction on $n \in \mathbb{N}_0$. For $n = 0$, we have

$$\begin{aligned} h_0^A \partial^{\mathrm{C}\,\mathrm{Diag}\,A} &= \mathrm{s}_0^{\mathrm{Diag}\,A} f_0^A (\mathrm{d}_0^{\mathrm{Diag}\,A} - \mathrm{d}_1^{\mathrm{Diag}\,A}) = \mathrm{s}_0^{\mathrm{Diag}\,A} \mathrm{d}_0^{\mathrm{Diag}\,A} - \mathrm{s}_0^{\mathrm{Diag}\,A} \mathrm{d}_1^{\mathrm{Diag}\,A} = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} \\ &= \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_n^A. \end{aligned}$$

Now we assume that $n \geq 1$ and that the assumpted relation holds in all lower dimensions. Then we have

$$\begin{aligned} & h_n^A \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} = (h_{n-1}^{\mathrm{P}^{(2)}A} + (-1)^n \mathrm{s}_n^{\mathrm{Diag}\,A} f_n^{\mathrm{P}^{(2)}A}) \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\ & = h_{n-1}^{\mathrm{P}^{(2)}A} \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + (-1)^n \mathrm{s}_n^{\mathrm{Diag}\,A} f_n^{\mathrm{P}^{(2)}A} \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \\ & = h_{n-1}^{\mathrm{P}^{(2)}A} \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + (-1)^n \mathrm{s}_n^{\mathrm{Diag}\,A} \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} f_{n-1}^{\mathrm{P}^{(2)}A} \\ & = (\mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_{n-1}^{\mathrm{P}^{(2)}A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-2}^{\mathrm{P}^{(2)}A}) + (-1)^n (\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \mathrm{s}_{n-1}^{\mathrm{Diag}\,A} + (-1)^n \mathrm{id}) f_{n-1}^{\mathrm{P}^{(2)}A} \\ & = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_{n-1}^{\mathrm{P}^{(2)}A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-2}^{\mathrm{P}^{(2)}A} + (-1)^n \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \mathrm{s}_{n-1}^{\mathrm{Diag}\,A} f_{n-1}^{\mathrm{P}^{(2)}A} + f_{n-1}^{\mathrm{P}^{(2)}A} \\ & = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-2}^{\mathrm{P}^{(2)}A} + (-1)^n \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} \mathrm{s}_{n-1}^{\mathrm{Diag}\,A} f_{n-1}^{\mathrm{P}^{(2)}A} \\ & = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} (h_{n-2}^{\mathrm{P}^{(2)}A} + (-1)^{n-1} \mathrm{s}_{n-1}^{\mathrm{Diag}\,A} f_{n-1}^{\mathrm{P}^{(2)}A}) = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-1}^A \end{aligned}$$

as well as, by proposition (2.31)(c),

$$\begin{aligned} h_n^A \mathrm{d}_{n+1}^{\mathrm{Diag}\,A} &= (h_{n-1}^{\mathrm{P}^{(2)}A} + (-1)^n \mathrm{s}_n^{\mathrm{Diag}\,A} f_n^{\mathrm{P}^{(2)}A}) \mathrm{d}_{n+1}^{\mathrm{Diag}\,A} = h_{n-1}^{\mathrm{P}^{(2)}A} \mathrm{d}_{n+1}^{\mathrm{Diag}\,A} + (-1)^n \mathrm{s}_n^{\mathrm{Diag}\,A} f_n^{\mathrm{P}^{(2)}A} \mathrm{d}_{n+1}^{\mathrm{Diag}\,A} \\ &= \mathrm{d}_n^{\mathrm{Diag}\,A} h_{n-1}^A + (-1)^n \mathrm{s}_n^{\mathrm{Diag}\,A} \mathrm{d}_{n+1}^{\mathrm{Diag}\,A} f_n^A = \mathrm{d}_n^{\mathrm{Diag}\,A} h_{n-1}^A + (-1)^n f_n^A. \end{aligned}$$

Hence we can conclude

$$\begin{aligned} h_n^A \partial^{\mathrm{C}\,\mathrm{Diag}\,A} &= h_n^A (\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + (-1)^{n+1} \mathrm{d}_{n+1}^{\mathrm{Diag}\,A}) = h_n^A \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + (-1)^{n+1} h_n^A \mathrm{d}_{n+1}^{\mathrm{Diag}\,A} \\ &= (\mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-1}^A) + (-1)^{n+1} (\mathrm{d}_n^{\mathrm{Diag}\,A} h_{n-1}^A + (-1)^n f_n^A) \\ &= \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - \partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-1}^A + (-1)^{n+1} \mathrm{d}_n^{\mathrm{Diag}\,A} h_{n-1}^A - f_n^A \\ &= -\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} h_{n-1}^A - (-1)^n \mathrm{d}_n^{\mathrm{Diag}\,A} h_{n-1}^A + \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_n^A \\ &= -(\partial^{\mathrm{C}\,\mathrm{Diag}\,\mathrm{P}^{(2)}A} + (-1)^n \mathrm{d}_n^{\mathrm{Diag}\,A}) h_{n-1}^A + \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_n^A \\ &= -\partial^{\mathrm{C}\,\mathrm{Diag}\,A} h_{n-1}^A + \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_n^A, \end{aligned}$$

that is, $h_n^A \partial^{\mathrm{C}\,\mathrm{Diag}\,A} + \partial^{\mathrm{C}\,\mathrm{Diag}\,A} h_{n-1}^A = \mathrm{id}_{\mathrm{C}_n\,\mathrm{Diag}\,A} - f_n^A$. □

(3.37) Theorem (generalised Eilenberg-Zilber theorem of Dold, Puppe and Cartier, cf. [9, Satz 2.9])**.** We have

$$\mathrm{C}\,\mathrm{Diag}\, A \simeq \mathrm{Tot}\,\mathrm{C}^{(2)} A.$$

Proof. By theorem (3.36), we have

$$\mathrm{M}\,\mathrm{Diag}\, A \simeq \mathrm{Tot}\,\mathrm{M}^{(2)} A.$$

Since the normalisation theorem states a homotopy equivalence between the associated (double) complexes and the Moore (double) complexes, cf. theorem (2.28) and theorem (3.24), and since the total complex functor preserves homotopy equivalences due to proposition (3.19), this implies by theorem (3.36) that

$$\mathrm{C}\,\mathrm{Diag}\, A \simeq \mathrm{M}\,\mathrm{Diag}\, A \simeq \mathrm{Tot}\,\mathrm{M}^{(2)} A \simeq \mathrm{Tot}\,\mathrm{C}^{(2)} A.$$ □

QUILLEN mentions the following corollary in [28] as well-known.

(3.38) Corollary. There exists a spectral sequence E with $E^1_{p,n-p} \cong \mathrm{H}_{n-p}(\mathrm{C}A_{p,-})$ that converges to the homology group $\mathrm{H}_n(\mathrm{C}\operatorname{Diag} A)$, where $p \in [0,n]$, $n \in \mathbb{N}_0$.

Proof. By the generalised Eilenberg-Zilber theorem (3.37), we have $\mathrm{C}\operatorname{Diag} A \simeq \operatorname{Tot}\mathrm{C}^{(2)}A$ and hence

$$\mathrm{H}_n(\mathrm{C}\operatorname{Diag} A) \cong \mathrm{H}_n(\operatorname{Tot}\mathrm{C}^{(2)}A)$$

for all $n \in \mathbb{N}_0$. The spectral sequence E of the "columnwise" filtered double complex $\mathrm{C}^{(2)}A$ has the entries

$$E^1_{p,n-p} \cong \mathrm{H}_{n-p}(\mathrm{C}^{(2)}_{p,-}A) = \mathrm{H}_{n-p}(\mathrm{C}A_{p,-})$$

for $p \in [0,n]$, $n \in \mathbb{N}_0$. □

(3.39) Corollary. We suppose given a bisimplicial set X, a commutative ring R and an R-module M.

(a) There exists a spectral sequence E with $E^1_{p,n-p} \cong \mathrm{H}_{n-p}(X_{p,-}, M; R)$ that converges to the homology group $\mathrm{H}_n(\operatorname{Diag} X, M; R)$, where $p \in [0,n]$, $n \in \mathbb{N}_0$.

(b) There exists a spectral sequence E with $E_1^{p,n-p} \cong \mathrm{H}^{n-p}(X_{p,-}, M; R)$ that converges to the cohomology group $\mathrm{H}^n(\operatorname{Diag} X, M; R)$, where $p \in [0,n]$, $n \in \mathbb{N}_0$.

Proof.

(a) We apply corollary (3.38) to $RX \otimes_R M$. Then we obtain

$$\begin{aligned}\mathrm{H}_n(\mathrm{C}\operatorname{Diag}(RX \otimes_R M)) &= \mathrm{H}_n(\mathrm{C}((\operatorname{Diag} RX) \otimes_R M)) = \mathrm{H}_n((\mathrm{C}\operatorname{Diag} RX) \otimes_R M)\\ &= \mathrm{H}_n((\mathrm{C}R\operatorname{Diag} X) \otimes_R M) = \mathrm{H}_n(\mathrm{C}(\operatorname{Diag} X; R) \otimes_R M)\\ &= \mathrm{H}_n(\operatorname{Diag} X, M; R)\end{aligned}$$

for $n \in \mathbb{N}_0$, and

$$\begin{aligned}\mathrm{H}_{n-p}(\mathrm{C}(RX \otimes_R M)_{p,-}) &= \mathrm{H}_{n-p}(\mathrm{C}(RX_{p,-} \otimes_R M) = \mathrm{H}_{n-p}((\mathrm{C}RX_{p,-}) \otimes_R M)\\ &= \mathrm{H}_{n-p}(\mathrm{C}(X_{p,-}; R) \otimes_R M) = \mathrm{H}_{n-p}(X_{p,-}, M; R)\end{aligned}$$

for $p \in [0,n]$, $n \in \mathbb{N}_0$.

(b) By the generalised Eilenberg-Zilber theorem (3.37), we have

$$\mathrm{C}(\operatorname{Diag} X; R) = \mathrm{C}R\operatorname{Diag} X = \mathrm{C}\operatorname{Diag} RX \simeq \operatorname{Tot}\mathrm{C}^{(2)}RX = \operatorname{Tot}\mathrm{C}^{(2)}(X; R)$$

and hence

$$\begin{aligned}\mathrm{H}^n(\operatorname{Diag} X, M; R) &= \mathrm{H}^n({}_R(\mathrm{C}(\operatorname{Diag} X; R), M)) \cong \mathrm{H}^n({}_R(\operatorname{Tot}\mathrm{C}^{(2)}(X; R), M))\\ &= \mathrm{H}^n(\operatorname{Tot}{}_R(\mathrm{C}^{(2)}(X; R), M))\end{aligned}$$

for $n \in \mathbb{N}_0$. The spectral sequence of the "columnwise" filtered double complex ${}_R(\mathrm{C}^{(2)}(X; R), M)$ has

$$E_1^{p,n-p} \cong \mathrm{H}^{n-p}({}_R(\mathrm{C}^{(2)}_{p,-}(X; R), M)) = \mathrm{H}^{n-p}({}_R(\mathrm{C}(X_{p,-}; R), M)) = \mathrm{H}^{n-p}(X_{p,-}, M; R)$$

for $p \in [0,n]$, $n \in \mathbb{N}_0$. ([1]) □

[1] The seeming non-duality in the proofs of (a) and (b) is due to the fact that cohomology of *co*simplicial objects has not been defined.

Chapter IV

Simplicial groups

We want to define homology groups for a given simplicial group. Thereto, we generalise the classifying simplicial set notion for groups given in chapter II, §5. Indeed, there are two known possibilities to define a classifying simplicial set. We show that both are equivalent by an algebraic proof (cf. theorem (4.32)).
References for this chapter are [8], [17], [20], [26], [29, §8].

§1 The Moore complex of a simplicial group

The Moore complex, introduced for objects in abelian categories in chapter II, §3, can be defined in the category of groups. Here, a *complex* M (*bounded below at* 0) of groups means a sequence

$$M = (\dots \xrightarrow{\partial} M_2 \xrightarrow{\partial} M_1 \xrightarrow{\partial} M_0)$$

of groups M_n for $n \in \mathbb{N}_0$ and group homomorphisms ∂ such that $\partial\partial = 1$, where 1 denotes the constant group homomorphism. A morphism $M \xrightarrow{\varphi} N$ of complexes of groups consists of group homomorphisms $\varphi_n \colon M_n \to N_n$ for $n \in \mathbb{N}_0$ such that $\varphi_n \partial = \partial \varphi_{n-1}$ for all $n \in \mathbb{N}$. The category of complexes of groups is denoted by $\mathbf{C}(\mathbf{Grp})$.
Given a complex M of groups we define, as usual, $\mathrm{Z}_n M := \mathrm{Ker}(M_n \xrightarrow{\partial} M_{n-1})$ and $\mathrm{B}_n M := \mathrm{Im}(M_{n+1} \xrightarrow{\partial} M_n)$ for $n \in \mathbb{N}_0$, where $M_{-1} = \{1\}$. A complex M of groups is said to be *normal* if $\mathrm{B}_n M$ is normal in $\mathrm{Z}_n M$, written $\mathrm{B}_n M \trianglelefteq \mathrm{Z}_n M$, for all $n \in \mathbb{N}_0$. If M is normal, we define $\mathrm{H}_n M := \mathrm{Z}_n M / \mathrm{B}_n M$ for $n \in \mathbb{N}_0$.

(4.1) Remark. We let G be a simplicial group. There is a complex of groups

$$\mathrm{M}G := (\dots \xrightarrow{\partial} \mathrm{M}_2 G \xrightarrow{\partial} \mathrm{M}_1 G \xrightarrow{\partial} \mathrm{M}_0 G),$$

where each entry $\mathrm{M}_n G$ is given by

$$\mathrm{M}_n G := \bigcap_{k \in [1,n]} \mathrm{Ker}\, \mathrm{d}_k \text{ for } n \in \mathbb{N}_0$$

and where the differentials are given by $\partial := \mathrm{d}_0|_{\mathrm{M}_n G}^{\mathrm{M}_{n-1} G}$ for all $n \in \mathbb{N}$.

Proof. We have

$$g_n \mathrm{d}_0 \mathrm{d}_k = g_n \mathrm{d}_{k+1} \mathrm{d}_0 = 1 \mathrm{d}_0 = 1$$

for all $g_n \in \mathrm{M}_n G$, $k \in [0, n-1]$, $n \in \mathbb{N}$. Hence $(\mathrm{M}_n G)\mathrm{d}_0 \leq \mathrm{M}_{n-1} G$ for all $n \in \mathbb{N}$ and $\partial\partial = 1$ for all $n \in \mathbb{N}$, $n \geq 2$. □

(4.2) Definition (Moore complex of a simplicial group)**.** We suppose given a simplicial group G. The complex

$$\mathrm{M}G = (\dots \xrightarrow{\partial} \mathrm{M}_2 G \xrightarrow{\partial} \mathrm{M}_1 G \xrightarrow{\partial} \mathrm{M}_0 G)$$

given as in remark (4.1) by $\mathrm{M}_n G := \bigcap_{k \in [1,n]} \mathrm{Ker}(\mathrm{d}_k)$ for $n \in \mathbb{N}_0$ and $\partial := \mathrm{d}_0|_{\mathrm{M}_n G}^{\mathrm{M}_{n-1} G}$ for $n \in \mathbb{N}$ is called the *Moore complex* of G.

(4.3) Proposition.

(a) Given simplicial groups G and H and a simplicial group homomorphism $G \xrightarrow{\varphi} H$, there exists an induced complex morphism

$$\mathrm{M}G \xrightarrow{\mathrm{M}\varphi} \mathrm{M}H$$

given by $\mathrm{M}_n\varphi := \varphi_n|_{\mathrm{M}_n G}^{\mathrm{M}_n H}$ for all $n \in \mathbb{N}_0$.

(b) The construction in (a) yields a functor

$$\mathbf{sGrp} \xrightarrow{\mathrm{M}} \mathbf{C}(\mathbf{Grp}).$$

Proof.

(a) We have

$$g_n\varphi_n\mathrm{d}_k = g_n\mathrm{d}_k\varphi_{n-1} = 1\varphi_{n-1} = 1$$

for all $g_n \in \mathrm{M}_nG$, $k \in [1, n]$, that is $g_n\varphi_n \in \mathrm{M}_nH$ for all $n \in \mathbb{N}_0$.

(b) We let G, H, K be simplicial groups and we let $G \xrightarrow{\varphi} H$ and $H \xrightarrow{\psi} K$ be simplicial group homomorphisms. Then we get

$$(\mathrm{M}_n\varphi)(\mathrm{M}_n\psi) = (\varphi_n|_{\mathrm{M}_nG}^{\mathrm{M}_nH})(\psi_n|_{\mathrm{M}_nH}^{\mathrm{M}_nK}) = (\varphi_n\psi_n)|_{\mathrm{M}_nG}^{\mathrm{M}_nK} = (\varphi\psi)_n|_{\mathrm{M}_nG}^{\mathrm{M}_nK} = \mathrm{M}_n(\varphi\psi)$$

and

$$\mathrm{M}_n\mathrm{id}_G = (\mathrm{id}_G)_n|_{\mathrm{M}_nG}^{\mathrm{M}_nG} = \mathrm{id}_{G_n}|_{\mathrm{M}_nG}^{\mathrm{M}_nG} = \mathrm{id}_{\mathrm{M}_nG}$$

for all $n \in \mathbb{N}_0$, that is $(\mathrm{M}\varphi)(\mathrm{M}\psi) = \mathrm{M}(\varphi\psi)$ and $\mathrm{Mid}_G = \mathrm{id}_{\mathrm{M}G}$. □

(4.4) Lemma. For every simplicial group G, we have $\mathrm{B}_n\mathrm{M}G \trianglelefteq G_n$ for all $n \in \mathbb{N}_0$.

Proof. We consider group elements $h \in \mathrm{B}_n\mathrm{M}G$ and $y \in G_n$ and we let $g \in \mathrm{M}_{n+1}G$ be such that $h = g\partial = g\mathrm{d}_0$. Then we get

$$((y\mathrm{s}_0)g(y^{-1}\mathrm{s}_0))\mathrm{d}_k = (y\mathrm{s}_0\mathrm{d}_k)(g\mathrm{d}_k)(y^{-1}\mathrm{s}_0\mathrm{d}_k) = (y\mathrm{s}_0\mathrm{d}_k)(y^{-1}\mathrm{s}_0\mathrm{d}_k) = (yy^{-1})\mathrm{s}_0\mathrm{d}_k = 1$$

for all $k \in [1, n+1]$. Furthermore,

$$((y\mathrm{s}_0)g(y^{-1}\mathrm{s}_0))\mathrm{d}_0 = (y\mathrm{s}_0\mathrm{d}_0)(g\mathrm{d}_0)(y^{-1}\mathrm{s}_0\mathrm{d}_0) = yhy^{-1}.$$

Thus $(y\mathrm{s}_0)g(y^{-1}\mathrm{s}_0) \in \mathrm{M}_{n+1}G$ is a preimage of yhy^{-1}, that is, $yhy^{-1} \in \mathrm{B}_n\mathrm{M}G$. Since $h \in \mathrm{B}_n\mathrm{M}G$ and $y \in G_n$ were chosen arbitrarily, this implies $\mathrm{B}_n\mathrm{M}G \trianglelefteq G_n$ for all $n \in \mathbb{N}_0$. □

(4.5) Corollary. The Moore complex $\mathrm{M}G$ of a simplicial group G is normal.

Proof. The assertion follows directly from lemma (4.4), because if $\mathrm{B}_n\mathrm{M}G$ is normal in G_n, then it is in particular normal in the subgroup $\mathrm{Z}_n\mathrm{M}G \leq G_n$ for all $n \in \mathbb{N}_0$. □

By the preceeding corollary, we are able to define homology groups of the Moore complex of a given simplicial group.

(4.6) Definition (homotopy groups of a simplicial group)**.** For $n \in \mathbb{N}_0$ we call

$$\pi_n(G) := \mathrm{H}_n\mathrm{M}G$$

the *n-th homotopy group* of a given simplicial group G.

(4.7) Lemma. We suppose given a simplicial group G and group elements $x, y \in G_n$ for $n \in \mathbb{N}$. If $x \in \operatorname{Ker} \mathrm{d}_0$ and $y \in \mathrm{M}_nG$, then $[x, y] \in \mathrm{B}_n\mathrm{M}G$.

Proof. We have

$$
\begin{aligned}
&((x\mathrm{s}_0)(x^{-1}\mathrm{s}_1)(y\mathrm{s}_0)(x\mathrm{s}_1)(x^{-1}\mathrm{s}_0)(y^{-1}\mathrm{s}_0))\mathrm{d}_k = (x\mathrm{s}_0\mathrm{d}_k)(x^{-1}\mathrm{s}_1\mathrm{d}_k)(y\mathrm{s}_0\mathrm{d}_k)(x\mathrm{s}_1\mathrm{d}_k)(x^{-1}\mathrm{s}_0\mathrm{d}_k)(y^{-1}\mathrm{s}_0\mathrm{d}_k)\\
&= \left\{\begin{array}{ll} x(x^{-1}\mathrm{d}_0\mathrm{s}_0)y(x\mathrm{d}_0\mathrm{s}_0)x^{-1}y^{-1} & \text{for } k = 0,\\ xx^{-1}yxx^{-1}y^{-1} & \text{for } k = 1,\\ (x\mathrm{d}_1\mathrm{s}_0)x^{-1}(y\mathrm{d}_1\mathrm{s}_0)x(x^{-1}\mathrm{d}_1\mathrm{s}_0)(y^{-1}\mathrm{d}_1\mathrm{s}_0) & \text{for } k = 2,\\ (x\mathrm{d}_{k-1}\mathrm{s}_0)(x^{-1}\mathrm{d}_{k-1}\mathrm{s}_1)(y\mathrm{d}_{k-1}\mathrm{s}_0)(x\mathrm{d}_{k-1}\mathrm{s}_1)(x^{-1}\mathrm{d}_{k-1}\mathrm{s}_0)(y^{-1}\mathrm{d}_{k-1}\mathrm{s}_0) & \text{for } k \in [3, n+1] \end{array}\right\}\\
&= \begin{cases} [x, y] & \text{for } k = 0,\\ 1 & \text{for } k \in [1, n+1]. \end{cases}
\end{aligned}
$$

Hence $(x\mathrm{s}_0)(x^{-1}\mathrm{s}_1)(y\mathrm{s}_0)(x\mathrm{s}_1)(x^{-1}\mathrm{s}_0)(y^{-1}\mathrm{s}_0) \in \bigcap_{k=1}^{n+1} \operatorname{Ker} \mathrm{d}_k = \mathrm{M}_{n+1}G$ and thus

$$[x, y] = ((x\mathrm{s}_0)(x^{-1}\mathrm{s}_1)(y\mathrm{s}_0)(x\mathrm{s}_1)(x^{-1}\mathrm{s}_0)(y^{-1}\mathrm{s}_0))\partial \in \mathrm{B}_n\mathrm{M}G. \qquad \square$$

(4.8) Corollary. Given a simplicial group G, its n-th homotopy group $\pi_n(G)$ for $n \in \mathbb{N}$ is abelian.

Proof. By lemma (4.7), we have $[x, y] \in \mathrm{B}_n\mathrm{M}G$ and hence $(x\mathrm{B}_n\mathrm{M}G)(y\mathrm{B}_n\mathrm{M}G) = (y\mathrm{B}_n\mathrm{M}G)(x\mathrm{B}_n\mathrm{M}G)$ for all $x, y \in \mathrm{Z}_n\mathrm{M}G$, $n \in \mathbb{N}$. □

§2 Semidirect product decomposition

Given a simplicial group G, the group G_n, $n \in \mathbb{N}_0$, can be decomposed as a certain iterated semidirect product of the Moore complex entries in dimension less or equal then n.
For further information, we refer to the article [5] by Carrasco and Cegarra.

(4.9) Lemma. We let G be a simplicial group. Then we have a split short exact sequence

$$\bigcap_{i\in[k,n]} \operatorname{Ker} \mathrm{d}_i \longrightarrow \bigcap_{i\in[k+1,n]} \operatorname{Ker} \mathrm{d}_i \xrightarrow{\mathrm{d}_k} \bigcap_{i\in[k,n-1]} \operatorname{Ker} \mathrm{d}_i$$

for all $n \in \mathbb{N}$, $k \in [1, n]$, where the middle term is a subgroup of G_n.

Proof. We let $n \in \mathbb{N}$ and $k \in [1, n]$ be given. Furthermore, we suppose given $g_n \in \bigcap_{i\in[k+1,n]} \operatorname{Ker} \mathrm{d}_i$. Then

$$g_n\mathrm{d}_k\mathrm{d}_j = g_n\mathrm{d}_{j+1}\mathrm{d}_k = 1\mathrm{d}_k = 1$$

for all $j \in [k, n-1]$, that is, $g_n\mathrm{d}_k \in \bigcap_{i\in[k,n-1]} \operatorname{Ker} \mathrm{d}_i$ and thus $\bigcap_{i\in[k+1,n]} \operatorname{Ker} \mathrm{d}_i \to \bigcap_{i\in[k,n-1]} \operatorname{Ker} \mathrm{d}_i$, $g_n \mapsto g_n\mathrm{d}_k$ is a well-defined group homomorphism. Its kernel is given by $\bigcap_{i\in[k,n]} \operatorname{Ker} \mathrm{d}_i$.
Now for every $g_{n-1} \in \bigcap_{i\in[k,n-1]} \operatorname{Ker} \mathrm{d}_i$, we have

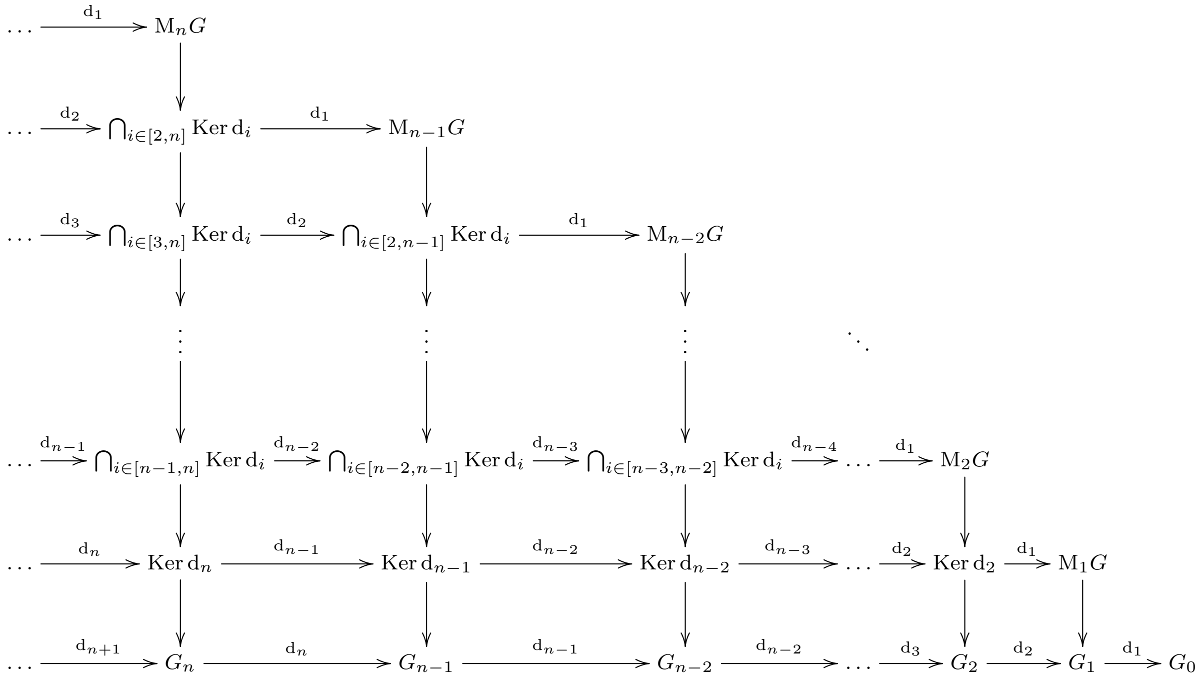
$$g_{n-1}\mathrm{s}_{k-1}\mathrm{d}_i = g_{n-1}\mathrm{d}_{i-1}\mathrm{s}_{k-1} = 1\mathrm{s}_{k-1} = 1$$

for $i \in [k+1, n]$ as well as

$$g_{n-1}\mathrm{s}_{k-1}\mathrm{d}_k = g_{n-1},$$

that is, $\bigcap_{i\in[k+1,n]} \operatorname{Ker} \mathrm{d}_i \to \bigcap_{i\in[k,n-1]} \operatorname{Ker} \mathrm{d}_i, g_n \mapsto g_n\mathrm{d}_k$ is a retraction with coretraction $\bigcap_{i\in[k,n-1]} \operatorname{Ker} \mathrm{d}_i \to \bigcap_{i\in[k+1,n]} \operatorname{Ker} \mathrm{d}_i, g_{n-1} \mapsto g_{n-1}\mathrm{s}_{k-1}$. □

(4.10) Remark. If G is a simplicial group, we have a (non commutative) diagram



In this diagram, every composition of a vertical inclusion with a horizontal morphism forms a split short exact sequence. We call this the *decomposition diagram* of G.

(4.11) Corollary. We let G be a simplicial group and $n \in \mathbb{N}_0$ a non-negative integer. Then the group of n-simplices G_n is isomorphic to an iterated semidirect product with 2^n factors that all derive from the Moore complex.

(4.12) Example. From the decomposition diagram for some simplicial group G we can read off:

$G_0 \cong \mathrm{M}_0 G$,

$G_1 \cong \mathrm{M}_1 G \rtimes \mathrm{M}_0 G$,

$G_2 \cong \mathrm{Ker}\,\mathrm{d}_2 \rtimes G_1 \cong (\mathrm{M}_2 G \rtimes \mathrm{M}_1 G) \rtimes (\mathrm{M}_1 G \rtimes \mathrm{M}_0 G)$,

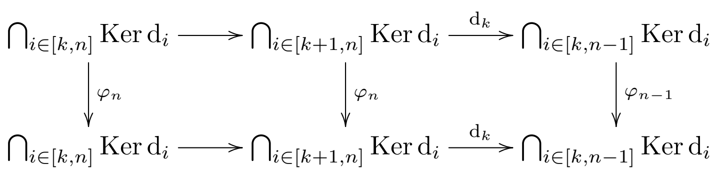
$G_3 \cong \mathrm{Ker}\,\mathrm{d}_3 \rtimes G_2 \cong ((\mathrm{M}_3 G \rtimes \mathrm{M}_2 G) \rtimes (\mathrm{M}_2 G \rtimes \mathrm{M}_1 G)) \rtimes ((\mathrm{M}_2 G \rtimes \mathrm{M}_1 G) \rtimes (\mathrm{M}_1 G \rtimes \mathrm{M}_0 G))$.

To obtain the decomposition of $\mathrm{Ker}\,\mathrm{d}_n$ from that of G_{n-1}, one has to increase every index in the decomposition of G_{n-1} by 1. Since $G_n = \mathrm{Ker}\,\mathrm{d}_n \rtimes G_{n-1}$, one has to "concatenate" both iterated semidirect products to obtain the decomposition of G_n. Thus the "sequence of the indices" is given by

$0, 10, 2110, 32212110, 4332322132212110, 54434332433232214332322132212110, \ldots$

(4.13) Remark. We let G, H be simplicial groups and $G \xrightarrow{\varphi} H$ a simplicial group homomorphism. Furthermore, we suppose given $n \in \mathbb{N}$. If $\mathrm{M}_n\varphi$ is an isomorphism and φ_k is an isomorphism for all $k \in [0, n-1]$, then φ_n is also an isomorphism.

Proof. Since φ_k is an isomorphism for all $k \in [0, n-1]$, we have induced isomorphisms on the kernels and their intersections, that is $\bigcap_{i\in[k,n-1]} \mathrm{Ker}\,\mathrm{d}_i \to \bigcap_{i\in[k,n-1]} \mathrm{Ker}\,\mathrm{d}_i, g_{n-1} \mapsto g_{n-1}\varphi_{n-1}$ is an isomorphism for all $k \in [1, n]$. We will show that the induced morphism $\bigcap_{i\in[k,n]} \mathrm{Ker}\,\mathrm{d}_i \to \bigcap_{i\in[k,n]} \mathrm{Ker}\,\mathrm{d}_i, g_n \mapsto g_n\varphi_n$ for $k \in [1, n+1]$ has to be an isomorphism, too. Thereto, we proceed by induction on $k \in [1, n+1]$. The induction basis is our assumption that $\mathrm{M}_n\varphi$ is an isomorphism. So we let $k \in [1, n]$ be given and we assume $\bigcap_{i\in[k,n]} \mathrm{Ker}\,\mathrm{d}_i \to \bigcap_{i\in[k,n]} \mathrm{Ker}\,\mathrm{d}_i, g_n \mapsto g_n\varphi_n$ to be an isomorphism. By lemma (4.9), we have a morphism of short exact sequences



The five lemma yields the assertion. By induction, $\bigcap_{i\in[k,n]} \operatorname{Ker} \mathrm{d}_i \to \bigcap_{i\in[k,n]} \operatorname{Ker} \mathrm{d}_i, g_n \mapsto g_n\varphi_n$ is an isomorphism for all $k \in [1, n+1]$, and in particular $\varphi_n \colon G_n \to H_n$ is an isomorphism. □

(4.14) Lemma. We let G, H be simplicial groups and $G \xrightarrow{\varphi} H$ be a simplicial group homomorphism. Then $\mathrm{M}\varphi$ is an isomorphism if and only if φ is an isomorphism.

Proof. If φ is an isomorphism, then $\mathrm{M}\varphi$ is an isomorphism by the functoriality of the Moore complex functor. So let us assume that $\mathrm{M}\varphi$ is an isomorphism of complexes. We show by induction on $n \in \mathbb{N}_0$ that φ_n is an isomorphism of groups. For $n = 0$, this holds since $\varphi_0 = \mathrm{M}_0\varphi$. Now we suppose given $n \in \mathbb{N}$ and we assume that φ_k is an isomorphism for $k \in [0, n-1]$. Since $\mathrm{M}_n\varphi$ is an isomorphism, too, by remark (4.13) we can conclude that φ_n has to be an isomorphism. Thus, by induction, φ_n is an isomorphism for all $n \in \mathbb{N}_0$ and so φ is an isomorphism of simplicial groups. □

§3 The coskeleton of a group

Every group P can be interpreted as the constant simplicial group $\operatorname{Const} P$. We show that the functor Const has a left adjoint.

(4.15) Proposition. The functor $\mathbf{sGrp} \xrightarrow{\pi_0} \mathbf{Grp}$ is left adjoint to $\mathbf{Grp} \xrightarrow{\operatorname{Const}} \mathbf{sGrp}$ and

$$\pi_0 \circ \operatorname{Const} \cong \mathrm{id}_{\mathbf{Grp}}.$$

Proof. Given $P \in \operatorname{Ob} \mathbf{Grp}$, we have

$$\mathrm{M}(\operatorname{Const} P) = (\ldots \longrightarrow \{1\} \longrightarrow \{1\} \longrightarrow P)$$

and therefore

$$\pi_0 \operatorname{Const} P = \mathrm{H}_0\mathrm{M}(\operatorname{Const} P) = \mathrm{Z}_0\mathrm{M}(\operatorname{Const} P)/\mathrm{B}_0\mathrm{M}(\operatorname{Const} P) = P/\{1\}.$$

For a group homomorphism $f \colon P \to Q$, we furthermore have

$$\pi_0 \operatorname{Const} f = \mathrm{H}_0\mathrm{M}(\operatorname{Const} f) = (p\{1\} \mapsto (pf)\{1\}).$$

To construct the counit $\pi_0 \circ \operatorname{Const} \xrightarrow{\eta} \mathrm{id}_{\mathbf{Grp}}$, we let $\eta_P \colon P/\{1\} \to P, p\{1\} \mapsto p$ for every $P \in \operatorname{Ob} \mathbf{Grp}$. Thus $\pi_0 \circ \operatorname{Const} \xrightarrow{\eta} \mathrm{id}_{\mathbf{Grp}}$ is a natural isotransformation, since η_P is an isomorphism of groups for all $P \in \operatorname{Ob} \mathbf{Grp}$ and since for all morphism $P \xrightarrow{f} Q$ in $\mathbf{Grp}$ we obtain a commutative diagram

$$\begin{array}{ccc} P/\{1\} & \xrightarrow{\eta_P} & P \\ {\scriptstyle \pi_0 \operatorname{Const} f}\downarrow & & \downarrow{\scriptstyle f} \\ Q/\{1\} & \xrightarrow{\eta_Q} & Q \end{array}$$

In particular, $\pi_0 \circ \operatorname{Const} \cong \mathrm{id}_{\mathbf{Grp}}$.
In order to show that $\pi_0 \dashv \operatorname{Const}$, we have to construct the unit $\mathrm{id}_{\mathbf{sGrp}} \xrightarrow{\varepsilon} \operatorname{Const} \circ \pi_0$. For this we let $G \in \operatorname{Ob} \mathbf{sGrp}$ be a simplicial group and we denote by $\nu = \nu_G \colon G_0 \to \pi_0 G$ the canonical epimorphism. We define for all $n \in \mathbb{N}_0$ a group homomorphism $(\varepsilon_G)_n \colon G_n \to \pi_0 G$ by $(\varepsilon_G)_n := \mathrm{d}_{\lfloor n,1 \rfloor}\nu$. To show that $G \xrightarrow{\varepsilon_G} \operatorname{Const} \pi_0 G$ is a simplicial group homomorphism, we have to prove the commutativity of the diagrams

$$\begin{array}{ccc} G_n & \xrightarrow{\mathrm{d}_k} & G_{n-1} \\ {\scriptstyle (\varepsilon_G)_n}\downarrow & & \downarrow{\scriptstyle (\varepsilon_G)_{n-1}} \\ \pi_0 G & \xrightarrow{\mathrm{id}_{\pi_0 G}} & \pi_0 G \end{array}$$

for all $k \in [0, n]$, $n \in \mathbb{N}$, and

$$\begin{array}{ccc} G_{n+1} & \xleftarrow{\mathrm{s}_k} & G_n \\ {\scriptstyle (\varepsilon_G)_{n+1}}\downarrow & & \downarrow{\scriptstyle (\varepsilon_G)_n} \\ \pi_0 G & \xleftarrow{\mathrm{id}_{\pi_0 G}} & \pi_0 G \end{array}$$

for all $k \in [0, n]$, $n \in \mathbb{N}_0$.
First, we consider the faces and proceed by an induction on $n \in \mathbb{N}$. We consider the case $n = 1$. Since the assertion holds trivially for d_1, we have to show that $\mathrm{d}_0(\varepsilon_G)_0 = (\varepsilon_G)_1$, that is, that $\mathrm{d}_0\nu = \mathrm{d}_1\nu$. But for all $g_1 \in G_1$ we have $g_1(g_1^{-1}\mathrm{d}_1\mathrm{s}_0) \in \mathrm{M}_1 G$, since

$$(g_1(g_1^{-1}\mathrm{d}_1\mathrm{s}_0))\mathrm{d}_1 = (g_1\mathrm{d}_1)(g_1^{-1}\mathrm{d}_1\mathrm{s}_0\mathrm{d}_1) = (g_1\mathrm{d}_1)(g_1^{-1}\mathrm{d}_1) = 1,$$

and $(g_1\mathrm{d}_0)(g_1^{-1}\mathrm{d}_1) = (g_1(g_1^{-1}\mathrm{d}_1\mathrm{s}_0))\mathrm{d}_0 \in \mathrm{B}_0\mathrm{M}G$, so that

$$g_1\mathrm{d}_0\nu = (g_1\mathrm{d}_0)\mathrm{B}_0\mathrm{M}G = (g_1\mathrm{d}_1)\mathrm{B}_0\mathrm{M}G = g_1\mathrm{d}_1\nu.$$

Now we consider a natural number $n \in \mathbb{N}$ with $n > 1$ and we assume that $(\varepsilon_G)_{n-1} = \mathrm{d}_k(\varepsilon_G)_{n-2}$ holds for all $k \in [0, n-1]$. This implies

$$\mathrm{d}_k(\varepsilon_G)_{n-1} = \mathrm{d}_k\mathrm{d}_{n-1}(\varepsilon_G)_{n-2} = \mathrm{d}_n\mathrm{d}_k(\varepsilon_G)_{n-2} = \mathrm{d}_n(\varepsilon_G)_{n-1} = \mathrm{d}_n\mathrm{d}_{\lfloor n-1,1\rfloor}\nu = \mathrm{d}_{\lfloor n,1\rfloor}\nu = (\varepsilon_G)_n$$

for all $k \in [0, n-1]$, and for $k = n$ the commutativity holds for trivial reasons. By induction we get the desired commutativity for all diagrams with the faces. But now the commutativity of the diagrams with the degeneracies follows as well since

$$\mathrm{s}_k(\varepsilon_G)_{n+1} = \mathrm{s}_k\mathrm{d}_k(\varepsilon_G)_n = (\varepsilon_G)_n$$

for all $k \in [0, n]$, $n \in \mathbb{N}_0$.
To show naturality of $(\varepsilon_G)_{G \in \operatorname{Ob}\mathbf{sGrp}}$, we let $G \xrightarrow{\varphi} H$ be a morphism of simplicial groups. We obtain

$$\varphi_n(\varepsilon_H)_n = \varphi_n\mathrm{d}_{\lfloor n,1\rfloor}\nu = \mathrm{d}_{\lfloor n,1\rfloor}\varphi_0\nu = \mathrm{d}_{\lfloor n,1\rfloor}\nu(\pi_0\varphi) = (\varepsilon_G)_n(\pi_0\varphi) \text{ for all } n \in \mathbb{N}_0$$

and thus a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varepsilon_G} & \operatorname{Const}\pi_0 G \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\operatorname{Const}\pi_0\varphi} \\ H & \xrightarrow{\varepsilon_H} & \operatorname{Const}\pi_0 H \end{array}$$

Thus we have a natural transformation $\mathrm{id}_{\mathbf{sGrp}} \xrightarrow{\varepsilon} \operatorname{Const} \circ \pi_0$.
Finally, we have to show that ε and η are unit and counit. Indeed,

$$(\varepsilon_{\operatorname{Const} P})_n(\operatorname{Const}\eta_P)_n = \mathrm{id}_P\nu_{\operatorname{Const} P}\eta_P = \nu_{\operatorname{Const} P}\eta_P = \mathrm{id}_P = (\mathrm{id}_{\operatorname{Const} P})_n$$

holds for all $n \in \mathbb{N}_0$, $P \in \operatorname{Ob}\mathbf{Grp}$, as well as

$$(g_0\mathrm{B}_0\mathrm{M}G)(\pi_0\varepsilon_G)(\eta_{\pi_0 G}) = ((g_0(\varepsilon_G)_0)\mathrm{B}_0\mathrm{M}(\operatorname{Const}\pi_0 G))\eta_{\pi_0 G} = (g_0\nu_G\{1\})\eta_{\pi_0 G} = g_0\nu_G = g_0\mathrm{B}_0\mathrm{M}G$$

for all $G \in \operatorname{Ob}\mathbf{sGrp}$, $g_0 \in G_0$. □

(4.16) Definition (0th coskeleton)**.** For every group P we define the *coskeleton* of P to be the simplicial group $\operatorname{Cosk} P := \operatorname{Const} P$. The functor

$$\mathbf{Grp} \xrightarrow{\operatorname{Cosk}} \mathbf{sGrp}$$

is called the (0*th*) *coskeleton*.

§4 The Kan classifying functor

We have already seen that we can define the homology of a group as the homology of the classifying simplicial set. In this section, we want to generalise this procedure to simplicial groups, introducing Kan's classifying functor

$$\mathbf{sGrp} \xrightarrow{\overline{\mathrm{W}}} \mathbf{sSet},$$

which generalises the classifying simplicial set functor

$$\mathbf{Grp} \xrightarrow{\mathrm{B}} \mathbf{sSet}$$

in the sense that $\mathrm{B} = \overline{\mathrm{W}} \circ \operatorname{Cosk}$. After this, we construct Kan's loop group functor G as a left adjoint to $\overline{\mathrm{W}}$. The reader is refered to [20].

(4.17) Definition (nerve of a simplicial group)**.** We define the *nerve* $\mathrm{N}G$ of a simplicial group G to be the bisimplicial set which corresponds to the nerve of the group object in **sSet** arising from G. Analogously for the morphisms in **sGrp**.

$$\begin{array}{ccc} \mathbf{sGrp} & \xrightarrow{\mathrm{N}} & \mathbf{s}^2\mathbf{Set} \\ \cong \updownarrow & & \updownarrow \cong \\ \mathbf{Grp}(\mathbf{sSet}) & \xrightarrow{\mathrm{N}} & \mathbf{s}(\mathbf{sSet}) \end{array}$$

(4.18) Remark (the nerve of a simplicial group is build componentwise)**.** The nerve of a simplicial group G is given by $\mathrm{N}_{m,-}G = \mathrm{N}G_m$ for all $m \in \mathbb{N}_0$ and $\mathrm{N}_{\theta,-}G = \mathrm{N}G_\theta$ for all morphisms $\theta \in \operatorname{Mor} \mathbf{\Delta}$.

Proof. Follows from example (1.27)(c) and definition (1.32). □

(4.19) Remark. There is a functor

$$\mathbf{sGrp} \xrightarrow{\overline{\mathrm{W}}} \mathbf{sSet}$$

isomorphic to $\operatorname{Tot} \circ \mathrm{N}$ that is given on objects by $\overline{\mathrm{W}}_n G := \bigtimes_{j \in \lfloor n-1,0 \rfloor} G_j$ and

$$(g_j)_{j \in \lfloor n-1,0 \rfloor} \overline{\mathrm{W}}_\theta G := (\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[i]}})_{i \in \lfloor m-1,0 \rfloor}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, where $\theta \in {}_{\mathbf{\Delta}}([m],[n])$, $m, n \in \mathbb{N}_0$, $G \in \operatorname{Ob} \mathbf{sGrp}$, and on morphisms by $\overline{\mathrm{W}}_n \varphi := \bigtimes_{j \in \lfloor n-1,0 \rfloor} \varphi_j$ for $n \in \mathbb{N}_0$, $\varphi \in {}_{\mathbf{sGrp}}(G, H)$, $G, H \in \operatorname{Ob} \mathbf{sGrp}$.

Proof. We let G be a simplicial group and compute $\operatorname{Tot} \mathrm{N}G$. The set of n-simplices is given by

$$\begin{aligned}
&\operatorname{Tot}_n \mathrm{N}G = \{(x_q)_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} \mathrm{N}_{q,n-q} G \mid x_q \mathrm{d}^{\mathrm{h}}_q = x_{q-1} \mathrm{d}^{\mathrm{v}}_0 \text{ for } q \in \lfloor n,1 \rfloor\} \\
&= \{((g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} G_q^{\times(n-q)} \mid (g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor} \mathrm{d}^{\mathrm{h}}_q = (g_{q-1,j})_{j \in \lfloor n-q,0 \rfloor} \mathrm{d}^{\mathrm{v}}_0 \text{ for } q \in \lfloor n,1 \rfloor\} \\
&= \{((g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} G_q^{\times(n-q)} \mid (g_{q,j} \mathrm{d}_q)_{j \in \lfloor n-q-1,0 \rfloor} = (g_{q-1,j+1})_{j \in \lfloor n-q-1,0 \rfloor} \text{ for } q \in \lfloor n,1 \rfloor\} \\
&= \{((g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} G_q^{\times(n-q)} \mid g_{q,j} \mathrm{d}_q = g_{q-1,j+1} \text{ for } j \in \lfloor n-q-1,0 \rfloor, q \in \lfloor n,1 \rfloor\} \\
&= \{((g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} G_q^{\times(n-q)} \mid g_{q,j} = g_{q+1,j-1} \mathrm{d}_{q+1} \text{ for } j \in \lfloor n-q-1,1 \rfloor, q \in \lfloor n-1,0 \rfloor\} \\
&= \{((g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} G_q^{\times(n-q)} \mid g_{q,j} = g_{q+j,0} \mathrm{d}_{\lfloor q+j,q+1 \rfloor} \text{ for } j \in \lfloor n-q-1,1 \rfloor, q \in \lfloor n-1,0 \rfloor\} \\
&= \{((g_{q,j})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} G_q^{\times(n-q)} \mid g_{q,j} = g_{q+j,0} \mathrm{d}_{\lfloor q+j,q+1 \rfloor} \text{ for } j \in \lfloor n-q-1,0 \rfloor, q \in \lfloor n,0 \rfloor\} \\
&= \{((g_{q+j,0} \mathrm{d}_{\lfloor q+j,q+1 \rfloor})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} \mid g_{q+j,0} \in G_{q+j} \text{ for } j \in \lfloor n-q-1,0 \rfloor, q \in \lfloor n,0 \rfloor\} \\
&= \{((g_j \mathrm{d}_{\lfloor j,q+1 \rfloor})_{j \in \lfloor n-1,q \rfloor})_{q \in \lfloor n,0 \rfloor} \mid g_j \in G_j \text{ for } j \in \lfloor n-1,q \rfloor, q \in \lfloor n,0 \rfloor\}
\end{aligned}$$

for $n \in \mathbb{N}_0$. For an element $((g_j \mathrm{d}_{\lfloor j,q+1 \rfloor})_{j \in \lfloor n-1,q \rfloor})_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n \mathrm{N}G$, we compute

$$\begin{aligned}
&((g_j \mathrm{d}_{\lfloor j,q+1 \rfloor})_{j \in \lfloor n-1,q \rfloor})_{q \in \lfloor n,0 \rfloor} (\operatorname{Tot}_\theta \mathrm{N}G) = ((g_{q+j} \mathrm{d}_{\lfloor q+j,q+1 \rfloor})_{j \in \lfloor n-q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor} (\operatorname{Tot}_\theta \mathrm{N}G) \\
&= ((g_{p\theta+j} \mathrm{d}_{\lfloor p\theta+j,p\theta+1 \rfloor})_{j \in \lfloor n-p\theta-1,0 \rfloor} (\mathrm{N}G)_{\mathrm{Spl}_p(\theta)})_{p \in \lfloor m,0 \rfloor} \\
&= ((g_{p\theta+j} \mathrm{d}_{\lfloor p\theta+j,p\theta+1 \rfloor})_{j \in \lfloor n-p\theta-1,0 \rfloor} (\mathrm{N}G)_{\mathrm{Spl}_{\leq p}(\theta),n-p\theta} (\mathrm{N}G)_{p,\mathrm{Spl}_{\geq p}(\theta)})_{p \in \lfloor m,0 \rfloor} \\
&= ((g_{p\theta+j} \mathrm{d}_{\lfloor p\theta+j,p\theta+1 \rfloor} G_{\mathrm{Spl}_{\leq p}(\theta)})_{j \in \lfloor n-p\theta-1,0 \rfloor} (\mathrm{N}G)_{p,\mathrm{Spl}_{\geq p}(\theta)})_{p \in \lfloor m,0 \rfloor} \\
&= ((\prod_{j \in \lfloor (i+1)\mathrm{Spl}_{\geq p}(\theta)-1, i\mathrm{Spl}_{\geq p}(\theta) \rfloor} g_{p\theta+j} \mathrm{d}_{\lfloor p\theta+j,p\theta+1 \rfloor} G_{\mathrm{Spl}_{\leq p}(\theta)})_{i \in \lfloor m-p-1,0 \rfloor})_{p \in \lfloor m,0 \rfloor}
\end{aligned}$$

$$= ((\prod_{j \in \lfloor (i+1+p)\theta - p\theta - 1, (i+p)\theta - p\theta \rfloor} g_{p\theta + j} G_{\delta^{\lceil p\theta+1, p\theta+j \rceil}} G_{\mathrm{Spl}_{\leq p}(\theta)})_{i \in \lfloor m-p-1, 0 \rfloor})_{p \in \lfloor m, 0 \rfloor}$$
$$= ((\prod_{j \in \lfloor (i+1)\theta - p\theta - 1, i\theta - p\theta \rfloor} g_{p\theta + j} G_{\mathrm{Spl}_{\leq p}(\theta) \delta^{\lceil p\theta+1, p\theta+j \rceil}})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor}$$
$$= ((\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\mathrm{Spl}_{\leq p}(\theta) \delta^{\lceil p\theta+1, j \rceil}})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor} = ((\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[p]}})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor}$$
$$= ((\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\delta^{\lceil p+1, i \rceil} \theta|^{[j]}_{[i]}})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor} = ((\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[i]}} G_{\delta^{\lceil p+1, i \rceil}})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor}$$
$$= ((\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[i]}} \mathrm{d}_{\lfloor i, p+1 \rfloor})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor}$$

for $\theta \in \mathbf{\Delta}([m], [n])$, $m, n \in \mathbb{N}_0$.
Thus, by transport of structure, $\overline{\mathrm{W}}G$ with $\overline{\mathrm{W}}_n G = \bigtimes_{j \in \lfloor n-1, 0 \rfloor} G_j$ for all $n \in \mathbb{N}_0$ becomes a reduced simplicial set isomorphic to $\mathrm{Tot}\,\mathrm{N}G$ via the bijections

$$(f_G)_n \colon \mathrm{Tot}_n \mathrm{N}G \to \overline{\mathrm{W}}G, ((g_j \mathrm{d}_{\lfloor j, q+1 \rfloor})_{j \in \lfloor n-1, q \rfloor})_{q \in \lfloor n, 0 \rfloor} \mapsto (g_j)_{j \in \lfloor n-1, 0 \rfloor}.$$

To prove the formula for $\overline{\mathrm{W}}G$ on the morphisms of $\mathbf{\Delta}$, we suppose given $\theta \in \mathbf{\Delta}([m], [n])$ for $m, n \in \mathbb{N}_0$. We obtain

$$\begin{aligned}(g_j)_{j \in \lfloor n-1, 0 \rfloor} \overline{\mathrm{W}}_\theta G &= (g_j)_{j \in \lfloor n-1, 0 \rfloor} (f_G)_n^{-1} (\mathrm{Tot}_\theta \mathrm{N}G)(f_G)_m = ((g_j \mathrm{d}_{\lfloor j, q+1 \rfloor})_{j \in \lfloor n-1, q \rfloor})_{q \in \lfloor n, 0 \rfloor} (\mathrm{Tot}_\theta \mathrm{N}G)(f_G)_m \\ &= ((\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[i]}} \mathrm{d}_{\lfloor i, p+1 \rfloor})_{i \in \lfloor m-1, p \rfloor})_{p \in \lfloor m, 0 \rfloor} (f_G)_m \\ &= (\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[i]}})_{i \in \lfloor m-1, 0 \rfloor}.\end{aligned}$$

Furthermore, given another simplicial group H and a simplicial group homomorphism $G \xrightarrow{\varphi} H$, we have, again by transport of structure,

$$\begin{aligned}(g_j)_{j \in \lfloor n-1, 0 \rfloor} (\overline{\mathrm{W}}_n \varphi) &= (g_j)_{j \in \lfloor n-1, 0 \rfloor} (f_G)_n^{-1} (\mathrm{Tot}_n \mathrm{N}\varphi)(f_H)_n \\ &= ((g_j \mathrm{d}_{\lfloor j, q+1 \rfloor})_{j \in \lfloor n-1, q \rfloor})_{q \in \lfloor n, 0 \rfloor} (\mathrm{Tot}_n \mathrm{N}\varphi)(f_H)_n \\ &= ((g_j \mathrm{d}_{\lfloor j, q+1 \rfloor})_{j \in \lfloor n-1, q \rfloor} \mathrm{N}_{q, n-q} \varphi)_{q \in \lfloor n, 0 \rfloor} (f_H)_n \\ &= ((g_j \mathrm{d}_{\lfloor j, q+1 \rfloor} \varphi_q)_{j \in \lfloor n-1, q \rfloor})_{q \in \lfloor n, 0 \rfloor} (f_H)_n \\ &= ((g_j \varphi_j \mathrm{d}_{\lfloor j, q+1 \rfloor})_{j \in \lfloor n-1, q \rfloor})_{q \in \lfloor n, 0 \rfloor} (f_H)_n = (g_j \varphi_j)_{j \in \lfloor n-1, 0 \rfloor}\end{aligned}$$

for all $(g_j)_{j \in \lfloor n-1, 0 \rfloor} \in \overline{\mathrm{W}}_n G$, that is,

$$\overline{\mathrm{W}}_n \varphi = \bigtimes_{j \in \lfloor n-1, 0 \rfloor} \varphi_j \text{ for all } n \in \mathbb{N}_0.$$

Altogether, we have constructed a functor $\mathbf{sGrp} \xrightarrow{\overline{\mathrm{W}}} \mathbf{sSet}$ and a natural isotransformation $\mathrm{Tot} \circ \mathrm{N} \xrightarrow{f} \overline{\mathrm{W}}$. □

(4.20) Definition (Kan classifying simplicial set)**.** We let G be a simplicial group. The reduced simplicial set $\overline{\mathrm{W}}G$ given as in remark (4.19) by

$$\overline{\mathrm{W}}_n G = \bigtimes_{j \in \lfloor n-1, 0 \rfloor} G_j \text{ for every } n \in \mathbb{N}_0$$

and $(g_j)_{j \in \lfloor n-1, 0 \rfloor} \overline{\mathrm{W}}_\theta G = (\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j G_{\theta|^{[j]}_{[i]}})_{i \in \lfloor m-1, 0 \rfloor}$ for $(g_j)_{j \in \lfloor n-1, 0 \rfloor} \in \overline{\mathrm{W}}_n G$, $\theta \in \mathbf{\Delta}([m], [n])$, is called the *Kan classifying simplicial set* of G.

(4.21) Proposition. We let G be a simplicial group. The faces and degeneracies of its Kan classifying simplicial set $\overline{\mathrm{W}}G$ are given by

$$(g_j)_{j \in \lfloor n-1, 0 \rfloor} \mathrm{d}_k = \begin{cases} (g_{j+1} \mathrm{d}_0)_{j \in \lfloor n-2, 0 \rfloor} & \text{if } k = 0, \\ (g_{j+1} \mathrm{d}_k)_{j \in \lfloor n-2, k \rfloor} \cup ((g_k \mathrm{d}_k) g_{k-1}) \cup (g_j)_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (g_j)_{j \in \lfloor n-2, 0 \rfloor} & \text{if } k = n, \end{cases}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, $k \in [0,n]$, $n \in \mathbb{N}$, and

$$(g_j)_{j \in \lfloor n-1,0 \rfloor} \mathrm{s}_k = (g_{j-1} \mathrm{s}_k)_{j \in \lfloor n,k+1 \rfloor} \cup (1) \cup (g_j)_{j \in \lfloor k-1,0 \rfloor}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, $k \in [0,n]$, $n \in \mathbb{N}_0$.

Proof. We have

$$\begin{aligned}
((g_j)_{j \in \lfloor n-1,0 \rfloor} \mathrm{d}_k)_i &= ((g_j)_{j \in \lfloor n-1,0 \rfloor} \overline{\mathrm{W}}_{\delta^k} G)_i = \prod_{j \in \lfloor (i+1)\delta^k - 1, i\delta^k \rfloor} g_j G_{\delta^k|^{[j]}_{[i]}} \\
&= \begin{cases} \prod_{j \in \lfloor (i+1)\delta^k - 1, i\delta^k \rfloor} g_j G_{\delta^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor n-2,k \rfloor, \\ \prod_{j \in \lfloor k\delta^k - 1, (k-1)\delta^k \rfloor} g_j G_{\delta^k|^{[j]}_{[k-1]}} & \text{for } i = k-1, \\ \prod_{j \in \lfloor (i+1)\delta^k - 1, i\delta^k \rfloor} g_j G_{\delta^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor k-2,0 \rfloor \end{cases} \\
&= \begin{cases} \prod_{j \in \lfloor i+2-1, i+1 \rfloor} g_j G_{\delta^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor n-2,k \rfloor, \\ \prod_{j \in \lfloor k+1-1, k-1 \rfloor} g_j G_{\delta^k|^{[j]}_{[k-1]}} & \text{for } i = k-1, \\ \prod_{j \in \lfloor i+1-1, i \rfloor} g_j G_{\delta^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor k-2,0 \rfloor \end{cases} \\
&= \begin{cases} g_{i+1} G_{\delta^k|^{[i+1]}_{[i]}} & \text{for } i \in \lfloor n-2,k \rfloor, \\ (g_k G_{\delta^k|^{[k]}_{[k-1]}})(g_{k-1} G_{\delta^k|^{[k-1]}_{[k-1]}}) & \text{for } i = k-1, \\ g_i G_{\delta^k|^{[i]}_{[i]}} & \text{for } i \in \lfloor k-2,0 \rfloor \end{cases} \\
&= \begin{cases} g_{i+1} G_{\delta^k} & \text{for } i \in \lfloor n-2,k \rfloor, \\ (g_k G_{\delta^k})(g_{k-1} G_{\mathrm{id}_{[k-1]}}) & \text{for } i = k-1, \\ g_i G_{\mathrm{id}_{[i]}} & \text{for } i \in \lfloor k-2,0 \rfloor \end{cases} = \begin{cases} g_{i+1} \mathrm{d}_k & \text{for } i \in \lfloor n-2,k \rfloor, \\ (g_k \mathrm{d}_k) g_{k-1} & \text{for } i = k-1, \\ g_i & \text{for } i \in \lfloor k-2,0 \rfloor \end{cases}
\end{aligned}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, $i \in \lfloor n-2,0 \rfloor$, $k \in [0,n]$, $n \in \mathbb{N}$, and

$$\begin{aligned}
((g_j)_{j \in \lfloor n-1,0 \rfloor} \mathrm{s}_k)_i &= ((g_j)_{j \in \lfloor n-1,0 \rfloor} \overline{\mathrm{W}}_{\sigma^k} G)_i = \prod_{j \in \lfloor (i+1)\sigma^k - 1, i\sigma^k \rfloor} g_j G_{\sigma^k|^{[j]}_{[i]}} \\
&= \begin{cases} \prod_{j \in \lfloor (i+1)\sigma^k - 1, i\sigma^k \rfloor} g_j G_{\sigma^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor n,k+1 \rfloor, \\ \prod_{j \in \lfloor (k+1)\sigma^k - 1, k\sigma^k \rfloor} g_j G_{\sigma^k|^{[j]}_{[k]}} & \text{for } i = k, \\ \prod_{j \in \lfloor (i+1)\sigma^k - 1, i\sigma^k \rfloor} g_j G_{\sigma^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor k-1,0 \rfloor \end{cases} \\
&= \begin{cases} \prod_{j \in \lfloor i+1-1-1, i-1 \rfloor} g_j G_{\sigma^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor n,k+1 \rfloor \\ \prod_{j \in \lfloor k+1-1-1, k \rfloor} g_j G_{\sigma^k|^{[j]}_{[k]}} & \text{for } i = k, \\ \prod_{j \in \lfloor i+1-1, i \rfloor} g_j G_{\sigma^k|^{[j]}_{[i]}} & \text{for } i \in \lfloor k-1,0 \rfloor \end{cases} \\
&= \begin{cases} g_{i-1} G_{\sigma^k|^{[i-1]}_{[i]}} & \text{for } i \in \lfloor n,k+1 \rfloor, \\ 1 & \text{for } i = k, \\ g_i G_{\sigma^k|^{[i]}_{[i]}} & \text{for } i \in \lfloor k-1,0 \rfloor \end{cases} = \begin{cases} g_{i-1} G_{\sigma^k} & \text{for } i \in \lfloor n,k+1 \rfloor, \\ 1 & \text{for } i = k, \\ g_i G_{\mathrm{id}_{[i]}} & \text{for } i \in \lfloor k-1,0 \rfloor \end{cases} \\
&= \begin{cases} g_{i-1} \mathrm{s}_k & \text{for } i \in \lfloor n,k+1 \rfloor, \\ 1 & \text{for } i = k, \\ g_i & \text{for } i \in \lfloor k-1,0 \rfloor \end{cases}
\end{aligned}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, $i \in \lfloor n,0 \rfloor$, $k \in [0,n]$, $n \in \mathbb{N}_0$. □

(4.22) Example. For a group G, we have $\overline{\mathrm{W}} \operatorname{Cosk} G = \mathrm{B}G$ and hence

$$\mathrm{H}_n(\overline{\mathrm{W}} \operatorname{Cosk} G, M; R) = \mathrm{H}_n(\mathrm{B}G, M; R) \cong \mathrm{H}_n(G, M; R)$$

resp.

$$\mathrm{H}^n(\overline{\mathrm{W}}\,\mathrm{Cosk}\,G, M; R) = \mathrm{H}^n(\mathrm{B}G, M; R) \cong \mathrm{H}^n(G, M; R)$$

for $n \in \mathbb{N}_0$ and a module M over a commutative ring R.

Now we want to construct a left adjoint for $\overline{\mathrm{W}}$.

(4.23) Remark. We let X be a reduced simplicial set and we let $X \longrightarrow \overline{X}$ be a simplicial bijection given by $X_n \to \overline{X}_n, x_n \mapsto \overline{x}_n$ for all $n \in \mathbb{N}_0$. For every $n \in \mathbb{N}_0$, we let $\mathrm{G}_n X$ be the free group

$$\mathrm{G}_n X := \langle \overline{x_{n+1}} \mid x_{n+1} \in X_{n+1}, \overline{x_n \mathrm{s}_n} = 1 \text{ for all } x_n \in X_n \rangle_{\mathbf{Grp}} \cong \langle \overline{x_{n+1}} \mid x_{n+1} \in X_{n+1} \setminus (\mathrm{Im}\,\mathrm{s}_n) \rangle_{\mathbf{Grp}}.$$

Further, for $\theta \in \mathbf{\Delta}([m],[n])$, $m, n \in \mathbb{N}_0$, we define a group homomorphism $\mathrm{G}_\theta X \colon \mathrm{G}_n X \to \mathrm{G}_m X$ on the generating set $\overline{X}_{n+1}$ of $\mathrm{G}_n X$ by setting

$$\overline{x_{n+1}}(\mathrm{G}_\theta X) := \overline{x_{n+1}(\mathrm{P}_\theta X)}\,(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} \text{ for all } x_{n+1} \in X_{n+1}.$$

Then $\mathrm{G}X$ is a simplicial group.

Proof. First of all, we note that if $x_{n+1} = x_n \mathrm{s}_n$ for some $x_n \in X_n$, then

$$\overline{x_{n+1}(\mathrm{P}_\theta X)}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} = \overline{x_n\mathrm{s}_n(\mathrm{P}_\theta X)}(\overline{x_n\mathrm{s}_n\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} = \overline{x_n\mathrm{s}_n(\mathrm{P}_\theta X)}(\overline{x_n\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} = 1$$

for $\theta \in \mathbf{\Delta}([m],[n])$, $m, n \in \mathbb{N}_0$. Now, given morphisms $\theta \in \mathbf{\Delta}([m],[n])$, $\rho \in \mathbf{\Delta}([n],[p])$ for $m, n, p \in \mathbb{N}_0$, we note that $(\mathrm{P}_\rho X)\mathrm{d}_{n+1} = \mathrm{d}_{p+1} X_\rho$ by proposition (2.31)(c) and obtain thus

$$\begin{aligned}
\overline{x_{p+1}}(\mathrm{G}_\rho X)(\mathrm{G}_\theta X) &= (\overline{x_{p+1}(\mathrm{P}_\rho X)}(\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_\rho X)})^{-1})(\mathrm{G}_\theta X) \\
&= \overline{x_{p+1}(\mathrm{P}_\rho X)}(\mathrm{G}_\theta X)(\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_\rho X)}(\mathrm{G}_\theta X))^{-1} \\
&= \overline{x_{p+1}(\mathrm{P}_\rho X)(\mathrm{P}_\theta X)}(\overline{x_{p+1}(\mathrm{P}_\rho X)\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} \\
&\quad \cdot (\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_\rho X)(\mathrm{P}_\theta X)}(\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_\rho X)\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1})^{-1} \\
&= \overline{x_{p+1}(\mathrm{P}_\rho X)(\mathrm{P}_\theta X)}(\overline{x_{p+1}(\mathrm{P}_\rho X)\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} \\
&\quad \cdot \overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_\rho X)\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)}(\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_\rho X)(\mathrm{P}_\theta X)})^{-1} \\
&= \overline{x_{p+1}(\mathrm{P}_{\theta\rho} X)}(\overline{x_{p+1}\mathrm{d}_{p+1}X_\rho\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1} \\
&\quad \cdot \overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p\mathrm{d}_{p+1}X_\rho\mathrm{s}_n(\mathrm{P}_\theta X)}(\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_{\theta\rho} X)})^{-1} \\
&= \overline{x_{p+1}(\mathrm{P}_{\theta\rho} X)}(\overline{x_{p+1}\mathrm{d}_{p+1}\mathrm{s}_p(\mathrm{P}_{\theta\rho} X)})^{-1} = \overline{x_{p+1}}(\mathrm{G}_{\theta\rho} X)
\end{aligned}$$

for $x_{p+1} \in X_{p+1}$ as well as

$$\overline{x_{n+1}}(\mathrm{G}_{\mathrm{id}_{[n]}} X) = \overline{x_{n+1}(\mathrm{P}_{\mathrm{id}_{[n]}} X)}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_{\mathrm{id}_{[n]}} X)})^{-1} = \overline{x_{n+1}\mathrm{id}_{\mathrm{P}_n X}}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n\mathrm{id}_{\mathrm{P}_n X}})^{-1} = \overline{x_{n+1}}$$

for $x_{n+1} \in X_{n+1}$. Hence $\mathrm{G}X$ is a simplicial group. □

(4.24) Definition (Kan loop group)**.** We let X be a reduced simplicial set and we let $X \longrightarrow \overline{X}$ be a simplicial bijection given by $X_n \to \overline{X}_n, x_n \mapsto \overline{x}_n$ for all $n \in \mathbb{N}_0$. The simplicial group $\mathrm{G}X$ given as in remark (4.23) by

$$\mathrm{G}_n X = \langle \overline{x_{n+1}} \mid x_{n+1} \in X_{n+1}, \overline{x_n \mathrm{s}_n} = 1 \text{ for all } x_n \in X_n \rangle_{\mathbf{Grp}}$$

and

$$\mathrm{G}_\theta X \colon \mathrm{G}_n X \to \mathrm{G}_m X, \overline{x_{n+1}} \mapsto \overline{x_{n+1}(\mathrm{P}_\theta X)}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1}$$

for $\theta \in \mathbf{\Delta}([m],[n])$, $m, n \in \mathbb{N}_0$, is called the *(Kan) loop group* of X.

In the following, we always assume given a simplicial bijection $X \longmapsto \overline{X}$ when referring to the construction $\mathrm{G}X$.

(4.25) Proposition. We let X be a reduced simplicial set. The faces $\mathrm{d}_k \colon \mathrm{G}_n X \to \mathrm{G}_{n-1} X$ for $k \in [0, n]$, $n \in \mathbb{N}$, and the degeneracies $\mathrm{s}_k \colon \mathrm{G}_n X \to \mathrm{G}_{n+1} X$ for $k \in [0, n]$, $n \in \mathbb{N}_0$, in the loop group $\mathrm{G}X$ are given by

$$\overline{x_{n+1}}\mathrm{d}_k = \begin{cases} \overline{x_{n+1}\mathrm{d}_k} & \text{for } k \in [0, n-1], \\ \overline{x_{n+1}\mathrm{d}_n}(\overline{x_{n+1}\mathrm{d}_{n+1}})^{-1} & \text{for } k = n \end{cases}$$

and

$$\overline{x_{n+1}}\mathrm{s}_k = \overline{x_{n+1}\mathrm{s}_k} \text{ for } k \in [0, n],$$

where $x_{n+1} \in X_{n+1}$.

Proof. We have

$$\begin{aligned} \overline{x_{n+1}}\mathrm{d}_k &= \overline{x_{n+1}}(\mathrm{G}_{\delta^k} X) = \overline{x_{n+1}\mathrm{P}_{\delta^k} X}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n\mathrm{P}_{\delta^k} X})^{-1} = \overline{x_{n+1}\mathrm{d}_k}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n\mathrm{d}_k})^{-1} \\ &= \left.\begin{cases} \overline{x_{n+1}\mathrm{d}_k}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{d}_k\mathrm{s}_{n-1}})^{-1} & \text{for } k \in [0, n-1], \\ \overline{x_{n+1}\mathrm{d}_n}(\overline{x_{n+1}\mathrm{d}_{n+1}})^{-1} & \text{for } k = n \end{cases}\right\} \\ &= \left.\begin{cases} \overline{x_{n+1}\mathrm{d}_k} & \text{for } k \in [0, n-1], \\ \overline{x_{n+1}\mathrm{d}_n}(\overline{x_{n+1}\mathrm{d}_{n+1}})^{-1} & \text{for } k = n \end{cases}\right\} \end{aligned}$$

for $x_{n+1} \in X_{n+1}$, $n \in \mathbb{N}$, and

$$\begin{aligned} \overline{x_{n+1}}\mathrm{s}_k &= \overline{x_{n+1}}(\mathrm{G}_{\sigma^k} X) = \overline{x_{n+1}\mathrm{P}_{\sigma^k} X}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n\mathrm{P}_{\sigma^k} X})^{-1} = \overline{x_{n+1}\mathrm{s}_k}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n\mathrm{s}_k})^{-1} \\ &= \overline{x_{n+1}\mathrm{s}_k}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_k\mathrm{s}_{n+1}})^{-1} = \overline{x_{n+1}\mathrm{s}_k} \end{aligned}$$

for $x_{n+1} \in X_{n+1}$, $n \in \mathbb{N}_0$. □

(4.26) Definition (simplicial free group)**.** A simplicial group F is called a *simplicial free group*, if F_n is a free group with a free generating system $X_n \subseteq F_n$ for every $n \in \mathbb{N}_0$ and $X_n\mathrm{s}_k \subseteq X_{n+1}$ for every $n \in \mathbb{N}_0$, $k \in [0, n]$.

(4.27) Proposition. The Kan loop group $\mathrm{G}X$ of a reduced simplicial set X is a simplicial free group.

Proof. This follows from proposition (4.25). □

(4.28) Proposition.

(a) We let $X \xrightarrow{f} Y$ be a simplicial map between reduced simplicial sets X and Y. Then we have an induced morphism

$$\mathrm{G}X \xrightarrow{\mathrm{G}f} \mathrm{G}Y,$$

given by

$$\overline{x_{n+1}}(\mathrm{G}_n f) = \overline{x_{n+1}(\mathrm{P}_n f)} = \overline{x_{n+1} f_{n+1}} \text{ for } x_{n+1} \in X_{n+1}, n \in \mathbb{N}_0.$$

(b) The construction in (a) yields a functor

$$\mathbf{sSet}_0 \xrightarrow{\mathrm{G}} \mathbf{sGrp}.$$

Proof.

(a) We have

$$\begin{aligned} \overline{x_{n+1}}(\mathrm{G}_\theta X)(\mathrm{G}_m f) &= (\overline{x_{n+1}(\mathrm{P}_\theta X)}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1})(\mathrm{G}_m f) \\ &= (\overline{x_{n+1}(\mathrm{P}_\theta X)}(\mathrm{G}_m f))(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)}(\mathrm{G}_m f))^{-1} \\ &= \overline{x_{n+1}(\mathrm{P}_\theta X)(\mathrm{P}_m f)}(\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)(\mathrm{P}_m f)})^{-1} \\ &= \overline{x_{n+1}(\mathrm{P}_n f)(\mathrm{P}_\theta Y)}(\overline{x_{n+1}(\mathrm{P}_n f)\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta Y)})^{-1} \\ &= \overline{x_{n+1}(\mathrm{P}_n f)}(\mathrm{G}_\theta Y) = \overline{x_{n+1}}(\mathrm{G}_n f)(\mathrm{G}_\theta Y), \end{aligned}$$

for $x_{n+1} \in X_{n+1}$ and thus $(\mathrm{G}_\theta X)(\mathrm{G}_m f) = (\mathrm{G}_n f)(\mathrm{G}_\theta Y)$ for $\theta \in \mathbf{\Delta}([m], [n])$. Hence $\mathrm{G}f$ is a simplicial group homomorphism $\mathrm{G}X \longrightarrow \mathrm{G}Y$.

(b) We let X, Y, Z be reduced simplicial sets and $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ be simplicial maps. Then

$$\overline{x_{n+1}}(\mathrm{G}_n f)(\mathrm{G}_n g) = \overline{x_{n+1}(\mathrm{P}_n f)}(\mathrm{G}_n g) = \overline{x_{n+1}(\mathrm{P}_n f)(\mathrm{P}_n g)} = \overline{x_{n+1}(\mathrm{P}_n (fg))} = \overline{x_{n+1}}(\mathrm{G}_n (fg))$$

and

$$\overline{x_{n+1}}(\mathrm{G}_n \mathrm{id}_X) = \overline{x_{n+1}(\mathrm{P}_n \mathrm{id}_X)} = \overline{x_{n+1} \mathrm{id}_{\mathrm{P}_n X}} = \overline{x_{n+1}}$$

for all $x_{n+1} \in X_{n+1}$, $n \in \mathbb{N}_0$. Thus $(\mathrm{G}f)(\mathrm{G}g) = \mathrm{G}(fg)$ and $\mathrm{G}\mathrm{id}_X = \mathrm{id}_{\mathrm{G}X}$, and hence G is a functor. □

(4.29) Theorem (cf. [20, Proposition 10.5])**.** The functor $\mathbf{sSet}_0 \xrightarrow{\mathrm{G}} \mathbf{sGrp}$ is left adjoint to $\mathbf{sGrp} \xrightarrow{\overline{\mathrm{W}}} \mathbf{sSet}_0$.

Proof. We let $X \in \operatorname{Ob} \mathbf{sSet}_0$ be a reduced simplicial set, $H \in \operatorname{Ob} \mathbf{sGrp}$ a simplicial group and $\mathrm{G}X \xrightarrow{\varphi} H$ a simplicial group homomorphism. We define $(\varphi\Phi_{X,H})_n \colon X_n \to \overline{\mathrm{W}}_n H, x_n \mapsto (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}}\varphi_j)_{j \in \lfloor n-1,0 \rfloor}$ for $n \in \mathbb{N}_0$. We suppose given $\theta \in \mathbf{\Delta}([m],[n])$ for $m, n \in \mathbb{N}_0$. Then

$$\begin{aligned}
x_n(\varphi\Phi_{X,H})_n(\overline{\mathrm{W}}_\theta H) &= (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}}\varphi_j)_{j \in \lfloor n-1,0 \rfloor}(\overline{\mathrm{W}}_\theta H) = (\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}}\varphi_j H_{\theta|^{[j]}_{[i]}}))_{i \in \lfloor m-1,0 \rfloor} \\
&= (\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}}(\mathrm{G}_{\theta|^{[j]}_{[i]}} X)\varphi_i))_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}}(\mathrm{G}_{\theta|^{[j]}_{[i]}} X)))\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}(\mathrm{P}_{\theta|^{[j]}_{[i]}} X)}(\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor}\mathrm{d}_{j+1}\mathrm{s}_j(\mathrm{P}_{\theta|^{[j]}_{[i]}} X)})^{-1}))\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,j+1 \rfloor} X_{\sigma^j} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})}})^{-1}))\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,j+1 \rfloor} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})\sigma^j}})^{-1}))\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= (((\prod_{j \in \lfloor (i+1)\theta-1, i\theta+1 \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,j+1 \rfloor} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})\sigma^j}})^{-1})) \\
&\qquad \cdot (\overline{x_n \mathrm{d}_{\lfloor n,i\theta+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[i\theta]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,i\theta+1 \rfloor} X_{\mathrm{Sh}(\theta|^{[i\theta]}_{[i]})\sigma^{i\theta}}})^{-1}))\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= (((\prod_{j \in \lfloor (i+1)\theta-1, i\theta+1 \rfloor} (\overline{x_n \mathrm{d}_{\lfloor n,j+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[j]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,j+1 \rfloor} X_{\mathrm{Sh}(\theta|^{[j-1]}_{[i]})}})^{-1})) \\
&\qquad \cdot (\overline{x_n \mathrm{d}_{\lfloor n,i\theta+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[i\theta]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,i\theta+1 \rfloor} X_{\mathrm{Sh}(\theta|^{[i\theta]}_{[i]})\sigma^{i\theta}}})^{-1}))\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\overline{x_n \mathrm{d}_{\lfloor n,(i+1)\theta-1+2 \rfloor} X_{\mathrm{Sh}(\theta|^{[(i+1)\theta-1]}_{[i]})}}(\overline{x_n \mathrm{d}_{\lfloor n,i\theta+1 \rfloor} X_{\mathrm{Sh}(\theta|^{[i\theta]}_{[i]})\sigma^{i\theta}}})^{-1})\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\overline{x_n X_{\delta\lceil (i+1)\theta+1,n \rceil} X_{\theta|^{[(i+1)\theta]}_{[i+1]}}}(\overline{x_n \mathrm{d}_{\lfloor n,i\theta+1 \rfloor} X_{\sigma^i(\theta|^{[i\theta]}_{[i]})}})^{-1})\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\overline{x_n X_{\theta|^{[(i+1)\theta]}_{[i+1]}\delta\lceil (i+1)\theta+1,n \rceil}}(\overline{x_n \mathrm{d}_{\lfloor n,i\theta+1 \rfloor} X_{\theta|^{[i\theta]}_{[i]}} X_{\sigma^i}})^{-1})\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\overline{x_n X_{\theta|^{[(i+1)\theta]}_{[i+1]}\delta\lceil (i+1)\theta+1,n \rceil}}(\overline{x_n \mathrm{d}_{\lfloor n,i\theta+1 \rfloor}(X_{\theta|^{[i\theta]}_{[i]}})\mathrm{s}_i})^{-1})\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= ((\overline{x_n X_{\theta|^{[(i+1)\theta]}_{[i+1]}\delta\lceil (i+1)\theta+1,n \rceil}})\varphi_i)_{i \in \lfloor m-1,0 \rfloor} = (\overline{x_n X_{\delta\lceil i+2,m \rceil \theta}}\varphi_i)_{i \in \lfloor m-1,0 \rfloor} \\
&= (\overline{x_n X_\theta X_{\delta\lceil i+2,m \rceil}}\varphi_i)_{i \in \lfloor m-1,0 \rfloor} = (\overline{x_n X_\theta \mathrm{d}_{\lfloor m,i+2 \rfloor}}\varphi_i)_{i \in \lfloor m-1,0 \rfloor} = x_n X_\theta(\varphi\Phi_{X,H})_m
\end{aligned}$$

for all $x_n \in X_n$, that is, the diagram

$$\begin{array}{ccc}
X_n & \xrightarrow{X_\theta} & X_m \\
{\scriptstyle (\varphi\Phi_{X,H})_n}\downarrow & & \downarrow{\scriptstyle (\varphi\Phi_{X,H})_m} \\
\overline{\mathrm{W}}_n H & \xrightarrow{\overline{\mathrm{W}}_\theta H} & \overline{\mathrm{W}}_m H
\end{array}$$

commutes. Thus the maps $(\varphi\Phi_{X,H})_n$ for $n \in \mathbb{N}_0$ yield a simplicial map

$$X \xrightarrow{\varphi\Phi_{X,H}} \overline{\mathrm{W}}H.$$

Since $\varphi \in {}_{\mathbf{sGrp}}(\mathrm{G}X, H)$ was arbitrary, we have a well-defined map

$$\Phi_{X,H} \colon {}_{\mathbf{sGrp}}(\mathrm{G}X, H) \to {}_{\mathbf{sSet}_0}(X, \overline{\mathrm{W}}H).$$

We claim that the maps $\Phi_{X,H}$ for $X \in \operatorname{Ob}\mathbf{sSet}_0$, $H \in \operatorname{Ob}\mathbf{sGrp}$, yield a natural transformation

$${}_{\mathbf{sGrp}}(\mathrm{G}-, =) \longrightarrow {}_{\mathbf{sSet}_0}(-, \overline{\mathrm{W}}=).$$

Indeed, given reduced simplicial sets $X, Y \in \operatorname{Ob}\mathbf{sSet}_0$, simplicial groups $H, K \in \operatorname{Ob}\mathbf{sGrp}$, a simplicial map $Y \xrightarrow{e} X$ and a simplicial group homomorphism $H \xrightarrow{\psi} K$, we have

$$\begin{aligned} y_n e_n (\varphi\Phi_{X,H})_n (\overline{\mathrm{W}}_n \psi) &= (\overline{y_n e_n \mathrm{d}_{\lfloor n,j+2\rfloor} \varphi_j})_{j \in \lfloor n-1,0\rfloor} (\overline{\mathrm{W}}_n \psi) = (\overline{y_n e_n \mathrm{d}_{\lfloor n,j+2\rfloor} \varphi_j \psi_j})_{j \in \lfloor n-1,0\rfloor} \\ &= (\overline{y_n \mathrm{d}_{\lfloor n,j+2\rfloor} e_{j+1} \varphi_j \psi_j})_{j \in \lfloor n-1,0\rfloor} = (\overline{y_n \mathrm{d}_{\lfloor n,j+2\rfloor}} (\mathrm{G}_j e) \varphi_j \psi_j)_{j \in \lfloor n-1,0\rfloor} \\ &= (\overline{y_n \mathrm{d}_{\lfloor n,j+2\rfloor}} ((\mathrm{G}e)\varphi\psi)_j)_{j \in \lfloor n-1,0\rfloor} = y_n (((\mathrm{G}e)\varphi\psi)\Phi_{Y,K})_n \end{aligned}$$

for $y_n \in Y_n$, $n \in \mathbb{N}_0$. Hence the diagram

$$\begin{array}{ccc} {}_{\mathbf{sGrp}}(\mathrm{G}X, H) & \xrightarrow{\Phi_{X,H}} & {}_{\mathbf{sSet}_0}(X, \overline{\mathrm{W}}H) \\ \downarrow {\scriptstyle (\mathrm{G}e)(-)\psi} & & \downarrow {\scriptstyle e(-)(\overline{\mathrm{W}}\psi)} \\ {}_{\mathbf{sGrp}}(\mathrm{G}Y, K) & \xrightarrow{\Phi_{Y,K}} & {}_{\mathbf{sSet}_0}(X, \overline{\mathrm{W}}K) \end{array}$$

commutes, and we have a natural transformation

$${}_{\mathbf{sGrp}}(\mathrm{G}-, =) \xrightarrow{\Phi} {}_{\mathbf{sSet}_0}(-, \overline{\mathrm{W}}=).$$

It remains to show that Φ is an isomorphism. To this end, for a given reduced simplicial set X, a given simplicial group H and a given simplicial map $X \xrightarrow{f} \overline{\mathrm{W}}H$, we define a group homomorphisms $(f\Psi_{X,H})_n \colon \mathrm{G}_n X \to H$ by $\overline{x_{n+1}}(f\Psi_{X,H})_n := (x_{n+1} f_{n+1})_n$ for $x_{n+1} \in X_{n+1}$, $n \in \mathbb{N}_0$. Given $\theta \in {}_{\mathbf{\Delta}}([m],[n])$ for $m, n \in \mathbb{N}_0$, we obtain

$$\begin{aligned} \overline{x_{n+1}}(\mathrm{G}_\theta X)(f\Psi_{X,H})_m &= (\overline{x_{n+1}(\mathrm{P}_\theta X)} (\overline{x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n(\mathrm{P}_\theta X)})^{-1})(f\Psi_{X,H})_m \\ &= (x_{n+1} X_{\mathrm{Sh}\theta} f_{m+1})_m (x_{n+1}\mathrm{d}_{n+1}\mathrm{s}_n X_{\mathrm{Sh}\theta} f_{m+1})_m^{-1} \\ &= (x_{n+1} f_{n+1} (\overline{\mathrm{W}}_{\mathrm{Sh}\theta} H))_m (x_{n+1} f_{n+1} \mathrm{d}_{n+1}\mathrm{s}_n (\overline{\mathrm{W}}_{\mathrm{Sh}\theta} H))_m^{-1} \\ &= \Big(\prod_{j \in \lfloor (m+1)(\mathrm{Sh}\theta)-1, m(\mathrm{Sh}\theta)\rfloor} (x_{n+1} f_{n+1})_j H_{(\mathrm{Sh}\theta)|_{[m]}^{[j]}}\Big) \\ &\quad \cdot \Big(\prod_{j \in \lfloor (m+1)(\mathrm{Sh}\theta)-1, m(\mathrm{Sh}\theta)\rfloor} (x_{n+1} f_{n+1} \mathrm{d}_{n+1}\mathrm{s}_n)_j H_{(\mathrm{Sh}\theta)|_{[m]}^{[j]}}\Big)^{-1} \\ &= \Big(\prod_{j \in \lfloor n, m\theta\rfloor} (x_{n+1} f_{n+1})_j H_{\theta|_{[m]}^{[j]}}\Big)\Big(\prod_{j \in \lfloor n, m\theta\rfloor} (x_{n+1} f_{n+1} \mathrm{d}_{n+1}\mathrm{s}_n)_j H_{\theta|_{[m]}^{[j]}}\Big)^{-1} \\ &= \Big(\prod_{j \in \lfloor n, m\theta\rfloor} (x_{n+1} f_{n+1})_j H_{\theta|^{[j]}}\Big)\Big(\prod_{j \in \lfloor n-1, m\theta\rfloor} (x_{n+1} f_{n+1})_j H_{\theta|^{[j]}}\Big)^{-1} \\ &= \prod_{j \in \lfloor n, m\theta\rfloor} (x_{n+1} f_{n+1})_j H_{\theta|^{[j]}} \prod_{j \in \lfloor m\theta, n-1\rfloor} (x_{n+1} f_{n+1})_j^{-1} H_{\theta|^{[j]}} \\ &= (x_{n+1} f_{n+1})_n H_{\theta|^{[n]}} = (x_{n+1} f_{n+1})_n H_\theta = \overline{x_{n+1}}(f\Psi_{X,H})_n H_\theta \end{aligned}$$

for $x_{n+1} \in X_{n+1}$, that is, the diagram

$$\begin{array}{ccc} \mathrm{G}_n X & \xrightarrow{\mathrm{G}_\theta X} & \mathrm{G}_m X \\ \downarrow {\scriptstyle (f\Psi_{X,H})_n} & & \downarrow {\scriptstyle (f\Psi_{X,H})_m} \\ H_n & \xrightarrow{H_\theta} & H_m \end{array}$$

commutes. Hence the group homomorphisms $(f\Psi_{X,H})_n$ for $n \in \mathbb{N}_0$ yield a simplicial group homomorphism

$$f\Psi_{X,H} \colon \mathrm{G}X \to H,$$

and since $f \in {}_{\mathbf{sSet}_0}(X, \overline{\mathrm{W}}H)$ was arbitrary, we have a well-defined map

$$\Psi_{X,H} \colon {}_{\mathbf{sSet}_0}(X, \overline{\mathrm{W}}H) \to {}_{\mathbf{sGrp}}(\mathrm{G}X, H).$$

We have to show that $\Phi_{X,H}$ and $\Psi_{X,H}$ are mutually inverse maps for each reduced simplicial set $X \in \operatorname{Ob} \mathbf{sSet}_0$ and each simplicial group $H \in \operatorname{Ob} \mathbf{sGrp}$. Indeed, it holds that

$$\overline{x_{n+1}}(\varphi\Phi_{X,H}\Psi_{X,H})_n = (x_{n+1}(\varphi\Phi_{X,H})_{n+1})_n = \overline{x_{n+1}\mathrm{d}_{\lfloor n+1,n+2\rfloor}}\varphi_n = \overline{x_{n+1}}\varphi_n$$

for all $x_{n+1} \in X_{n+1}$, $n \in \mathbb{N}_0$, that is, $\varphi\Phi_{X,H}\Psi_{X,H} = \varphi$ for every simplicial group homomorphism $\varphi \in {}_{\mathbf{sGrp}}(\mathrm{G}X, H)$. Moreover,

$$\begin{aligned} x_n(f\Psi_{X,H}\Phi_{X,H})_n &= (\overline{x_n\mathrm{d}_{\lfloor n,j+2\rfloor}}(f\Psi_{X,H})_j)_{j\in\lfloor n-1,0\rfloor} = ((x_n\mathrm{d}_{\lfloor n,j+2\rfloor}f_{j+1})_j)_{j\in\lfloor n-1,0\rfloor} \\ &= ((x_nf_n\mathrm{d}_{\lfloor n,j+2\rfloor})_j)_{j\in\lfloor n-1,0\rfloor} = x_nf_n \end{aligned}$$

for all $x_n \in X_n$, $n \in \mathbb{N}_0$, that is, $f\Psi_{X,H}\Phi_{X,H} = f$ for every simplicial map $f \in {}_{\mathbf{sSet}_0}(X, \overline{\mathrm{W}}H)$. This implies

$$\Phi_{X,H}\Psi_{X,H} = \mathrm{id}_{{}_{\mathbf{sGrp}}(\mathrm{G}X,H)} \text{ and } \Psi_{X,H}\Phi_{X,H} = \mathrm{id}_{{}_{\mathbf{sSet}_0}(X,\overline{\mathrm{W}}H)} \text{ for all } X \in \operatorname{Ob}\mathbf{sSet}_0, H \in \operatorname{Ob}\mathbf{sGrp},$$

and hence the maps $\Psi_{X,H}$ for $X \in \operatorname{Ob} \mathbf{sSet}_0$, $H \in \operatorname{Ob} \mathbf{sGrp}$ yield a natural transformation

$${}_{\mathbf{sSet}_0}(-, \overline{\mathrm{W}}=) \xrightarrow{\Psi} {}_{\mathbf{sGrp}}(\mathrm{G}-, =)$$

inverse to Φ. Thus ${}_{\mathbf{sGrp}}(\mathrm{G}-, =) \cong {}_{\mathbf{sSet}_0}(-, \overline{\mathrm{W}}=)$, that is, $\mathrm{G} \dashv \overline{\mathrm{W}}$. □

§5 The classifying simplicial set of a simplicial group

Here we introduce the second possible notation for a classifying simplicial set of a simplicial group, namely the diagonal of its nerve. It will be shown that the Kan classifying simplicial set and the diagonal nerve construction are simplicially homotopy equivalent.

(4.30) Proposition (diagonal of the nerve of a simplicial group)**.** The diagonal of the nerve of a given simplicial group G is a simplicial set with $\operatorname{Diag}_n \mathrm{N}G = G_n^{\times n}$ and where the faces and degeneracies are given by

$$\mathrm{d}_k = \bigtimes_{i\in\lfloor n-1,k+1\rfloor} \mathrm{d}_k \times (\mathrm{d}_k \times \mathrm{d}_k)\mathrm{m} \times \bigtimes_{i\in\lfloor k-2,0\rfloor} \mathrm{d}_k \text{ for all } k \in [0,n], n \in \mathbb{N},$$

and

$$\mathrm{s}_k = \bigtimes_{i\in\lfloor n-1,k\rfloor} \mathrm{s}_k \times \mathrm{n} \times \bigtimes_{i\in\lfloor k-1,0\rfloor} \mathrm{s}_k \text{ for all } k \in [0,n], n \in \mathbb{N}_0.$$

Proof. Follows from proposition (1.36) and proposition (3.6). □

(4.31) Proposition.

(a) We have a natural transformation

$$\operatorname{Diag} \circ \mathrm{N} \xrightarrow{D} \overline{\mathrm{W}}$$

given by $(D_G)_n = \bigtimes_{i\in\lfloor n-1,0\rfloor} \mathrm{d}_{\lfloor n,i+1\rfloor} \colon G_n^{\times n} \to \bigtimes_{i\in\lfloor n-1,0\rfloor} G_i$ for all $n \in \mathbb{N}_0$, $G \in \operatorname{Ob} \mathbf{sGrp}$.

(b) The natural transformation D is a retraction. A corresponding coretraction is given by

$$\overline{\mathrm{W}} \xrightarrow{S} \operatorname{Diag} \circ \mathrm{N},$$

where S_G is recursively given by

$$(S_G)_n\colon \bigtimes_{i\in\lfloor n-1,0\rfloor} G_i \to G_n^{\times n},\, (g_i)_{i\in\lfloor n-1,0\rfloor} \mapsto (y_i)_{i\in\lfloor n-1,0\rfloor}$$

with

$$y_i := \prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-1\rceil}) \in G_n$$

for each $i\in\lfloor n-1,0\rfloor$, $n\in\mathbb{N}_0$, $G\in\operatorname{Ob}\mathbf{sGrp}$.

Proof.

(a) First, we have to show that for a simplicial group G the maps $(D_G)_n$ for $n\in\mathbb{N}_0$ are compatible with the faces and degeneracies of G. In fact, we obtain

$$\begin{aligned}
\mathrm{d}_k(D_G)_{n-1} &= (\bigtimes_{i\in\lfloor n-1,k+1\rfloor}\mathrm{d}_k \times (\mathrm{d}_k\times\mathrm{d}_k)\mathrm{m}\times \bigtimes_{i\in\lfloor k-2,0\rfloor}\mathrm{d}_k)(\bigtimes_{i\in\lfloor n-2,0\rfloor}\mathrm{d}_{\lfloor n-1,i+1\rfloor})\\
&= \bigtimes_{i\in\lfloor n-1,k+1\rfloor}\mathrm{d}_k\mathrm{d}_{\lfloor n-1,i\rfloor}\times(\mathrm{d}_k\times\mathrm{d}_k)\mathrm{m}\mathrm{d}_{\lfloor n-1,k\rfloor}\times\bigtimes_{i\in\lfloor k-2,0\rfloor}\mathrm{d}_k\mathrm{d}_{\lfloor n-1,i+1\rfloor}\\
&= \bigtimes_{i\in\lfloor n-1,k+1\rfloor}\mathrm{d}_k\mathrm{d}_{\lfloor n-1,i\rfloor}\times(\mathrm{d}_k\times\mathrm{d}_k)\mathrm{d}_{\lfloor n-1,k\rfloor}\mathrm{m}\times\bigtimes_{i\in\lfloor k-2,0\rfloor}\mathrm{d}_k\mathrm{d}_{\lfloor n-1,i+1\rfloor}\\
&= \bigtimes_{i\in\lfloor n-1,k+1\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor}\mathrm{d}_k\times(\mathrm{d}_{\lfloor n,k+1\rfloor}\mathrm{d}_k\times\mathrm{d}_{\lfloor n,k\rfloor})\mathrm{m}\times\bigtimes_{i\in\lfloor k-2,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor}\\
&= (\bigtimes_{i\in\lfloor n-1,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor})(\bigtimes_{i\in\lfloor n-1,k+1\rfloor}\mathrm{d}_k\times(\mathrm{d}_k\times\mathrm{id})\mathrm{m}\times\bigtimes_{i\in\lfloor k-2,0\rfloor}\mathrm{id}) = (D_G)_n\mathrm{d}_k
\end{aligned}$$

for all $k\in[0,n]$, $n\in\mathbb{N}$, and

$$\begin{aligned}
\mathrm{s}_k(D_G)_{n+1} &= (\bigtimes_{i\in\lfloor n-1,k\rfloor}\mathrm{s}_k\times\mathrm{n}\times\bigtimes_{i\in\lfloor k-1,0\rfloor}\mathrm{s}_k)(\bigtimes_{i\in\lfloor n,0\rfloor}\mathrm{d}_{\lfloor n+1,i+1\rfloor})\\
&= \bigtimes_{i\in\lfloor n-1,k\rfloor}\mathrm{s}_k\mathrm{d}_{\lfloor n+1,i+2\rfloor}\times\mathrm{n}\times\bigtimes_{i\in\lfloor k-1,0\rfloor}\mathrm{s}_k\mathrm{d}_{\lfloor n+1,i+1\rfloor}\\
&= \bigtimes_{i\in\lfloor n-1,k\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor}\mathrm{s}_k\times\mathrm{n}\times\bigtimes_{i\in\lfloor k-1,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor}\\
&= (\bigtimes_{i\in\lfloor n-1,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor})(\bigtimes_{i\in\lfloor n-1,k\rfloor}\mathrm{s}_k\times\mathrm{n}\times\bigtimes_{i\in\lfloor k-1,0\rfloor}\mathrm{id}) = (D_G)_n\mathrm{s}_k
\end{aligned}$$

for all $k\in[0,n]$, $n\in\mathbb{N}_0$, that is

$$\operatorname{Diag}\mathrm{N}G \xrightarrow{D_G} \overline{\mathrm{W}}G$$

is a simplicial map for each $G\in\operatorname{Ob}\mathbf{sGrp}$.([1])

$$\begin{array}{ccccc}
\operatorname{Diag}_{n+1}\mathrm{N}G & \xleftarrow{\mathrm{s}_k} & \operatorname{Diag}_n\mathrm{N}G & \xrightarrow{\mathrm{d}_k} & \operatorname{Diag}_{n-1}G\\
\downarrow{\scriptstyle (D_G)_{n+1}} & & \downarrow{\scriptstyle (D_G)_n} & & \downarrow{\scriptstyle (D_G)_{n-1}}\\
\overline{\mathrm{W}}_{n+1}G & \xleftarrow{\mathrm{s}_k} & \overline{\mathrm{W}}_nG & \xrightarrow{\mathrm{d}_k} & \overline{\mathrm{W}}_{n-1}G
\end{array}$$

Now we let G and H be simplicial groups and we let $G\xrightarrow{\varphi}H$ be a simplicial group homomorphism. We get

$$(D_G)_n(\overline{\mathrm{W}}\varphi) = (\bigtimes_{i\in\lfloor n-1,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor})(\bigtimes_{i\in\lfloor n-1,0\rfloor}\varphi_i) = \bigtimes_{i\in\lfloor n-1,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor}\varphi_i = \bigtimes_{i\in\lfloor n-1,0\rfloor}\varphi_n\mathrm{d}_{\lfloor n,i+1\rfloor}$$

[1]The morphism D_G can be obtained as composite $D_G = \phi_{\mathrm{N}G}f_G$, where ϕ is the natural transformation between the functors Diag and Tot from $\mathbf{s^2Set}$ to $\mathbf{sSet}$, cf. (3.15), and where f is the isomorphism from $\operatorname{Tot}\circ\mathrm{N}$ to $\overline{\mathrm{W}}$, cf. remark (4.19). This yields an alternative proof of the fact that D_G is a simplicial map.

$$= \varphi_n^{\times n}(\bigtimes_{i \in \lfloor n-1,0 \rfloor} \mathrm{d}_{\lfloor n,i+1 \rfloor}) = (\mathrm{Diag}_n \mathrm{N}\varphi)(D_H)_n$$

for all $n \in \mathbb{N}_0$. This implies that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Diag}\,\mathrm{N}G & \xrightarrow{D_G} & \overline{\mathrm{W}}G \\ {\scriptstyle \mathrm{Diag}\,\mathrm{N}\varphi}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{W}}\varphi} \\ \mathrm{Diag}\,\mathrm{N}H & \xrightarrow{D_H} & \overline{\mathrm{W}}H \end{array}$$

that is, $(D_G)_{G \in \mathrm{Ob}\,\mathbf{sGrp}}$ is natural in G and thus $\mathrm{Diag} \circ \mathrm{N} \xrightarrow{D} \overline{\mathrm{W}}$ is a natural transformation.

(b) Again, we have to show that the maps $(S_G)_n$ for $n \in \mathbb{N}_0$ commute with the faces and degeneracies of a simplicial group G.

First, we consider the faces: We let $n \in \mathbb{N}$ and $k \in [0, n]$. For an n-tuple $(g_i)_{i \in \lfloor n-1,0 \rfloor} \in \bigtimes_{i \in \lfloor n-1,0 \rfloor} G_i$ we compute

$$(g_i)_{i \in \lfloor n-1,0 \rfloor} \mathrm{d}_k (S_G)_{n-1} = (f_i)_{i \in \lfloor n-2,0 \rfloor} (S_G)_{n-1} = (x_i)_{i \in \lfloor n-2,0 \rfloor},$$

where

$$f_i := \begin{cases} g_{i+1}\mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (g_k \mathrm{d}_k) g_{k-1} & \text{for } i = k-1, \\ g_i & \text{for } i \in \lfloor k-2, 0 \rfloor \end{cases}$$

and

$$x_i := \prod_{j \in \lceil i+1, n-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \text{ for each } i \in \lfloor n-2, 0 \rfloor.$$

On the other hand, we get

$$(g_i)_{i \in \lfloor n-1,0 \rfloor} (S_G)_n \mathrm{d}_k = (y_i)_{i \in \lfloor n-1,0 \rfloor} \mathrm{d}_k = (x'_i)_{i \in \lfloor n-2,0 \rfloor}$$

with

$$y_i := \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil}) \text{ for } i \in \lfloor n-1, 0 \rfloor$$

and

$$x'_i := \begin{cases} y_{i+1}\mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (y_k \mathrm{d}_k)(y_{k-1} \mathrm{d}_k) & \text{for } i = k-1, \\ y_i \mathrm{d}_k & \text{for } i \in \lfloor k-2, 0 \rfloor. \end{cases}$$

We have to show that $x_i = x'_i$ for all $i \in \lfloor n-2, 0 \rfloor$. To this end, we proceed by induction on i.

For $i \in \lfloor n-2, k \rfloor$, we calculate

$$\begin{aligned} x_i &= \prod_{j \in \lceil i+1, n-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, n-2 \rceil} (x'^{-1}_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, n-2 \rceil} (y_{j+1}^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (g_{j+1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, n-2 \rceil} (y_{j+1}^{-1} \mathrm{d}_{\lfloor j+1,i+2 \rfloor} \mathrm{d}_k \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (g_{j+1} \mathrm{d}_{\lfloor j+1,i+2 \rfloor} \mathrm{d}_k \mathrm{s}_{\lceil i,n-2 \rceil}) \end{aligned}$$

$$= \prod_{j\in\lceil i+1,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_{\lfloor j+1,i+2\rfloor}\mathrm{s}_{\lceil i+1,j\rceil}\mathrm{d}_k) \prod_{j\in\lfloor n-2,i\rfloor} (g_{j+1}\mathrm{d}_{\lfloor j+1,i+2\rfloor}\mathrm{s}_{\lceil i+1,n-1\rceil}\mathrm{d}_k)$$
$$= \Big(\prod_{j\in\lceil i+1,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_{\lfloor j+1,i+2\rfloor}\mathrm{s}_{\lceil i+1,j\rceil}) \prod_{j\in\lfloor n-2,i\rfloor} (g_{j+1}\mathrm{d}_{\lfloor j+1,i+2\rfloor}\mathrm{s}_{\lceil i+1,n-1\rceil})\Big)\mathrm{d}_k$$
$$= \Big(\prod_{j\in\lceil i+2,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+2\rfloor}\mathrm{s}_{\lceil i+1,j-1\rceil}) \prod_{j\in\lfloor n-1,i+1\rfloor} (g_j\mathrm{d}_{\lfloor j,i+2\rfloor}\mathrm{s}_{\lceil i+1,n-1\rceil})\Big)\mathrm{d}_k = y_{i+1}\mathrm{d}_k.$$

For $i = k-1$, we have

$$x_{k-1} = \prod_{j\in\lceil k,n-2\rceil} (x_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor n-2,k-1\rfloor} (f_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-2\rceil})$$
$$= \prod_{j\in\lceil k,n-2\rceil} (x_j'^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor n-2,k-1\rfloor} (f_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-2\rceil})$$
$$= \prod_{j\in\lceil k,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor n-2,k\rfloor} (g_{j+1}\mathrm{d}_k\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-2\rceil}) \cdot ((g_k\mathrm{d}_k)g_{k-1})\mathrm{s}_{\lceil k-1,n-2\rceil}$$
$$= \prod_{j\in\lceil k,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_{\lfloor j+1,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor n-2,k-2\rfloor} (g_{j+1}\mathrm{d}_{\lfloor j+1,k\rfloor}\mathrm{s}_{\lceil k-1,n-2\rceil})$$
$$= \prod_{j\in\lceil k+1,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-2\rceil}) \prod_{j\in\lfloor n-1,k-1\rfloor} (g_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-2\rceil})$$
$$= (y_k\mathrm{d}_k) \prod_{j\in\lceil k,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-2\rceil}) \prod_{j\in\lfloor n-1,k-1\rfloor} (g_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-2\rceil})$$
$$= (y_k\mathrm{d}_k) \prod_{j\in\lceil k,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lfloor n-1,k-1\rfloor} (g_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-1\rceil}\mathrm{d}_k)$$
$$= (y_k\mathrm{d}_k)\Big(\prod_{j\in\lceil k,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor n-1,k-1\rfloor} (g_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,n-1\rceil})\Big)\mathrm{d}_k$$
$$= (y_k\mathrm{d}_k)(y_{k-1}\mathrm{d}_k).$$

For $i \in \lfloor k-2, 0\rfloor$, we finally get

$$x_i = \prod_{j\in\lceil i+1,n-2\rceil} (x_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-2,i\rfloor} (f_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil})$$
$$= \prod_{j\in\lceil i+1,n-2\rceil} (x_j'^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-2,i\rfloor} (f_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil})$$
$$= (\prod_{j\in\lceil i+1,k-2\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))((y_ky_{k-1})^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})$$
$$\cdot (\prod_{j\in\lceil k,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))(\prod_{j\in\lfloor n-2,k\rfloor} (y_{j+1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil}))$$
$$\cdot (((g_k\mathrm{d}_k)g_{k-1})\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil})(\prod_{j\in\lfloor k-2,i\rfloor} (g_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil}))$$
$$= (\prod_{j\in\lceil i+1,k-2\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))(y_{k-1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})(y_k^{-1}\mathrm{d}_{\lfloor k,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})$$
$$\cdot (\prod_{j\in\lceil k,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_{\lfloor j+1,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))(\prod_{j\in\lfloor n-2,k\rfloor} (g_{j+1}\mathrm{d}_{\lfloor j+1,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil}))$$
$$\cdot (g_k\mathrm{d}_{\lfloor k,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil})(g_{k-1}\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil})(\prod_{j\in\lfloor k-2,i\rfloor} (g_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil}))$$
$$= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k-1,n-2\rceil} (y_{j+1}^{-1}\mathrm{d}_{\lfloor j+1,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})$$
$$\cdot \prod_{j\in\lfloor n-2,k-1\rfloor} (g_{j+1}\mathrm{d}_{\lfloor j+1,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-2\rceil})$$

$$
\begin{aligned}
&= \prod_{j \in \lceil i+1,k-1 \rceil} (y_j^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lceil k,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-2 \rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1,k \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \prod_{j \in \lfloor k-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \\
&= \prod_{j \in \lceil i+1,k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{d}_{k-j+i} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lceil k,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-2 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \\
&= \prod_{j \in \lceil i+1,k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil} \mathrm{d}_k) \prod_{j \in \lceil k,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil} \mathrm{d}_k) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-2 \rceil}) \\
&= \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil} \mathrm{d}_k) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil} \mathrm{d}_k) \\
&= \Big(\prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil}) \Big) \mathrm{d}_k = y_i \mathrm{d}_k.
\end{aligned}
$$

Hence

$$
x_i = \begin{cases} y_{i+1} \mathrm{d}_k & \text{for } i \in \lfloor n-2,k \rfloor, \\ (y_k \mathrm{d}_k)(y_{k-1} \mathrm{d}_k) & \text{for } i = k-1, \\ y_i \mathrm{d}_k & \text{for } i \in \lfloor k-2,0 \rfloor \end{cases} \Bigg\} = x'_i
$$

for $i \in \lfloor n-2,0 \rfloor$, and therefore

$$
(g_i)_{i \in \lfloor n-1,0 \rfloor} \mathrm{d}_k (S_G)_{n-1} = (g_i)_{i \in \lfloor n-1,0 \rfloor} (S_G)_n \mathrm{d}_k.
$$

We conclude that $\mathrm{d}_k (S_G)_{n-1} = (S_G)_n \mathrm{d}_k$ for all $k \in [0,n]$, $n \in \mathbb{N}$.

Next, we come to the degeneracies.

We let $n \in \mathbb{N}_0$, $k \in [0,n]$ and $(g_i)_{i \in \lfloor n-1,0 \rfloor} \in \bigtimes_{i \in \lfloor n-1,0 \rfloor} G_i$. We compute

$$
(g_i)_{i \in \lfloor n-1,0 \rfloor} \mathrm{s}_k (S_G)_{n+1} = (h_i)_{i \in \lfloor n,0 \rfloor} (S_G)_{n+1} = (z_i)_{i \in \lfloor n,0 \rfloor},
$$

where

$$
h_i := \begin{cases} g_{i-1} \mathrm{s}_k & \text{for } i \in \lfloor n,k+1 \rfloor, \\ 1 & \text{for } i = k, \\ g_i & \text{for } i \in \lfloor k-1,0 \rfloor \end{cases}
$$

and

$$
z_i := \prod_{j \in \lceil i+1,n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \text{ for each } i \in \lfloor n,0 \rfloor.
$$

Further, we get

$$
(g_i)_{i \in \lfloor n-1,0 \rfloor} (S_G)_n \mathrm{s}_k = (y_i)_{i \in \lfloor n-1,0 \rfloor} \mathrm{s}_k = (z'_i)_{i \in \lfloor n,0 \rfloor}
$$

with

$$
y_i := \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil}) \text{ for } i \in \lfloor n-1,0 \rfloor
$$

and

$$
z'_i := \begin{cases} y_{i-1} \mathrm{s}_k & \text{for } i \in \lfloor n,k+1 \rfloor, \\ 1 & \text{for } i = k, \\ y_i \mathrm{s}_k & \text{for } i \in \lfloor k-1,0 \rfloor. \end{cases}
$$

Thus we have to show that $z_i = z'_i$ for every $i \in \lfloor n,0 \rfloor$. To this end, we perform an induction on $i \in \lfloor n,0 \rfloor$.

For $i \in \lfloor n, k+1 \rfloor$, we have

$$\begin{aligned}
z_i &= \prod_{j \in \lceil i+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} (y_{j-1}^{-1} \mathrm{d}_{\lfloor j-1,i \rfloor} \mathrm{s}_k \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (g_{j-1} \mathrm{d}_{\lfloor j-1,i \rfloor} \mathrm{s}_k \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} (y_{j-1}^{-1} \mathrm{d}_{\lfloor j-1,i \rfloor} \mathrm{s}_{\lceil i-1,j-2 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor n,i \rfloor} (g_{j-1} \mathrm{d}_{\lfloor j-1,i \rfloor} \mathrm{s}_{\lceil i-1,n-1 \rceil} \mathrm{s}_k) \\
&= \prod_{j \in \lceil i, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor n-1,i-1 \rfloor} (g_j \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,n-1 \rceil} \mathrm{s}_k) \\
&= \Big(\prod_{j \in \lceil i, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,j-1 \rceil}) \prod_{j \in \lfloor n-1,i-1 \rfloor} (g_j \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,n-1 \rceil}) \Big) \mathrm{s}_k = y_{i-1} \mathrm{s}_k.
\end{aligned}$$

For $i = k$, we compute

$$\begin{aligned}
z_k &= \prod_{j \in \lceil k+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n,k \rfloor} (h_j \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n,k \rfloor} (h_j \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n,k+1 \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{d}_{\lfloor j-1,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n,k+1 \rfloor} (g_{j-1} \mathrm{d}_{\lfloor j-1,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,j-1 \rceil}) \prod_{j \in \lfloor n,k+1 \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,j-1 \rceil}) \prod_{j \in \lfloor n,k+1 \rfloor} (h_j \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,n \rceil}) \\
&= z_{k+1}^{-1} \prod_{j \in \lceil k+2, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,j-1 \rceil}) \prod_{j \in \lfloor n,k+1 \rfloor} (h_j \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,n \rceil}) = z_{k+1}^{-1} z_{k+1} = 1.
\end{aligned}$$

For $i \in \lfloor k-1, 0 \rfloor$, we get

$$\begin{aligned}
z_i &= \prod_{j \in \lceil i+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= (\prod_{j \in \lceil i+1, k-1 \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}))({z'_k}^{-1} \mathrm{d}_{\lfloor k,i+1 \rfloor} \mathrm{s}_{\lceil i,k-1 \rceil})(\prod_{j \in \lceil k+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil})) \\
&\quad \cdot (\prod_{j \in \lfloor n,k+1 \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}))(h_k \mathrm{d}_{\lfloor k,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil})(\prod_{j \in \lfloor k-1,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil})) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n,k+1 \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \prod_{j \in \lfloor k-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1,i+1 \rfloor} \mathrm{s}_{\lceil i,j \rceil})
\end{aligned}$$

$$
\begin{aligned}
&\cdot \prod_{j\in\lfloor n-1,k\rfloor} (g_j \mathrm{s}_k \mathrm{d}_{\lfloor j+1,i+1\rfloor} \mathrm{s}_{\lceil i,n\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{k-j+i} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j\rceil}) \\
&\quad \cdot \prod_{j\in\lfloor n-1,k\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \mathrm{s}_k) \prod_{j\in\lceil k,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \mathrm{s}_k) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n\rceil}) \\
&= \prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \mathrm{s}_k) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil} \mathrm{s}_k) \\
&= \Big(\prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil}) \Big) \mathrm{s}_k = y_i \mathrm{s}_k.
\end{aligned}
$$

Hence

$$
z_i = \left\{ \begin{array}{ll} y_{i-1}\mathrm{s}_k & \text{for } i \in \lfloor n, k+1 \rfloor, \\ 1 & \text{for } i = k, \\ y_i \mathrm{s}_k & \text{for } i \in \lfloor k-1, 0 \rfloor \end{array} \right\} = z'_i
$$

for $i \in \lfloor n, 0 \rfloor$, and therefore

$$(g_i)_{i\in\lfloor n-1,0\rfloor} \mathrm{s}_k (S_G)_{n+1} = (g_i)_{i\in\lfloor n-1,0\rfloor} (S_G)_n \mathrm{s}_k.$$

We conclude that $\mathrm{s}_k (S_G)_{n+1} = (S_G)_n \mathrm{s}_k$ for all $k \in [0, n]$, $n \in \mathbb{N}_0$.

Thus $(S_G)_{n\in\mathbb{N}}$ yields a simplicial map

$$\overline{\mathrm{W}}G \xrightarrow{S_G} \operatorname{Diag} \mathrm{N}G.$$

Now we shall show that $(S_G)_{G\in\operatorname{Ob}\mathbf{sGrp}}$ is a natural transformation. We let G, H be simplicial groups and $G \xrightarrow{\varphi} H$ be a simplicial group homomorphism. Further, we let $n \in \mathbb{N}_0$ be a non-negative integer and $(g_i)_{i\in\lfloor n-1,0\rfloor} \in \overline{\mathrm{W}}_n G$ be an element. We write $(y_i)_{i\in\lfloor n-1,0\rfloor}$ for the image of $(g_i)_{i\in\lfloor n-1,0\rfloor}$ under $(S_G)_n$ and we write $(z_i)_{i\in\lfloor n-1,0\rfloor}$ for the image of $(g_i\varphi_i)_{i\in\lfloor n-1,0\rfloor} = (g_i)_{i\in\lfloor n-1,0\rfloor}(\overline{\mathrm{W}}_n\varphi) \in \overline{\mathrm{W}}_n H$ under $(S_H)_n$. By induction on $i \in \lfloor n-1, 0 \rfloor$, we get

$$
\begin{aligned}
z_i &= \prod_{j\in\lceil i+1,n-1\rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \varphi_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil}) \\
&= \prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1} \varphi_n \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \varphi_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil}) \\
&= \prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \varphi_n) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil} \varphi_n) \\
&= \Big(\prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil}) \Big) \varphi_n = y_i \varphi_n.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(g_i)_{i\in\lfloor n-1,0\rfloor}(\overline{\mathrm{W}}_n\varphi)(S_H)_n &= (g_i\varphi_i)_{i\in\lfloor n-1,0\rfloor}(S_H)_n = (z_i)_{i\in\lfloor n-1,0\rfloor} = (y_i\varphi_n)_{i\in\lfloor n-1,0\rfloor} \\
&= (y_i)_{i\in\lfloor n-1,0\rfloor} \operatorname{Diag}_n \mathrm{N}\varphi = (g_i)_{i\in\lfloor n-1,0\rfloor}(S_G)_n \operatorname{Diag}_n \mathrm{N}\varphi.
\end{aligned}
$$

Since $(g_i)_{i\in\lfloor n-1,0\rfloor} \in \overline{\mathrm{W}}_n G$ and $n \in \mathbb{N}_0$ were chosen arbitrarily, this implies the commutativity of

$$
\begin{array}{ccc}
\overline{\mathrm{W}}G & \xrightarrow{S_G} & \operatorname{Diag}\mathrm{N}G \\
{\scriptstyle \overline{\mathrm{W}}\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Diag}\mathrm{N}\varphi} \\
\overline{\mathrm{W}}H & \xrightarrow{S_H} & \operatorname{Diag}\mathrm{N}H
\end{array}
$$

Finally, we have to prove that D_G is a retraction with coretraction S_G, that is

$$(S_G)_n(D_G)_n = \mathrm{id}_{\overline{\mathrm{W}}_n G} \text{ for all } n \in \mathbb{N}_0.$$

Again, we let $(y_i)_{i\in\lfloor n-1,0\rfloor}$ denote the image of an element $(g_i)_{i\in\lfloor n-1,0\rfloor} \in \times_{i\in\lfloor n-1,0\rfloor} G_i$ under $(S_G)_n$. Then we have

$$(g_i)_{i\in\lfloor n-1,0\rfloor}(S_G)_n(D_G)_n = (y_i)_{i\in\lfloor n-1,0\rfloor}(D_G)_n = (y_i \mathrm{d}_{\lfloor n,i+1\rfloor})_{i\in\lfloor n-1,0\rfloor}.$$

Induction on $i \in \lfloor n-1, 0\rfloor$ shows that

$$\begin{aligned} y_i \mathrm{d}_{\lfloor n,i+1\rfloor} &= \Big(\prod_{j\in\lceil i+1,n-1\rceil}(y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor n-1,i\rfloor}(g_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-1\rceil})\Big)\mathrm{d}_{\lfloor n,i+1\rfloor} \\ &= \prod_{j\in\lceil i+1,n-1\rceil}(y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}\mathrm{d}_{\lfloor n,i+1\rfloor})\prod_{j\in\lfloor n-1,i\rfloor}(g_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-1\rceil}\mathrm{d}_{\lfloor n,i+1\rfloor}) \\ &= \prod_{j\in\lceil i+1,n-1\rceil}(y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{d}_{\lfloor n-j+i,j+1-j+i\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}\mathrm{d}_{\lfloor j,i+1\rfloor})\prod_{j\in\lfloor n-1,i\rfloor}(g_j\mathrm{d}_{\lfloor j,i+1\rfloor}) \\ &= \prod_{j\in\lceil i+1,n-1\rceil}(y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{d}_{\lfloor n-j+i,i+1\rfloor})\prod_{j\in\lfloor n-1,i\rfloor}(g_j\mathrm{d}_{\lfloor j,i+1\rfloor}) \\ &= \prod_{j\in\lceil i+1,n-1\rceil}(y_j^{-1}\mathrm{d}_{\lfloor n,i+1\rfloor})\prod_{j\in\lfloor n-1,i\rfloor}(g_j\mathrm{d}_{\lfloor j,i+1\rfloor}) \\ &= \prod_{j\in\lceil i+1,n-1\rceil}(g_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor})\prod_{j\in\lfloor n-1,i\rfloor}(g_j\mathrm{d}_{\lfloor j,i+1\rfloor}) = g_i. \end{aligned}$$

This implies that $(S_G)_n(D_G)_n = \mathrm{id}_{\overline{\mathrm{W}}_n G}$ for all $n \in \mathbb{N}_0$. □

(4.32) Theorem. For each simplicial group G, the Kan classifying simplicial set $\overline{\mathrm{W}}G$ is a strong simplicial deformation retract of $\operatorname{Diag}\mathrm{N}G$. A strong simplicial deformation retraction is given by

$$\operatorname{Diag}\mathrm{N}G \xrightarrow{D_G} \overline{\mathrm{W}}G,$$

where $(D_G)_n = \times_{i\in\lfloor n-1,0\rfloor}\mathrm{d}_{\lfloor n,i+1\rfloor}$ for every $n \in \mathbb{N}_0$, $G \in \operatorname{Ob}\mathbf{sGrp}$; cf. proposition (4.31)(a).

Proof. For $n \in \mathbb{N}_0$, we define $H_n\colon \operatorname{Diag}_n \mathrm{N}G \times \Delta^1_n \to \operatorname{Diag}_n \mathrm{N}G$, $((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-k}) \mapsto (y_i)_{i\in\lfloor n-1,0\rfloor}$, where $k \in [0, n+1]$ and, recursively defined,

$$y_i := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, k-2\rfloor \cap \mathbb{N}_0, \\ \prod_{j\in\lceil i+1,k-2\rceil}(y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor k-1,i\rfloor}(g_{n,j}\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil}) & \text{for } i \in \lfloor k-2, 0\rfloor. \end{cases}$$

Cf. definition (2.3).
We will show that these maps yield a simplicial homotopy from $D_G S_G$ to $\mathrm{id}_{\operatorname{Diag}\mathrm{N}G}$, where S_G is given as in proposition (4.31), which is constant along S_G.
First, we show the compatibility with the faces. For $k \in [0, n]$, $l \in [0, n+1]$, $n \in \mathbb{N}_0$, $(g_{n,i})_{i\in\lfloor n-1,0\rfloor} \in \operatorname{Diag}_n \mathrm{N}G$, we have

$$\begin{aligned} ((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})\mathrm{d}_k H_{n-1} &= ((g_{n,i})_{i\in\lfloor n-1,0\rfloor}\mathrm{d}_k, \tau^{n+1-l}\mathrm{d}_k)H_{n-1} = ((f_i)_{i\in\lfloor n-2,0\rfloor}, \delta^k\tau^{n+1-l})H_{n-1} \\ &= \left.\begin{cases} ((f_i)_{i\in\lfloor n-2,0\rfloor}, \tau^{n-l})H_{n-1} & \text{for } k \geq l, \\ ((f_i)_{i\in\lfloor n-2,0\rfloor}, \tau^{n+1-l})H_{n-1} & \text{for } k < l \end{cases}\right\} = (x_i)_{i\in\lfloor n-2,0\rfloor}, \end{aligned}$$

where

$$f_i := \begin{cases} g_{n,i+1}\mathrm{d}_k & \text{for } i \in \lfloor n-2, k\rfloor, \\ (g_{n,k}\mathrm{d}_k)(g_{n,k-1}\mathrm{d}_k) & \text{for } i = k-1, \\ g_{n,i}\mathrm{d}_k & \text{for } i \in \lfloor k-2, 0\rfloor \end{cases}$$

for all $i \in \lfloor n-2, 0 \rfloor$ and

$$x_i := \begin{cases} \left.\begin{cases} f_i & \text{for } i \in \lfloor n-2, l-1 \rfloor, \\ \prod_{j \in \lceil i+1, l-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-2, i \rfloor} (f_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) & \text{for } i \in \lfloor l-2, 0 \rfloor \end{cases}\right\} & \text{if } k \geq l, \\ \left.\begin{cases} f_i & \text{for } i \in \lfloor n-2, l-2 \rfloor, \\ \prod_{j \in \lceil i+1, l-3 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-3, i \rfloor} (f_j \mathrm{d}_{\lfloor l-2, i+1 \rfloor} \mathrm{s}_{\lceil i, l-3 \rceil}) & \text{for } i \in \lfloor l-3, 0 \rfloor \end{cases}\right\} & \text{if } k < l \end{cases}$$

for all $i \in \lfloor n-2, 0 \rfloor$. On the other hand, we have

$$((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-l}) H_n \mathrm{d}_k = (y_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_k = (x'_i)_{i \in \lfloor n-2, 0 \rfloor}$$

with

$$y_i := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, l-1 \rfloor, \\ \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) & \text{for } i \in \lfloor l-2, 0 \rfloor \end{cases}$$

for $i \in \lfloor n-1, 0 \rfloor$ and

$$x'_i := \begin{cases} y_{i+1} \mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (y_k \mathrm{d}_k)(y_{k-1} \mathrm{d}_k) & \text{for } i = k-1, \\ y_i \mathrm{d}_k & \text{for } i \in \lfloor k-2, 0 \rfloor \end{cases}$$

for $i \in \lfloor n-2, 0 \rfloor$. We have to show that $x_i = x'_i$ for all $i \in \lfloor n-2, 0 \rfloor$. To this end, we consider three cases and we handle each one by induction on $i \in \lfloor n-2, 0 \rfloor$.
We suppose that $k \in \lfloor n, l \rfloor$.
For $i \in \lfloor n-2, k \rfloor$, we have

$$x_i = f_i = g_{n,i+1} \mathrm{d}_k = y_{i+1} \mathrm{d}_k = x'_i.$$

For $i = k-1$, we get

$$x_{k-1} = f_{k-1} = (g_{n,k} \mathrm{d}_k)(g_{n,k-1} \mathrm{d}_k) = (y_k \mathrm{d}_k)(y_{k-1} \mathrm{d}_k) = x'_{k-1}.$$

For $i \in \lfloor k-2, l-1 \rfloor$, we get

$$x_i = f_i = g_{n,i} \mathrm{d}_k = y_i \mathrm{d}_k = x'_i$$

Finally, for $i \in \lfloor l-2, 0 \rfloor$, we calculate

$$\begin{aligned} x_i &= \prod_{j \in \lceil i+1, l-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (f_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, l-2 \rceil} (x'^{-1}_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (f_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_k \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil} \mathrm{d}_k) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil} \mathrm{d}_k) \\ &= \Big(\prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \Big) \mathrm{d}_k = y_i \mathrm{d}_k = x'_i. \end{aligned}$$

Next, we suppose that $k = l-1$.
For $i \in \lfloor n-2, k \rfloor$, we have

$$x_i = f_i = g_{n,i+1} \mathrm{d}_k = y_{i+1} \mathrm{d}_k = x'_i.$$

For $i = k-1$, we compute

$$x_{k-1} = f_{k-1} = (g_{n,k}\mathrm{d}_k)(g_{n,k-1}\mathrm{d}_k) = (g_{n,k}\mathrm{d}_k)(g_{n,k-1}\mathrm{d}_k\mathrm{s}_{k-1}\mathrm{d}_k) = (y_k\mathrm{d}_k)(y_{k-1}\mathrm{d}_k) = x'_{k-1}.$$

For $i \in \lfloor k-2, 0\rfloor$, we get

$$\begin{aligned}
x_i &= \prod_{j\in\lceil i+1,k-2\rceil} (x_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k-2,i\rfloor} (f_j\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j\in\lceil i+1,k-2\rceil} ({x'_j}^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k-2,i\rfloor} (f_j\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j\in\lceil i+1,k-2\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k-2,i\rfloor} (g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lfloor k-1,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor k,i+1\rfloor}\mathrm{s}_{\lceil i,k-1\rceil}\mathrm{d}_k) \\
&= \Big(\prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor k,i+1\rfloor}\mathrm{s}_{\lceil i,k-1\rceil})\Big)\mathrm{d}_k = y_i\mathrm{d}_k = x'_i.
\end{aligned}$$

Finally, we suppose that $k \in \lfloor l-2, 0\rfloor$.
For $i \in \lfloor n-2, l-2\rfloor$, we see that

$$x_i = f_i = g_{n,i+1}\mathrm{d}_k = y_{i+1}\mathrm{d}_k = x'_i.$$

For $i \in \lfloor l-3, k\rfloor$, we have

$$\begin{aligned}
x_i &= \prod_{j\in\lceil i+1,l-3\rceil} (x_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-3,i\rfloor} (f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+1,l-3\rceil} ({x'_j}^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-3,i\rfloor} (f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+1,l-3\rceil} (y_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-3,i\rfloor} (g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+2,l-2\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j-1,i+1\rfloor}\mathrm{s}_{\lceil i,j-2\rceil}) \prod_{j\in\lfloor l-2,i+1\rfloor} (g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+2,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+2\rfloor}\mathrm{d}_k\mathrm{s}_{\lceil i,j-2\rceil}) \prod_{j\in\lfloor l-2,i+1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+2\rfloor}\mathrm{d}_k\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+2,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+2\rfloor}\mathrm{s}_{\lceil i+1,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lfloor l-2,i+1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+2\rfloor}\mathrm{s}_{\lceil i+1,l-2\rceil}\mathrm{d}_k) \\
&= \Big(\prod_{j\in\lceil i+2,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+2\rfloor}\mathrm{s}_{\lceil i+1,j-1\rceil}) \prod_{j\in\lfloor l-2,i+1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+2\rfloor}\mathrm{s}_{\lceil i+1,l-2\rceil})\Big)\mathrm{d}_k = y_{i+1}\mathrm{d}_k = x'_i.
\end{aligned}$$

For $i = k-1$, we have

$$\begin{aligned}
x_{k-1} &= \prod_{j\in\lceil k,l-3\rceil} (x_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor l-3,k-1\rfloor} (f_j\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}) \\
&= \prod_{j\in\lceil k,l-3\rceil} ({x'_j}^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor l-3,k-1\rfloor} (f_j\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}) \\
&= (\prod_{j\in\lceil k,l-3\rceil} (y_{j+1}\mathrm{d}_k\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}))(\prod_{j\in\lfloor l-3,k\rfloor} (g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})) \\
&\quad \cdot (g_{n,k}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})(g_{n,k-1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}) \\
&= \prod_{j\in\lceil k,l-3\rceil} (y_{j+1}\mathrm{d}_k\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor l-3,k-2\rfloor} (g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}) \\
&= \prod_{j\in\lceil k+1,l-2\rceil} (y_j\mathrm{d}_k\mathrm{d}_{\lfloor j-1,k\rfloor}\mathrm{s}_{\lceil k-1,j-2\rceil}) \prod_{j\in\lfloor l-2,k-1\rfloor} (g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{j\in\lceil k+1,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-2\rceil}) \prod_{j\in\lfloor l-2,k-1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}) \\
&= (y_k\mathrm{d}_k) \prod_{j\in\lceil k,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-2\rceil}) \prod_{j\in\lfloor l-2,k-1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}) \\
&= (y_k\mathrm{d}_k) \prod_{j\in\lceil k,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lfloor l-2,k-1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,k\rfloor}\mathrm{s}_{\lceil k-1,l-2\rceil}\mathrm{d}_k) \\
&= (y_k\mathrm{d}_k)\Big(\prod_{j\in\lceil k,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}) \prod_{j\in\lfloor l-2,k-1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,k\rfloor}\mathrm{s}_{\lceil k-1,l-2\rceil})\Big)\mathrm{d}_k = (y_k\mathrm{d}_k)(y_{k-1}\mathrm{d}_k) \\
&= x'_{k-1}.
\end{aligned}
$$

At last, for $i \in \lfloor k-2, 0\rfloor$, we get

$$
\begin{aligned}
x_i &= \prod_{j\in\lceil i+1,l-3\rceil} (x_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-3,i\rfloor} (f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+1,l-3\rceil} ({x'_j}^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-3,i\rfloor} (f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= (\prod_{j\in\lceil i+1,k-2\rceil} ({x'_j}^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))({x'_{k-1}}^{-1}\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})(\prod_{j\in\lceil k,l-3\rceil} ({x'_j}^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})) \\
&\quad \cdot (\prod_{j\in\lfloor l-3,k\rfloor} (f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}))(f_{k-1}\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})(\prod_{j\in\lfloor k-2,i\rfloor} (f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})) \\
&= (\prod_{j\in\lceil i+1,k-2\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))(y_{k-1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})(y_k^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil}) \\
&\quad \cdot (\prod_{j\in\lceil k,l-3\rceil} (y_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))(\prod_{j\in\lfloor l-3,k\rfloor} (g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}))(g_{n,k}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&\quad \cdot (g_{n,k-1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})(\prod_{j\in\lfloor k-2,i\rfloor} (g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k-1,l-3\rceil} (y_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \\
&\quad \cdot \prod_{j\in\lfloor l-3,k-1\rfloor} (g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k,l-2\rceil} (y_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j-1,i+1\rfloor}\mathrm{s}_{\lceil i,j-2\rceil}) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{d}_{k-j+i}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-2\rceil}) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lceil k,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil}\mathrm{d}_k) \\
&= \prod_{j\in\lceil i+1,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}\mathrm{d}_k) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil}\mathrm{d}_k) \\
&= \Big(\prod_{j\in\lceil i+1,l-2\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})\Big)\mathrm{d}_k = y_i\mathrm{d}_k = x'_i.
\end{aligned}
$$

Hence $x_i = x'_i$ for all $i \in \lfloor n-2, 0\rfloor$, regardless of k and l, and therefore

$$((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})\mathrm{d}_k H_{n-1} = ((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})H_n\mathrm{d}_k$$

for all $k \in [0, n]$, $l \in [0, n+1]$, $n \in \mathbb{N}_0$. We conclude that

$$\mathrm{d}_k H_{n-1} = H_n\mathrm{d}_k \text{ for all } k \in [0, n], n \in \mathbb{N}_0.$$

Now we consider the degeneracies. We let $n \in \mathbb{N}_0$, $k \in [0, n]$, $l \in [0, n+1]$, and $(g_{n,i})_{i \in \lfloor n-1, 0 \rfloor} \in \operatorname{Diag}_n \mathrm{N}G$. We compute

$$((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})\mathrm{s}_k H_{n+1} = ((g_{n,i})_{i\in\lfloor n-1,0\rfloor}\mathrm{s}_k, \tau^{n+1-l}\mathrm{s}_k)H_{n+1} = ((h_i)_{i\in\lfloor n-1,0\rfloor}, \sigma^k\tau^{n+1-l})H_{n+1}$$
$$= \left.\begin{cases} ((h_i)_{i\in\lfloor n-1,0\rfloor}, \tau^{n+2-l})H_{n+1} & \text{for } k \geq l, \\ ((h_i)_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})H_{n+1} & \text{for } k < l \end{cases}\right\} = (z_i)_{i\in\lfloor n,0\rfloor},$$

where

$$h_i := \begin{cases} g_{n,i-1}\mathrm{s}_k & \text{for } i \in \lfloor n, k+1\rfloor, \\ 1 & \text{for } i = k, \\ g_{n,i}\mathrm{s}_k & \text{for } i \in \lfloor k-1, 0\rfloor \end{cases}$$

and

$$z_i := \begin{cases} \left.\begin{cases} h_i & \text{for } i \in \lfloor n, l-1\rfloor, \\ \prod_{j\in\lceil i+1,l-2\rceil}(z_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-2,i\rfloor}(h_j\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil}) & \text{for } i \in \lfloor l-2, 0\rfloor \end{cases}\right\} & \text{if } k \geq l, \\ \left.\begin{cases} h_i & \text{for } i \in \lfloor n, l\rfloor, \\ \prod_{j\in\lceil i+1,l-1\rceil}(z_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-1,i\rfloor}(h_j\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil}) & \text{for } i \in \lfloor l-1, 0\rfloor \end{cases}\right\} & \text{if } k < l. \end{cases}$$

Furthermore, we have

$$((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})H_n\mathrm{s}_k = (y_i)_{i\in\lfloor n-1,0\rfloor}\mathrm{s}_k = (z'_i)_{i\in\lfloor n,0\rfloor},$$

where

$$y_i := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, l-1\rfloor, \\ \prod_{j\in\lceil i+1,l-2\rceil}(y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-2,i\rfloor}(g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil}) & \text{for } i \in \lfloor l-2, 0\rfloor \end{cases}$$

and

$$z'_i := \begin{cases} y_{i-1}\mathrm{s}_k & \text{for } i \in \lfloor n, k+1\rfloor, \\ 1 & \text{for } i = k, \\ y_i\mathrm{s}_k & \text{for } i \in \lfloor k-1, 0\rfloor. \end{cases}$$

Thus we have to show that $z_i = z'_i$ for every $i \in \lfloor n, 0\rfloor$. Again, we distinguish three cases, and in each one, we perform an induction on $i \in \lfloor n, 0\rfloor$.
We suppose that $k \in \lfloor n, l\rfloor$.
For $i \in \lfloor n, k+1\rfloor$, we calculate

$$z_i = h_i = g_{n,i-1}\mathrm{s}_k = y_{i-1}\mathrm{s}_k = z'_i.$$

Moreover, for $i = k$, we get

$$z_k = h_k = 1 = z'_k.$$

For $i \in \lfloor k-1, l-1\rfloor$, we have

$$z_i = h_i = g_{n,i}\mathrm{s}_k = y_i\mathrm{s}_k = z'_i.$$

Finally, for $i \in \lfloor l-2, 0\rfloor$, we get

$$z_i = \prod_{j\in\lceil i+1,l-2\rceil}(z_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-2,i\rfloor}(h_j\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})$$
$$= \prod_{j\in\lceil i+1,l-2\rceil}({z'_j}^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-2,i\rfloor}(h_j\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})$$

$$= \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l-1,i+1 \rfloor} \mathrm{s}_{\lceil i,l-2 \rceil})$$
$$= \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{k-j+i} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1,i+1 \rfloor} \mathrm{s}_{k-l+i+1} \mathrm{s}_{\lceil i,l-2 \rceil})$$
$$= \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1,i+1 \rfloor} \mathrm{s}_{\lceil i,l-2 \rceil} \mathrm{s}_k)$$
$$= \Big(\prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1,i+1 \rfloor} \mathrm{s}_{\lceil i,l-2 \rceil}) \Big) \mathrm{s}_k = y_i \mathrm{s}_k.$$

Now we suppose that $k = l - 1$.
For $i \in \lfloor n, k+1 \rfloor$, we calculate

$$z_i = h_i = g_{n,i-1} \mathrm{s}_k = y_{i-1} \mathrm{s}_k = z'_i.$$

Moreover, for $i = k$, we get

$$z_k = h_k \mathrm{d}_{k+1} \mathrm{s}_k = 1 = z'_k.$$

For $i \in \lfloor k-1, 0 \rfloor$, we get

$$z_i = \prod_{j \in \lceil i+1, k \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor k, i \rfloor} (h_j \mathrm{d}_{\lfloor k+1,i+1 \rfloor} \mathrm{s}_{\lceil i,k \rceil})$$
$$= \prod_{j \in \lceil i+1, k \rceil} (z_j'^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor k, i \rfloor} (h_j \mathrm{d}_{\lfloor k+1,i+1 \rfloor} \mathrm{s}_{\lceil i,k \rceil})$$
$$= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor k-1, i \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor k+1,i+1 \rfloor} \mathrm{s}_{\lceil i,k \rceil})$$
$$= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor k-1, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor k,i+1 \rfloor} \mathrm{s}_{\lceil i,k \rceil})$$
$$= \Big(\prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor k-1, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor k,i+1 \rfloor} \mathrm{s}_{\lceil i,k-1 \rceil}) \Big) \mathrm{s}_k = y_i \mathrm{s}_k.$$

At last, we suppose that $k \in \lfloor l-2, 0 \rfloor$.
For $i \in \lfloor n, l \rfloor$, we have

$$z_i = h_i = g_{n,i-1} \mathrm{s}_k = y_{i-1} \mathrm{s}_k = z'_i.$$

For $i \in \lfloor l-1, k+1 \rfloor$, we get

$$z_i = \prod_{j \in \lceil i+1, l-1 \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor l-1, i \rfloor} (h_j \mathrm{d}_{\lfloor l,i+1 \rfloor} \mathrm{s}_{\lceil i,l-1 \rceil})$$
$$= \prod_{j \in \lceil i+1, l-1 \rceil} (z_j'^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor l-1, i \rfloor} (h_j \mathrm{d}_{\lfloor l,i+1 \rfloor} \mathrm{s}_{\lceil i,l-1 \rceil})$$
$$= \prod_{j \in \lceil i+1, l-1 \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor l-1, i \rfloor} (g_{n,j-1} \mathrm{s}_k \mathrm{d}_{\lfloor l,i+1 \rfloor} \mathrm{s}_{\lceil i,l-1 \rceil})$$
$$= \prod_{j \in \lceil i, l-2 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1,i+1 \rfloor} \mathrm{s}_{\lceil i,j \rceil}) \prod_{j \in \lfloor l-2, i-1 \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l,i+1 \rfloor} \mathrm{s}_{\lceil i,l-1 \rceil})$$
$$= \prod_{j \in \lceil i, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_k \mathrm{s}_{\lceil i,j \rceil}) \prod_{j \in \lfloor l-2, i-1 \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1,i \rfloor} \mathrm{s}_k \mathrm{s}_{\lceil i,l-1 \rceil})$$
$$= \prod_{j \in \lceil i, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor l-2, i-1 \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1,i \rfloor} \mathrm{s}_{\lceil i-1,l-2 \rceil} \mathrm{s}_k)$$
$$= (\prod_{j \in \lceil i, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,j-1 \rceil}) \prod_{j \in \lfloor l-2, i-1 \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1,i \rfloor} \mathrm{s}_{\lceil i-1,l-2 \rceil})) \mathrm{s}_k = y_{i-1} \mathrm{s}_k = z'_i.$$

For $i = k$, we have

$$\begin{aligned}
z_k &= \prod_{j \in \lceil k+1, l-1 \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j, k+1 \rfloor} \mathrm{s}_{\lceil k, j-1 \rceil}) \prod_{j \in \lfloor l-1, k \rfloor} (h_j \mathrm{d}_{\lfloor l, k+1 \rfloor} \mathrm{s}_{\lceil k, l-1 \rceil}) \\
&= \prod_{j \in \lceil k+1, l-1 \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j, k+1 \rfloor} \mathrm{s}_{\lceil k, j-1 \rceil}) \prod_{j \in \lfloor l-1, k \rfloor} (h_j \mathrm{d}_{\lfloor l, k+1 \rfloor} \mathrm{s}_{\lceil k, l-1 \rceil}) \\
&= \prod_{j \in \lceil k+1, l-1 \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j, k+1 \rfloor} \mathrm{s}_{\lceil k, j-1 \rceil}) \prod_{j \in \lfloor l-1, k+1 \rfloor} (g_{n,j-1} \mathrm{s}_k \mathrm{d}_{\lfloor l, k+1 \rfloor} \mathrm{s}_{\lceil k, l-1 \rceil}) \\
&= \prod_{j \in \lceil k, l-2 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1, k+1 \rfloor} \mathrm{s}_{\lceil k, j \rceil}) \prod_{j \in \lfloor l-2, k \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l, k+1 \rfloor} \mathrm{s}_{\lceil k, l-1 \rceil}) \\
&= (y_k^{-1} \mathrm{s}_k \mathrm{d}_{k+1} \mathrm{s}_k) \Big(\prod_{j \in \lceil k+1, l-2 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1, k+1 \rfloor} \mathrm{s}_{\lceil k, j \rceil}) \Big) \Big(\prod_{j \in \lfloor l-2, k \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l, k+1 \rfloor} \mathrm{s}_{\lceil k, l-1 \rceil}) \Big) \\
&= (y_k^{-1} \mathrm{s}_k) \Big(\prod_{j \in \lceil k+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, k+1 \rfloor} \mathrm{s}_{\lceil k, j \rceil}) \Big) \Big(\prod_{j \in \lfloor l-2, k \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, k+1 \rfloor} \mathrm{s}_{\lceil k, l-1 \rceil}) \Big) \\
&= (y_k^{-1} \mathrm{s}_k) \Big(\prod_{j \in \lceil k+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, k+1 \rfloor} \mathrm{s}_{\lceil k, j-1 \rceil} \mathrm{s}_k) \Big) \Big(\prod_{j \in \lfloor l-2, k \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, k+1 \rfloor} \mathrm{s}_{\lceil k, l-2 \rceil} \mathrm{s}_k) \Big) \\
&= (y_k^{-1} \mathrm{s}_k) \bigg(\Big(\prod_{j \in \lceil k+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, k+1 \rfloor} \mathrm{s}_{\lceil k, j-1 \rceil}) \prod_{j \in \lfloor l-2, k \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, k+1 \rfloor} \mathrm{s}_{\lceil k, l-2 \rceil}) \Big) \mathrm{s}_k \bigg) = (y_k^{-1} \mathrm{s}_k)(y_k \mathrm{s}_k) = 1 \\
&= z'_k.
\end{aligned}$$

For $i \in \lfloor k-1, 0 \rfloor$, we get

$$\begin{aligned}
z_i &= \prod_{j \in \lceil i+1, l-1 \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-1, i \rfloor} (h_j \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \\
&= \prod_{j \in \lceil i+1, l-1 \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-1, i \rfloor} (h_j \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lceil k+1, l-1 \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \\
&\qquad \cdot \prod_{j \in \lfloor l-1, k+1 \rfloor} (g_{n,j-1} \mathrm{s}_k \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \prod_{j \in \lfloor k-1, i \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lceil k, l-2 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1, i+1 \rfloor} \mathrm{s}_{\lceil i, j \rceil}) \\
&\qquad \cdot \prod_{j \in \lfloor l-2, k \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \prod_{j \in \lfloor k-1, i \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lceil k, l-2 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1, i+1 \rfloor} \mathrm{s}_{\lceil i, j \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{s}_k \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{k-j+i} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lceil k, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lceil k, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil} \mathrm{s}_k) \\
&= \prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil} \mathrm{s}_k) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil} \mathrm{s}_k) \\
&= \Big(\prod_{j \in \lceil i+1, l-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \Big) \mathrm{s}_k = y_i \mathrm{s}_k = z'_i.
\end{aligned}$$

Hence $z_i = z'_i$ for all $i \in \lfloor n, 0 \rfloor$, regardless of k and l, and therefore

$$((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-l}) \mathrm{s}_k H_{n+1} = ((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-l}) H_n \mathrm{s}_k.$$

We conclude that $\mathrm{s}_k H_{n+1} = H_n \mathrm{s}_k$.
Altogether, we obtain a simplicial map

$$\operatorname{Diag} \mathrm{N}G \times \Delta^1 \xrightarrow{H} \operatorname{Diag} \mathrm{N}G.$$

We have

$$(g_{n,i})_{i \in \lfloor n-1,0 \rfloor}(D_G)_n(S_G)_n = (g_{n,i}\mathrm{d}_{\lfloor n,i+1 \rfloor})_{i \in \lfloor n-1,0 \rfloor}(S_G)_n = (y_i)_{i \in \lfloor n-1,0 \rfloor}$$

for all $(g_{n,i})_{i \in \lfloor n-1,0 \rfloor} \in \operatorname{Diag}_n \mathrm{N}G$, $n \in \mathbb{N}_0$, where

$$\begin{aligned} y_i &:= \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_{n,j}\mathrm{d}_{\lfloor n,j+1 \rfloor}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,n-1 \rceil}) \\ &= \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_{n,j}\mathrm{d}_{\lfloor n,i+1 \rfloor}\mathrm{s}_{\lceil i,n-1 \rceil}) \end{aligned}$$

for all $i \in \lfloor n-1, 0 \rfloor$. Hence the simplicial map H fulfills

$$((g_{n,i})_{i \in \lfloor n-1,0 \rfloor}, \tau^0)H_n = (g_{n,i})_{i \in \lfloor n-1,0 \rfloor}(D_G)_n(S_G)_n$$

and

$$((g_{n,i})_{i \in \lfloor n-1,0 \rfloor}, \tau^{n+1})H_n = (g_{n,i})_{i \in \lfloor n-1,0 \rfloor}$$

for each $(g_{n,i})_{i \in \lfloor n-1,0 \rfloor} \in \operatorname{Diag}_n \mathrm{N}G$, $n \in \mathbb{N}_0$, that is, H is a simplicial homotopy from $D_G S_G$ to $\mathrm{id}_{\operatorname{Diag} \mathrm{N}G}$.
In order to prove that $\overline{\mathrm{W}}G$ is a strong deformation retract of $\operatorname{Diag} \mathrm{N}G$, it remains to show that H is constant along S_G.
Concretely, this means the following. For $(g_i)_{i \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, we have

$$((g_i)_{i \in \lfloor n-1,0 \rfloor}(S_G)_n, \tau^{n+1-k})H_n = ((y_i)_{i \in \lfloor n-1,0 \rfloor}, \tau^{n+1-k})H_n = (z_i)_{i \in \lfloor n-1,0 \rfloor},$$

where

$$y_i := \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,n-1 \rceil})$$

and

$$z_i := \begin{cases} y_i & \text{for } i \in \lfloor n-1, k-1 \rfloor \cap \mathbb{N}_0, \\ \prod_{j \in \lceil i+1,k-2 \rceil} (z_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor k-2,i \rfloor} (y_j\mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) & \text{for } i \in \lfloor k-2, 0 \rfloor. \end{cases}$$

Now, we have to show that $z_i = y_i$ for all $i \in \lfloor n-1, 0 \rfloor$, $k \in [0, n+1]$. For $k \in \{n+1, 0\}$, this follows since H is a simplicial homotopy from $D_G S_G$ to $\mathrm{id}_{\operatorname{Diag} \mathrm{N}G}$ and since $S_G D_G S_G = S_G$. So we may assume that $k \in \lfloor n, 1 \rfloor$ and have to show that $z_i = y_i$ for every $i \in \lfloor k-2, 0 \rfloor$. But we have

$$\begin{aligned} y_i\mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil} &= \Big(\prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,n-1 \rceil}) \Big) \mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil} \\ &= \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}\mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,n-1 \rceil}\mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) \\ &= \prod_{j \in \lceil i+1,k-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}\mathrm{d}_{\lfloor k-1,j+1 \rfloor}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) \\ &\quad \cdot \prod_{j \in \lceil k,n-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,k-1 \rceil}\mathrm{s}_{\lceil k,j-1 \rceil}\mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) \\ &\quad \cdot \prod_{j \in \lfloor n-1,i \rfloor} (g_j\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,k-1 \rceil}\mathrm{s}_{\lceil k,n-1 \rceil}\mathrm{d}_{\lfloor k-1,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) \\ &= \prod_{j \in \lceil i+1,k-1 \rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{d}_{\lfloor i+k-1-j,i+1 \rfloor}\mathrm{s}_{\lceil i,j-1 \rceil}\mathrm{d}_{\lfloor j,i+1 \rfloor}\mathrm{s}_{\lceil i,k-2 \rceil}) \end{aligned}$$

$$\begin{aligned}
&\cdot \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, k-1 \rceil} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i+1, i+j-k \rceil} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&\cdot \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, k-1 \rceil} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i+1, i+n-k \rceil} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{d}_{\lfloor i+k-1-j, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_i \mathrm{s}_{\lceil i+1, i+j-k \rceil} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&\cdot \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_i \mathrm{s}_{\lceil i+1, i+n-k \rceil} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil}),
\end{aligned}$$

and this implies, by induction on $i \in \lfloor k-2, 0 \rfloor$, that

$$\begin{aligned}
z_i &= \prod_{j \in \lceil i+1, k-2 \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor k-2, i \rfloor} (y_j \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor k-2, i \rfloor} (y_j \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&= \prod_{j \in \lceil i+1, k-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor k-2, i+1 \rfloor} (y_j \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&\quad \cdot \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil}) \\
&= (\prod_{j \in \lceil i+1, k-2 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}))(y_{k-1}^{-1} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&\quad \cdot (\prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}))(\prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil})) \\
&= \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil}) = y_i
\end{aligned}$$

for all $i \in \lfloor k-2, 0 \rfloor$. □

Now we have to fix one of both notions. Since the diagonal nerve construction has a bisimplicial set as an intermediate result, we will define this to be the classifying simplicial set of a given simplicial group. The usage of this choice will be shown in the next section.

(4.33) Definition (classifying (bi)simplicial set of a simplicial group)**.** We let G be a simplicial group. We call $\mathrm{B}^{(2)}G := \mathrm{N}G$ the *classifying bisimplicial set* of G and $\mathrm{B}G := \mathrm{Diag}\,\mathrm{N}G$ the *classifying simplicial set* of G.

(4.34) Definition (homology and cohomology of simplicial groups)**.** We let G be a simplicial group, R be a commutative ring, M be an R-module and $n \in \mathbb{N}_0$ be a non-negative integer. The *n-th homology group* of G with coefficients in M over R is defined to be the n-th homology group of its classifying simplicial set, that is

$$\mathrm{H}_n(G, M; R) := \mathrm{H}_n(\mathrm{B}G, M; R).$$

Dually, we let

$$\mathrm{H}^n(G, M; R) := \mathrm{H}^n(\mathrm{B}G, M; R)$$

be the *n-th cohomology group* of G with coefficients in M over R. As in definition (2.18), we abbreviate

$$\begin{aligned}
&\mathrm{H}_n(G; R) := \mathrm{H}_n(G, R; R), \\
&\mathrm{H}_n(G, M) := \mathrm{H}_n(G, M; \mathbb{Z}), \\
&\mathrm{H}_n(G) := \mathrm{H}_n(G, \mathbb{Z}; \mathbb{Z}),
\end{aligned}$$

and

$$\mathrm{H}^n(G;R) := \mathrm{H}^n(G,R;R),$$
$$\mathrm{H}^n(G,M) := \mathrm{H}^n(G,M;\mathbb{Z}),$$
$$\mathrm{H}^n(G) := \mathrm{H}^n(G,\mathbb{Z};\mathbb{Z}).$$

We note that here, $\mathrm{H}_n(G,M;R)$ does *not* denote the homology of the underlying simplicial set of G; similarly for cohomology.

(4.35) Corollary. Suppose given a simplicial group G, a commutative ring R and an R-module M. Homology and cohomology of G can be computed as

$$\mathrm{H}_n(G,M;R) \cong \mathrm{H}_n(\overline{\mathrm{W}}G,M;R) \text{ resp. } \mathrm{H}^n(G,M;R) \cong \mathrm{H}^n(\overline{\mathrm{W}}G,M;R) \text{ for all } n \in \mathbb{N}_0.$$

Proof. Since $\mathrm{B}G = \operatorname{Diag}\mathrm{N}G \simeq \overline{\mathrm{W}}G$ by theorem (4.32), we have

$$\mathrm{H}_n(G,M;R) = \mathrm{H}_n(\mathrm{B}G,M;R) \cong \mathrm{H}_n(\overline{\mathrm{W}}G,M;R)$$

resp.

$$\mathrm{H}^n(G,M;R) = \mathrm{H}^n(\mathrm{B}G,M;R) \cong \mathrm{H}^n(\overline{\mathrm{W}}G,M;R)$$

for all $n \in \mathbb{N}_0$. □

§6 The Jardine spectral sequence

In this section we show a connection between the (co)homology of groups and the (co)homology of simplicial groups found by Jardine [19, Lemma 4.1.3].

(4.36) Theorem. We suppose given a simplicial group G, a commutative ring R and an R-module M.

(a) There exists a spectral sequence E with $E^1_{p,n-p} \cong \mathrm{H}_{n-p}(G_p,M;R)$ that converges to the homology group $\mathrm{H}_n(G,M;R)$ of the simplicial group G, where $p \in [0,n]$, $n \in \mathbb{N}_0$.

(b) There exists a spectral sequence E with $E_1^{p,n-p} \cong \mathrm{H}^{n-p}(G_p,M;R)$ that converges to the cohomology group $\mathrm{H}^n(G,M;R)$ of the simplicial group G, where $p \in [0,n]$, $n \in \mathbb{N}_0$.

Proof. We apply corollary (3.39) to $\mathrm{B}^{(2)}G$.

(a) For $n \in \mathbb{N}_0$, we have

$$\mathrm{H}_n(\operatorname{Diag}\mathrm{B}^{(2)}G,M;R) = \mathrm{H}_n(\mathrm{B}G,M;R) = \mathrm{H}_n(G,M;R),$$

and for $p \in [0,n]$, $n \in \mathbb{N}_0$, we obtain

$$\begin{aligned}\mathrm{H}_{n-p}(\mathrm{B}^{(2)}_{p,-}G,M;R) &= \mathrm{H}_{n-p}(\mathrm{N}_{p,-}G,M;R) = \mathrm{H}_{n-p}(\mathrm{N}G_p,M;R) = \mathrm{H}_{n-p}(\mathrm{B}G_p,M;R)\\ &= \mathrm{H}_{n-p}(G_p,M;R).\end{aligned}$$

(b) Dually. □

(4.37) Definition (Jardine spectral sequences)**.** We let G be a simplicial group. The spectral sequences exhibited in theorem (4.36) are called *Jardine spectral sequences* of G (in the case of homology resp. in the case of cohomology).

We obtain the following proposition as an immediate application of the Jardine spectral sequence in the case of cohomology.

(4.38) Proposition. We suppose given a simplicial group G such that G_n is finite for every $n \in \mathbb{N}_0$. Moreover, we let R be a commutative ring such that the additive group of R is a torsion-free abelian group. Then $\mathrm{H}^1(G;R) \cong 0$.

Proof. Since G_n is finite for all $n \in \mathbb{N}_0$, the first cohomology group $\mathrm{H}^1(G_n;R)$ is trivial for all $n \in \mathbb{N}_0$ (cf. [21, Aufgabe 49 (4)]). Hence $\mathrm{H}^0\mathrm{H}^1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}G;R)) \cong 0$. Furthermore, the "vertically" taken cohomology

$$\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}G;R)) \cong (R \xrightarrow{0} R \xrightarrow{\mathrm{id}} R \xrightarrow{0} R \xrightarrow{\mathrm{id}} \dots)$$

and thus $\mathrm{H}^1\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}G;R)) \cong 0$. This implies $\mathrm{H}^1(G;R) \cong 0$ by theorem (4.36). □

Chapter V

Crossed modules and categorical groups

In this chapter, we introduce the notion of a crossed module, which has been the starting point of this diploma thesis. We study some purely algebraic properties of crossed modules and introduce a second algebraic object, that of a categorical group. At the end, we give a proof of the Brown-Spencer theorem, which states that the categories of crossed modules resp. categorical groups are equivalent.

§1 Crossed Modules

A standard introduction is the survey [3].

(5.1) Definition (crossed modules and their morphisms)**.**

(a) A *crossed module* consists of a group G, a (left) G-group M and a group homomorphism $\mu\colon M \to G$, such that the following two axioms hold.

(CM1) We have $({}^{g}m)\mu = {}^{g}(m\mu)$ for all $m \in M$, $g \in G$ (that is, μ is a morphism of G-groups).

(CM2) We have ${}^{n\mu}m = {}^{n}m$ for all $m, n \in M$ (the so called *Peiffer identity*).

Here, the action of the elements of G on G resp. of M on M denotes in each case the conjugation. We call G resp. M the *group part* resp. the *module part* of the crossed module. The group homomorphism $\mu\colon M \to G$ is said to be the *structure morphism* of the crossed module.

Given a crossed module V with group part G, module part M and structure morphism μ, we write $\operatorname{Gp} V := G$, $\operatorname{Mp} V := M$ and $\upmu := \upmu^{V} := \mu$.

(b) We let V and W be crossed modules. A *morphism of crossed modules* between V and W is a pair of group homomorphisms $\varphi_0\colon \operatorname{Gp} V \to \operatorname{Gp} W$ and $\varphi_1\colon \operatorname{Mp} V \to \operatorname{Mp} W$ such that the diagram

$$\begin{array}{ccc} \operatorname{Mp} V & \xrightarrow{\upmu^{V}} & \operatorname{Gp} V \\ {\scriptstyle \varphi_1}\downarrow & & \downarrow{\scriptstyle \varphi_0} \\ \operatorname{Mp} W & \xrightarrow{\upmu^{W}} & \operatorname{Gp} W \end{array}$$

commutes, that is, $\varphi_1 \upmu^{W} = \upmu^{V} \varphi_0$, and such that $({}^{g}m)\varphi_1 = {}^{g\varphi_0}(m\varphi_1)$ holds for all $m \in \operatorname{Mp} V$, $g \in \operatorname{Gp} V$. The group homomorphism φ_0 resp. φ_1 is said to be the *group part* resp. the *module part* of the morphism of crossed modules.

Given a crossed module morphism $V \xrightarrow{\varphi} W$ with group part φ_0 and module part φ_1, we write $\operatorname{Gp} \varphi := \varphi_0$ and $\operatorname{Mp} \varphi := \varphi_1$.

Composition of morphisms of crossed modules is defined by the composition on the group parts and on the module parts.

(c) The *category of crossed modules* consisting of crossed modules as objects and morphisms of crossed modules as morphisms will be denoted by **CrMod**.

(5.2) Example.

(a) We let G be a group and $N \trianglelefteq G$ a normal subgroup. Then the inclusion $N \hookrightarrow G$ with the conjugation action of G on N is a crossed module.

(b) We suppose given a group H. The inner automorphism homomorphism $H \to \operatorname{Aut} H, h \mapsto (-)^h$, which assigns to every $h \in H$ the conjugation action of h on H, is a crossed module, where the action of $\operatorname{Aut} H$ on H is given by applying the inverse of an automorphism to the group elements of H.

(c) The trivial homomorphism $M \to G, m \mapsto 1$, where G is a group and M is an (abelian) G-module, yields a crossed module.

(d) Given a group G and a central extension E of G, the surjection $\pi \colon E \to G$ yields a crossed module, where the action of G on E is given by ${}^{f\pi}e := {}^{f}e$ for $e, f \in E$.

Proof.

(a) We denote the inclusion by $\iota \colon N \to G$. Then we have

$$({}^{g}n)\iota = {}^{g}n = {}^{g}(n\iota) \text{ for all } n \in N, g \in G,$$

and

$${}^{n'\iota}n = {}^{n'}n \text{ for all } n, n' \in N.$$

(b) The inner automorphism homomorphism of H is denoted by $\kappa \colon H \to \operatorname{Aut} H, h \mapsto (-)^h$. The automorphism group $\operatorname{Aut} H$ acts on H from the right by $h^{\alpha} = h\alpha$ and hence it acts on H from the left by ${}^{\alpha}h = h\alpha^{-1}$. This turns H into a $(\operatorname{Aut} H)$-group. We verify the crossed module axioms by elementwise argumentation. We have

$$\begin{aligned} x(({}^{\alpha}h)\kappa) &= x((h\alpha^{-1})\kappa) = x^{h\alpha^{-1}} = (h\alpha^{-1})^{-1}x(h\alpha^{-1}) = (h^{-1}\alpha^{-1})x(h\alpha^{-1}) = (h^{-1}(x\alpha)h)\alpha^{-1} \\ &= (x\alpha)^{h}\alpha^{-1} = x\alpha(h\kappa)\alpha^{-1} = x({}^{\alpha}(h\kappa)) \end{aligned}$$

for all $x \in H$, that is $({}^{\alpha}h)\kappa = {}^{\alpha}(h\kappa)$ for all $h \in H$, $\alpha \in \operatorname{Aut} H$, and

$${}^{k\kappa}h = h(k\kappa)^{-1} = h(k^{-1}\kappa) = h^{k^{-1}} = {}^{k}h$$

for all $h, k \in H$.

(c) We write $\mu \colon M \to G, m \mapsto 1$. Then we have

$$({}^{g}m)\mu = 1 = {}^{g}1 = {}^{g}(m\mu) \text{ for all } m \in M, g \in G,$$

and

$${}^{n\mu}m = {}^{1}m = m = nmn^{-1} = {}^{n}m \text{ for all } m, n \in M$$

since M is abelian.

(d) We let $f, f' \in E$ such that $f\pi = f'\pi$. Then $ff'^{-1} \in \operatorname{Ker} \pi \subseteq \operatorname{Z}(E)$ and hence

$${}^{f}e = {}^{ff'^{-1}f'}e = {}^{ff'^{-1}}({}^{f'}e) = {}^{f'}e \text{ for all } e \in E,$$

where $\operatorname{Z}(E)$ denotes the center of E. Since $\pi \colon E \to G$ is surjective, this implies that the definition ${}^{f\pi}e := {}^{f}e$ for $e, f \in E$ is well-defined. Moreover, since

$${}^{f\pi}({}^{f'\pi}e) = {}^{f}({}^{f'}e) = {}^{ff'}e = {}^{(ff')\pi}e = {}^{(f\pi)(f'\pi)}e$$

and

$${}^{1\pi}e = {}^{1}e = e$$

and

$${}^{f\pi}(ee') = {}^{f}(ee') = ({}^{f}e)({}^{f}e') = ({}^{f\pi}e)({}^{f\pi}e')$$

for all $e, e', f, f' \in E$, this defines a G-group action on E. Since the definition of this action is just the second crossed module axiom, it remains to show the first one. Indeed, we have

$${}^{f\pi}(e\pi) = ({}^{f}e)\pi = ({}^{f\pi}e)\pi \text{ for all } e, f \in E. \qquad \square$$

(5.3) Proposition (simple properties of crossed modules)**.** For a crossed module V the following identities hold.

(a) We have $\operatorname{Im}\mu^V \trianglelefteq \operatorname{Gp} V$.

(b) We have $\operatorname{Ker}\mu^V \leq \operatorname{Z}(\operatorname{Mp} V)$.

(c) The restricted action of $\operatorname{Im}\mu^V \trianglelefteq \operatorname{Gp} V$ on $\operatorname{Ker}\mu^V \trianglelefteq \operatorname{Mp} V$ is trivial, that is, an action of $\operatorname{Coker}\mu^V$ on $\operatorname{Ker}\mu^V$ is induced.

Proof.

(a) Given $h \in \operatorname{Im}\mu^V$, say, $h = m\mu^V$ for some $m \in \operatorname{Mp} V$, and $g \in \operatorname{Gp} V$, we get

$${}^{g}h = {}^{g}(m\mu^V) = ({}^{g}m)\mu^V \in \operatorname{Im}\mu^V,$$

and thus $\operatorname{Im}\mu^V$ is a normal subgroup in $\operatorname{Gp} V$.

(b) For $n \in \operatorname{Ker}\mu^V$ and $m \in \operatorname{Mp} V$, we have

$${}^{n}m = {}^{n\mu^V}m = {}^{1}m = m$$

and thus $nm = mn$. Hence $n \in \operatorname{Z}(\operatorname{Mp} V)$ and thus $\operatorname{Ker}\mu^V \leq \operatorname{Z}(\operatorname{Mp} V)$.

(c) We let $h \in \operatorname{Im}\mu^V$, say, $h = m\mu^V$ for some $m \in \operatorname{Mp} V$, and $n \in \operatorname{Ker}\mu^V$. Using (b), we obtain

$${}^{h}n = {}^{m\mu^V}n = {}^{m}n = n. \qquad \square$$

For some examples, where the group part resp. the module part of a crossed module are given by presentations with generators and relations, it would be hard to check the crossed module axioms for all elements of these groups. We will show that it is enough to verify the axioms for the generators in this case.

(5.4) Lemma. We let G be a group, M be a G-group and $\mu\colon M \to G$ be a group homomorphism.

(a) We let $g_1, g_2 \in G$ such that $({}^{g_1}m)\mu = {}^{g_1}(m\mu)$ and $({}^{g_2}m)\mu = {}^{g_2}(m\mu)$ for all $m \in M$. Then

$$({}^{g_1 g_2^{-1}}m)\mu = {}^{g_1 g_2^{-1}}(m\mu)$$

for all $m \in M$.

(b) We let $n_1, n_2 \in M$ such that ${}^{n_1\mu}m = {}^{n_1}m$ and ${}^{n_2\mu}m = {}^{n_2}m$ for all $m \in M$. Then

$${}^{(n_1 n_2^{-1})\mu}m = {}^{n_1 n_2^{-1}}m$$

for all $m \in M$.

Proof.

(a) We have

$$({}^{g_2^{-1}}m)\mu = {}^{g_2^{-1}g_2}(({}^{g_2^{-1}}m)\mu) = {}^{g_2^{-1}}({}^{g_2}(({}^{g_2^{-1}}m)\mu)) = {}^{g_2^{-1}}(({}^{g_2}({}^{g_2^{-1}}m))\mu) = {}^{g_2^{-1}}(({}^{g_2 g_2^{-1}}m)\mu) = {}^{g_2^{-1}}(m\mu)$$

and hence

$$({}^{g_1 g_2^{-1}}m)\mu = ({}^{g_1}({}^{g_2^{-1}}m))\mu = {}^{g_1}(({}^{g_2^{-1}}m)\mu) = {}^{g_1}({}^{g_2^{-1}}(m\mu)) = {}^{g_1 g_2^{-1}}(m\mu)$$

for all $m \in M$.

(b) We compute

$$\begin{aligned} {}^{n_2^{-1}\mu}m &= {}^{n_2^{-1}n_2}({}^{n_2^{-1}\mu}m) = {}^{n_2^{-1}}({}^{n_2}({}^{n_2^{-1}\mu}m)) = {}^{n_2^{-1}}({}^{n_2\mu}({}^{n_2^{-1}\mu}m)) = {}^{n_2^{-1}}({}^{(n_2\mu)(n_2^{-1}\mu)}m) \\ &= {}^{n_2^{-1}}({}^{(n_2 n_2^{-1})\mu}m) = {}^{n_2^{-1}}m \end{aligned}$$

and thus we obtain

$${}^{(n_1 n_2^{-1})\mu}m = {}^{(n_1\mu)(n_2^{-1}\mu)}m = {}^{n_1\mu}({}^{n_2^{-1}\mu}m) = {}^{n_1\mu}({}^{n_2^{-1}}m) = {}^{n_1}({}^{n_2^{-1}}m) = {}^{n_1 n_2^{-1}}m$$

for all $m \in M$. □

(5.5) Corollary. We let G be a group, M be a G-group and $\mu\colon M \to G$ be a group homomorphism. Furthermore, we let $A \subseteq G$ and $B \subseteq M$ be generating subsets, that is, $G = \langle A \rangle$ and $M = \langle B \rangle$. If we have $({}^a b)\mu = {}^a(b\mu)$ and ${}^{c\mu}b = {}^c b$ for all $a \in A$, $b, c \in B$, then there exists a crossed module V with $\operatorname{Gp} V = G$, $\operatorname{Mp} V = M$ and $\mu^V = \mu$.

Proof. We suppose given $a \in A$ and $c \in B$. Since $M = \langle B \rangle$ and $({}^a b)\mu = {}^a(b\mu)$ and ${}^{c\mu}b = {}^c b$ for all $b \in B$, the composites ${}^a(-)\,\mu$ and $\mu\,{}^a(-)$ coincide as maps on the generating subset B. Hence, they coincide as group homomorphism on $M = \langle B \rangle$, that is, we have

$$({}^a m)\mu = {}^a(m\mu) \text{ and } {}^{c\mu}m = {}^c m \text{ for } m \in M.$$

But $G = \langle A \rangle$ and $M = \langle B \rangle$ then imply

$$({}^g m)\mu = {}^g(m\mu) \text{ and } {}^{n\mu}m = {}^n m \text{ for all } g \in G, m, n \in M$$

by lemma (5.4)(a) and (b), that is, there is a crossed module V with $\operatorname{Gp} V = G$, $\operatorname{Mp} V = M$ and $\mu^V = \mu$. □

(5.6) Example. We let $G := \langle a \mid a^4 = 1 \rangle \cong C_4$ and $M := \langle b \mid b^4 = 1 \rangle \cong C_4$ be cyclic groups of order 4. Since $(a^2)^4 = 1$, there exists a group homomorphism $\mu\colon M \to G$ given on the generator $b \in M$ by $b\mu := a^2$. Moreover, M has a non-trivial group automorphism of order 2 sending b to the other element of M that has order 4, namely b^{-1}. Thus M is a G-group via ${}^a b := b^{-1}$. We show that these data deliver a crossed module V with module part $\operatorname{Mp} V = M$, group part $\operatorname{Gp} V = G$ and structure morphism $\mu^V = \mu$. Indeed, we have

$$({}^a b)\mu = b^{-1}\mu = (b\mu)^{-1} = (a^2)^{-1} = a^2 = b\mu = {}^a(b\mu)$$

and

$${}^{b\mu}b = {}^{a^2}b = {}^a({}^a b) = {}^a(b^{-1}) = ({}^a b)^{-1} = (b^{-1})^{-1} = b = {}^b b.$$

In the following, we denote the isomorphy type of this crossed module by $C_{4,4}^{2,-1}$.

(5.7) Notation. If V is a crossed module, then the module part $\operatorname{Mp} V$ acts on the group part $\operatorname{Gp} V$ by

$$mg := (m\mu^V)g \text{ and } gm := g(m\mu^V) \text{ for all } m \in \operatorname{Mp} V, g \in \operatorname{Gp} V.$$

We get for example

$${}^{(mg)}n = {}^{(m\mu^V)g}n = {}^{m\mu^V}({}^g n) = {}^m({}^g n)$$

and

$$gm = g(m\mu^V) = {}^g(m\mu^V)g = (({}^g m)\mu^V)g = ({}^g m)g$$

for $m, n \in \operatorname{Mp} V$, $g \in \operatorname{Gp} V$. Also note that $(mg)n = m(gn)$ for $m, n \in \operatorname{Mp} V$, $g \in \operatorname{Gp} V$.
Moreover, given another crossed module W and a morphism of crossed modules $V \xrightarrow{\varphi} W$, we often write $m\varphi$ and $g\varphi$ instead of $m(\operatorname{Mp}\varphi)$ and $g(\operatorname{Gp}\varphi)$. Using this, we get

$$\begin{aligned}(mg)\varphi &= (mg)(\operatorname{Gp}\varphi) = ((m\mu^V)g)(\operatorname{Gp}\varphi) = ((m\mu^V)(\operatorname{Gp}\varphi))(g(\operatorname{Gp}\varphi)) = (m(\mu^V(\operatorname{Gp}\varphi)))(g(\operatorname{Gp}\varphi)) \\ &= (m((\operatorname{Mp}\varphi)\mu^W))(g(\operatorname{Gp}\varphi)) = ((m(\operatorname{Mp}\varphi))\mu^W)(g(\operatorname{Gp}\varphi)) = (m(\operatorname{Mp}\varphi))(g(\operatorname{Gp}\varphi)) = (m\varphi)(g\varphi)\end{aligned}$$

for $m \in \operatorname{Mp} V$, $g \in \operatorname{Gp} V$.

§2 Categorical groups

We introduce the concept of an categorical group (cf. [16], [24]).

(5.8) Definition (categorical groups and their morphisms)**.**

(a) A *categorical group* is a (small) category C, such that $\operatorname{Ob} C$ and $\operatorname{Mor} C$ are groups and such that the multiplication maps $\mathrm{m}^{\operatorname{Ob} C}$ on $\operatorname{Ob} C$ and $\mathrm{m}^{\operatorname{Mor} C}$ on $\operatorname{Mor} C$ give a functor $C \times C \xrightarrow{\mathrm{m}^C} C$.

(b) We let C and D be categorical groups. A *categorical group homomorphism* is a functor $C \xrightarrow{\varphi} D$ such that $\operatorname{Ob}\varphi$ and $\operatorname{Mor}\varphi$ are group homomorphisms.

Composition of categorical group homomorphisms is given by the ordinary composition of functors.

(c) The *category of categorical groups* consisting of categorical groups as objects and categorical group homomorphisms as morphisms will be denoted by **cGrp**.

Given categorical groups C and D and a categorical group homomorphism $C \xrightarrow{\varphi} D$, we often abbreviate $o\varphi := o(\operatorname{Ob}\varphi)$ for $o \in \operatorname{Ob} C$ and $m\varphi := m(\operatorname{Mor}\varphi)$ for $m \in \operatorname{Mor} C$.

(5.9) Lemma. We let C be a categorical group.

(a) The source map $\mathrm{s}^C\colon \operatorname{Mor} C \to \operatorname{Ob} C$, the target map $\mathrm{t}^C\colon \operatorname{Mor} C \to \operatorname{Ob} C$, the identity map $\mathrm{e}^C\colon \operatorname{Ob} C \to \operatorname{Mor} C$ and the composition map $\mathrm{c}^C\colon \operatorname{Mor} C \,{}_{\mathrm{t}}\!\times_{\mathrm{s}} \operatorname{Mor} C \to \operatorname{Mor} C$ of C are group homomorphisms.

(b) The maps arising from the neutral resp. inverse elements $\mathrm{n}^{\operatorname{Ob} C}$ resp. $\mathrm{i}^{\operatorname{Ob} C}$ in $\operatorname{Ob} C$ and $\mathrm{n}^{\operatorname{Mor} C}$ resp. $\mathrm{i}^{\operatorname{Mor} C}$ in $\operatorname{Mor} C$ yield functors $C^{\times 0} \xrightarrow{\mathrm{n}^C} C$ resp. $C \xrightarrow{\mathrm{i}^C} C$.

Proof.

(a) Since $C \times C \xrightarrow{\mathrm{m}^C} C$ is a functor, we have

$$\mathrm{m}^{\operatorname{Mor} C}\mathrm{s}^C = (\operatorname{Mor}\mathrm{m}^C)\mathrm{s}^C = \mathrm{s}^{C\times C}(\operatorname{Ob}\mathrm{m}^C) = (\mathrm{s}^C \times \mathrm{s}^C)\mathrm{m}^{\operatorname{Ob} C}$$

and

$$\mathrm{m}^{\operatorname{Mor} C}\mathrm{t}^C = (\operatorname{Mor}\mathrm{m}^C)\mathrm{t}^C = \mathrm{t}^{C\times C}(\operatorname{Ob}\mathrm{m}^C) = (\mathrm{t}^C \times \mathrm{t}^C)\mathrm{m}^{\operatorname{Ob} C}$$

as well as

$$\mathrm{m}^{\operatorname{Ob} C}\mathrm{e}^C = (\operatorname{Ob}\mathrm{m}^C)\mathrm{e}^C = \mathrm{e}^{C\times C}(\operatorname{Mor}\mathrm{m}^C) = (\mathrm{e}^C \times \mathrm{e}^C)\mathrm{m}^{\operatorname{Mor} C}.$$

By considering the canonical isomorphism

$$\alpha\colon (\operatorname{Mor} C \,{}_{\mathrm{t}}\!\times_{\mathrm{s}} \operatorname{Mor} C) \times (\operatorname{Mor} C \,{}_{\mathrm{t}}\!\times_{\mathrm{s}} \operatorname{Mor} C) \to (\operatorname{Mor} C \times \operatorname{Mor} C) \,{}_{\mathrm{t}}\!\times_{\mathrm{s}} (\operatorname{Mor} C \times \operatorname{Mor} C),$$

we also have

$$\begin{aligned}\mathrm{m}^{\operatorname{Mor} C_{\mathrm{t}}\times_{\mathrm{s}}\operatorname{Mor} C}\,\mathrm{c}^C &= \alpha(\mathrm{m}^{\operatorname{Mor} C} \,{}_{\mathrm{t}}\!\times_{\mathrm{s}} \mathrm{m}^{\operatorname{Mor} C})\mathrm{c}^C = \alpha((\operatorname{Mor}\mathrm{m}^C) \,{}_{\mathrm{t}}\!\times_{\mathrm{s}} (\operatorname{Mor}\mathrm{m}^C))\mathrm{c}^C \\ &= \alpha\mathrm{c}^{C\times C}(\operatorname{Mor}\mathrm{m}^C) = (\mathrm{c}^C \times \mathrm{c}^C)(\operatorname{Mor}\mathrm{m}^C).\end{aligned}$$

Thus s^C, t^C, e^C and c^C are group homomorphisms.

(b) According to (a), the structure maps $\mathrm{s}^C\colon \operatorname{Mor} C \to \operatorname{Ob} C$, $\mathrm{t}^C\colon \operatorname{Mor} C \to \operatorname{Ob} C$, $\mathrm{e}^C\colon \operatorname{Ob} C \to \operatorname{Mor} C$ and $\mathrm{c}^C\colon \operatorname{Mor} C \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} \operatorname{Mor} C \to \operatorname{Mor} C$, which arise from the underlying category structure of C, are group homomorphisms. Hence we have

$$\begin{aligned}
&\mathrm{n}^{\operatorname{Mor} C}\mathrm{s}^C = (\mathrm{s}^C)^{\times 0}\mathrm{n}^{\operatorname{Ob} C} = \mathrm{s}^{C^{\times 0}}\mathrm{n}^{\operatorname{Ob} C},\\
&\mathrm{n}^{\operatorname{Mor} C}\mathrm{t}^C = (\mathrm{t}^C)^{\times 0}\mathrm{n}^{\operatorname{Ob} C} = \mathrm{t}^{C^{\times 0}}\mathrm{n}^{\operatorname{Ob} C},\\
&\mathrm{n}^{\operatorname{Ob} C}\mathrm{e}^C = (\mathrm{e}^C)^{\times 0}\mathrm{n}^{\operatorname{Ob} C} = \mathrm{e}^{C^{\times 0}}\mathrm{n}^{\operatorname{Mor} C}
\end{aligned}$$

and

$$(\mathrm{n}^{\operatorname{Mor} C} \,{}_{\mathrm{t}}{\times}_{\mathrm{s}}\, \mathrm{n}^{\operatorname{Mor} C})\mathrm{c}^C = \mathrm{n}^{\operatorname{Mor} C \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} \operatorname{Mor} C}\mathrm{c}^C = (\mathrm{c}^C)^{\times 0}\mathrm{n}^{\operatorname{Mor} C} = \mathrm{c}^{C^{\times 0}}\mathrm{n}^{\operatorname{Mor} C},$$

that is, n^C with $\operatorname{Ob}\mathrm{n}^C := \mathrm{n}^{\operatorname{Ob} C}$ and $\operatorname{Mor}\mathrm{n}^C := \mathrm{n}^{\operatorname{Mor} C}$ is a functor. Analogously we have

$$\begin{aligned}
&\mathrm{i}^{\operatorname{Mor} C}\mathrm{s}^C = \mathrm{s}^C\mathrm{i}^{\operatorname{Ob} C},\\
&\mathrm{i}^{\operatorname{Mor} C}\mathrm{t}^C = \mathrm{t}^C\mathrm{i}^{\operatorname{Ob} C},\\
&\mathrm{i}^{\operatorname{Ob} C}\mathrm{e}^C = \mathrm{e}^C\mathrm{i}^{\operatorname{Mor} C}
\end{aligned}$$

and

$$(\mathrm{i}^{\operatorname{Mor} C} \,{}_{\mathrm{t}}{\times}_{\mathrm{s}}\, \mathrm{i}^{\operatorname{Mor} C})\mathrm{c}^C = \mathrm{i}^{\operatorname{Mor} C \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} \operatorname{Mor} C}\mathrm{c}^C = \mathrm{c}^C\mathrm{i}^{\operatorname{Mor} C},$$

hence i^C with $\operatorname{Ob}\mathrm{i}^C := \mathrm{i}^{\operatorname{Ob} C}$ and $\operatorname{Mor}\mathrm{i}^C := \mathrm{i}^{\operatorname{Mor} C}$ is a functor. □

(5.10) Corollary. The categories **cGrp**, **GrpCat** and **CatGrp** are isomorphic.

Proof.

(a) We begin by constructing an isofunctor

$$\mathbf{cGrp} \xrightarrow{\texttt{GrpCat}} \mathbf{GrpCat}.$$

We let C be a categorical group. Then $\operatorname{Ob} C$ and $\operatorname{Mor} C$ are groups, that is, we have

$$\begin{aligned}
&(\mathrm{id}_G \times \mathrm{m}^G)\mathrm{m}^G = (\mathrm{m}^G \times \mathrm{id}_G)\mathrm{m}^G,\\
&(\mathrm{n}^G \times \mathrm{id}_G)\mathrm{m}^G = \mathrm{pr}_2 \text{ and } (\mathrm{id}_G \times \mathrm{n}^G)\mathrm{m}^G = \mathrm{pr}_1,\\
&(\,\mathrm{id}_G\ \mathrm{i}^G\,)\,\mathrm{m}^G = {*}\mathrm{n}^G = (\,\mathrm{i}^G\ \mathrm{id}_G\,)\,\mathrm{m}^G
\end{aligned}$$

for $G \in \{\operatorname{Ob} C, \operatorname{Mor} C\}$, cf. definition (1.26). Furthermore, by the definition of a categorical group (5.8), we have a functor m^C given by $\operatorname{Ob}\mathrm{m}^C = \mathrm{m}^{\operatorname{Ob} C}$ and $\operatorname{Mor}\mathrm{m}^C = \mathrm{m}^{\operatorname{Mor} C}$. Additionally, lemma (5.9)(b) tells us that there are functors n^C and i^C, where $\operatorname{Ob}\mathrm{n}^C = \mathrm{n}^{\operatorname{Ob} C}$, $\operatorname{Mor}\mathrm{n}^C = \mathrm{n}^{\operatorname{Mor} C}$, $\operatorname{Ob}\mathrm{i}^C = \mathrm{i}^{\operatorname{Ob} C}$ and $\operatorname{Mor}\mathrm{i}^C = \mathrm{i}^{\operatorname{Mor} C}$. This implies

$$\begin{aligned}
&(\mathrm{id}_C \times \mathrm{m}^C)\mathrm{m}^C = (\mathrm{m}^C \times \mathrm{id}_C)\mathrm{m}^C,\\
&(\mathrm{n}^C \times \mathrm{id}_C)\mathrm{m}^C = \mathrm{pr}_2 \text{ and } (\mathrm{id}_C \times \mathrm{n}^C)\mathrm{m}^C = \mathrm{pr}_1,\\
&(\,\mathrm{id}_C\ \mathrm{i}^C\,)\,\mathrm{m}^C = {*}\mathrm{n}^C = (\,\mathrm{i}^C\ \mathrm{id}_C\,)\,\mathrm{m}^C,
\end{aligned}$$

that is, C together with the functors m^C, n^C and i^C is a group object in **Cat**. Further, given categorical groups C and D and a categorical group homomorphism $C \xrightarrow{\varphi} D$, we have group homomorphisms $\operatorname{Ob}\varphi$ and $\operatorname{Mor}\varphi$. Hence

$$\mathrm{m}^{\operatorname{Ob} C}(\operatorname{Ob}\varphi) = (\operatorname{Ob}\varphi \times \operatorname{Ob}\varphi)\mathrm{m}^{\operatorname{Ob} D} \text{ and } \mathrm{m}^{\operatorname{Mor} C}(\operatorname{Mor}\varphi) = (\operatorname{Mor}\varphi \times \operatorname{Mor}\varphi)\mathrm{m}^{\operatorname{Mor} D}.$$

But since m^C and φ are functors, we already get

$$\mathrm{m}^C\varphi = (\varphi \times \varphi)\mathrm{m}^D,$$

that is, φ is a group homomorphism in **Cat** by proposition (1.29). Altogether, we obtain a functor

$$\mathbf{cGrp} \xrightarrow{\texttt{GrpCat}} \mathbf{GrpCat}$$

given on a categorical group C by $\texttt{GrpCat}(C) := C$ and on a categorical group homomorphism φ by $\texttt{GrpCat}(\varphi) := \varphi$.

Conversely, given a group object C in **Cat**, we have functors m^C, n^C and i^C such that

$$(\mathrm{id}_C \times \mathrm{m}^C)\mathrm{m}^C = (\mathrm{m}^C \times \mathrm{id}_C)\mathrm{m}^C,$$
$$(\mathrm{n}^C \times \mathrm{id}_C)\mathrm{m}^C = \mathrm{pr}_2 \text{ and } (\mathrm{id}_C \times \mathrm{n}^C)\mathrm{m}^C = \mathrm{pr}_1,$$
$$(\,\mathrm{id}_C\ \mathrm{i}^C\,)\,\mathrm{m}^C = *\mathrm{n}^C = (\,\mathrm{i}^C\ \mathrm{id}_C\,)\,\mathrm{m}^C.$$

In particular, we have

$$(\mathrm{id}_{\mathrm{Ob}\,C} \times (\mathrm{Ob}\,\mathrm{m}^C))(\mathrm{Ob}\,\mathrm{m}^C) = ((\mathrm{Ob}\,\mathrm{m}^C) \times \mathrm{id}_{\mathrm{Ob}\,C})(\mathrm{Ob}\,\mathrm{m}^C),$$
$$((\mathrm{Ob}\,\mathrm{n}^C) \times \mathrm{id}_{\mathrm{Ob}\,C})(\mathrm{Ob}\,\mathrm{m}^C) = \mathrm{pr}_2 \text{ and } (\mathrm{id}_{\mathrm{Ob}\,C} \times (\mathrm{Ob}\,\mathrm{n}^C))(\mathrm{Ob}\,\mathrm{m}^C) = \mathrm{pr}_1,$$
$$(\,\mathrm{id}_{\mathrm{Ob}\,C}\ \mathrm{Ob}\,\mathrm{i}^C\,)\,(\mathrm{Ob}\,\mathrm{m}^C) = *(\mathrm{Ob}\,\mathrm{n}^C) = (\,\mathrm{Ob}\,\mathrm{i}^C\ \mathrm{id}_{\mathrm{Ob}\,C}\,)\,(\mathrm{Ob}\,\mathrm{m}^C)$$

and

$$(\mathrm{id}_{\mathrm{Mor}\,C} \times (\mathrm{Mor}\,\mathrm{m}^C))(\mathrm{Mor}\,\mathrm{m}^C) = ((\mathrm{Mor}\,\mathrm{m}^C) \times \mathrm{id}_{\mathrm{Mor}\,C})(\mathrm{Mor}\,\mathrm{m}^C),$$
$$((\mathrm{Mor}\,\mathrm{n}^C) \times \mathrm{id}_{\mathrm{Mor}\,C})(\mathrm{Mor}\,\mathrm{m}^C) = \mathrm{pr}_2 \text{ and } (\mathrm{id}_{\mathrm{Mor}\,C} \times (\mathrm{Mor}\,\mathrm{n}^C))(\mathrm{Mor}\,\mathrm{m}^C) = \mathrm{pr}_1,$$
$$(\,\mathrm{id}_{\mathrm{Mor}\,C}\ \mathrm{Mor}\,\mathrm{i}^C\,)\,(\mathrm{Mor}\,\mathrm{m}^C) = *(\mathrm{Mor}\,\mathrm{n}^C) = (\,\mathrm{Mor}\,\mathrm{i}^C\ \mathrm{id}_{\mathrm{Mor}\,C}\,)\,(\mathrm{Mor}\,\mathrm{m}^C),$$

that is, $\mathrm{Ob}\,C$ and $\mathrm{Mor}\,C$ are groups with $\mathrm{m}^{\mathrm{Ob}\,C} := \mathrm{Ob}\,\mathrm{m}^C$, $\mathrm{n}^{\mathrm{Ob}\,C} := \mathrm{Ob}\,\mathrm{n}^C$, $\mathrm{i}^{\mathrm{Ob}\,C} := \mathrm{Ob}\,\mathrm{i}^C$ and $\mathrm{m}^{\mathrm{Mor}\,C} := \mathrm{Mor}\,\mathrm{m}^C$, $\mathrm{n}^{\mathrm{Mor}\,C} := \mathrm{Mor}\,\mathrm{n}^C$, $\mathrm{i}^{\mathrm{Mor}\,C} := \mathrm{Mor}\,\mathrm{i}^C$. Hence the underlying category of C together with the group structures on $\mathrm{Ob}\,C$ and $\mathrm{Mor}\,C$ and the functor m^C is a categorical group. Moreover, given group objects C and D in **Cat** and a group homomorphism $C \xrightarrow{\varphi} D$ in **Cat**, we have a functor φ such that

$$\mathrm{m}^C\varphi = (\varphi \times \varphi)\mathrm{m}^D.$$

Then in particular

$$\mathrm{m}^{\mathrm{Ob}\,C}(\mathrm{Ob}\,\varphi) = (\mathrm{Ob}\,\mathrm{m}^C)(\mathrm{Ob}\,\varphi) = (\mathrm{Ob}\,\varphi \times \mathrm{Ob}\,\varphi)(\mathrm{Ob}\,\mathrm{m}^D) = (\mathrm{Ob}\,\varphi \times \mathrm{Ob}\,\varphi)\mathrm{m}^{\mathrm{Ob}\,D}$$

and

$$\mathrm{m}^{\mathrm{Mor}\,C}(\mathrm{Mor}\,\varphi) = (\mathrm{Mor}\,\mathrm{m}^C)(\mathrm{Mor}\,\varphi) = (\mathrm{Mor}\,\varphi \times \mathrm{Mor}\,\varphi)(\mathrm{Mor}\,\mathrm{m}^D) = (\mathrm{Mor}\,\varphi \times \mathrm{Mor}\,\varphi)\mathrm{m}^{\mathrm{Mor}\,D},$$

that is, the maps $\mathrm{Ob}\,\varphi$ and $\mathrm{Mor}\,\varphi$ are group homomorphisms. Altogether, the functor $\texttt{GrpCat}$ is invertible with inverse

$$\mathbf{GrpCat} \xrightarrow{\texttt{cGrp}} \mathbf{cGrp},$$

where $\texttt{cGrp}$ is given on a group object C in **Cat** by $\texttt{cGrp}(C) := C$ and on a group homomorphism φ in **Cat** by $\texttt{cGrp}(\varphi) := \varphi$.

(b) As above, we construct an isofunctor

$$\mathbf{cGrp} \xrightarrow{\texttt{CatGrp}} \mathbf{CatGrp}.$$

We suppose given a categorical group C. Then in particular C is a category such that $\mathrm{Ob}\,C$ and $\mathrm{Mor}\,C$ are groups. According to lemma (5.9)(a), the categorical structure maps s^C, t^C, e^C and c^C are group homomorphisms. Now having a commutative diagram in **Grp** just means having a commutative diagram in **Set**, where all maps are group homomorphisms. Therefore, the groups $\mathrm{Ob}\,C$ and $\mathrm{Mor}\,C$ together with the group homomorphisms s^C, t^C, e^C, c^C define a category object C in **Grp**. Additionally, every categorical group homomorphism $C \xrightarrow{\varphi} D$ between categorical groups C and D is a functor such that $\operatorname{Ob}\varphi$ and $\operatorname{Mor}\varphi$ are group homomorphisms, that is, a functor in **Grp**. Thus we have a functor

$$\mathbf{cGrp} \xrightarrow{\texttt{CatGrp}} \mathbf{CatGrp}$$

given on a categorical group C by $\texttt{CatGrp}(C) := C$ and on a categorical group homomorphism φ by $\texttt{CatGrp}(\varphi) := \varphi$.

Let us conversely assume that we have a category object C in **Grp**. Then C is in particular a category object in **Set**, that is, an ordinary category. Since the structure maps $\mathrm{s}^C\colon \operatorname{Mor} C \to \operatorname{Ob} C$, $\mathrm{t}^C\colon \operatorname{Mor} C \to \operatorname{Ob} C$, $\mathrm{e}^C\colon \operatorname{Ob} C \to \operatorname{Mor} C$ and $\mathrm{c}^C\colon \operatorname{Mor} C \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} \operatorname{Mor} C \to \operatorname{Mor} C$ are group homomorphisms, we have

$$\begin{aligned}
&\mathrm{m}^{\operatorname{Mor} C}\mathrm{s}^C = (\mathrm{s}^C \times \mathrm{s}^C)\mathrm{m}^{\operatorname{Ob} C} = \mathrm{s}^{C\times C}\mathrm{m}^{\operatorname{Ob} C},\\
&\mathrm{m}^{\operatorname{Mor} C}\mathrm{t}^C = (\mathrm{t}^C \times \mathrm{t}^C)\mathrm{m}^{\operatorname{Ob} C} = \mathrm{t}^{C\times C}\mathrm{m}^{\operatorname{Ob} C},\\
&\mathrm{m}^{\operatorname{Ob} C}\mathrm{e}^C = (\mathrm{e}^C \times \mathrm{e}^C)\mathrm{m}^{\operatorname{Mor} C} = \mathrm{e}^{C\times C}\mathrm{m}^{\operatorname{Ob} C}.
\end{aligned}$$

By considering the canonical isomorphism

$$\alpha\colon (\operatorname{Mor} C \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} \operatorname{Mor} C) \times (\operatorname{Mor} C \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} \operatorname{Mor} C) \to (\operatorname{Mor} C \times \operatorname{Mor} C) \,{}_{\mathrm{t}}{\times}_{\mathrm{s}} (\operatorname{Mor} C \times \operatorname{Mor} C),$$

we also get

$$(\mathrm{m}^{\operatorname{Mor} C} \,{}_{\mathrm{t}}{\times}_{\mathrm{s}}\, \mathrm{m}^{\operatorname{Mor} C})\mathrm{c}^C = \alpha\mathrm{m}^{\operatorname{Mor} C_{\mathrm{t}}\times_{\mathrm{s}}\operatorname{Mor} C}\mathrm{c}^C = \alpha(\mathrm{c}^C \times \mathrm{c}^C)\mathrm{m}^{\operatorname{Mor} C} = \mathrm{c}^{C\times C}\mathrm{m}^{\operatorname{Mor} C}.$$

Hence we have a functor m^C defined by $\operatorname{Ob}\mathrm{m}^C := \mathrm{m}^{\operatorname{Ob} C}$ and $\operatorname{Mor}\mathrm{m}^C := \mathrm{m}^{\operatorname{Mor} C}$. Thus C is a categorical group. Additionally, every functor $C \xrightarrow{\varphi} D$ in **Grp** between category objects C and D in **Grp** is an ordinary functor, where $\operatorname{Ob}\varphi$ and $\operatorname{Mor}\varphi$ are group homomorphisms, that is, a categorical group homomorphism. Hence we have shown that `CatGrp` is an isofunctor with inverse

$$\mathbf{CatGrp} \xrightarrow{\texttt{cGrp}} \mathbf{cGrp},$$

where `cGrp` is given on a category object C in **Grp** by $\texttt{cGrp}(C) := C$ and on a functor φ in **Grp** by $\texttt{cGrp}(\varphi) := \varphi$. □

(5.11) Convention. In the following, we will often identify **cGrp**, **GrpCat** and **CatGrp** along the isofunctors given in corollary (5.10).

(5.12) Proposition. We let C be a categorical group.

(a) The composition in C is given by

$$(m,n)\mathrm{c} = m(m\mathrm{te})^{-1}n = m(n\mathrm{se})^{-1}n = n(m\mathrm{te})^{-1}m = n(n\mathrm{se})^{-1}m$$

for all composable morphisms $m, n \in \operatorname{Mor} C$.

(b) Every morphism m in C is an isomorphism. Its inverse is given by $(m\mathrm{te})m^{-1}(m\mathrm{se})$.

(c) We have $[\operatorname{Ker}\mathrm{t}, \operatorname{Ker}\mathrm{s}] \cong 1$.

Proof.

(a) We let $m, n \in \operatorname{Mor} C$ be composable morphisms, that is, such that $m\mathrm{t} = n\mathrm{s}$ holds. This condition implies that it suffices to show the equality of the first and the second resp. of the first and the last term. But since c and e are group homomorphisms, we get

$$(m,n)\mathrm{c} = (m\cdot 1, 1\cdot n)\mathrm{c} = (m(1\mathrm{e}), (m\mathrm{te})(m\mathrm{te})^{-1}n)\mathrm{c} = (m, m\mathrm{te})\mathrm{c}\,(1\mathrm{e}, (m\mathrm{te})^{-1}n)\mathrm{c} = m(m\mathrm{te})^{-1}n$$

and analogously

$$(m,n)\mathrm{c} = (1\cdot m, n\cdot 1)\mathrm{c} = ((n\mathrm{se})(n\mathrm{se})^{-1}m, n(1\mathrm{e}))\mathrm{c} = (n\mathrm{se}, n)\mathrm{c}\,((n\mathrm{se})^{-1}m, 1\mathrm{e})\mathrm{c} = n(n\mathrm{se})^{-1}m.$$

(b) We suppose given a morphism $m \in \operatorname{Mor} C$. Since

$$((m\mathrm{te})m^{-1}(m\mathrm{se}))\mathrm{s} = (m\mathrm{t})(m^{-1}\mathrm{s})(m\mathrm{s}) = m\mathrm{t}$$

and

$$((m\mathrm{te})m^{-1}(m\mathrm{se}))\mathrm{t} = (m\mathrm{t})(m^{-1}\mathrm{t})(m\mathrm{s}) = m\mathrm{s},$$

the morphisms m and $(m\mathrm{te})m^{-1}(m\mathrm{se})$ are composable in both directions. With (a) we compute

$$(m, (m\mathrm{te})m^{-1}(m\mathrm{se}))\mathrm{c} = m(m\mathrm{te})^{-1}(m\mathrm{te})m^{-1}(m\mathrm{se}) = m\mathrm{se}$$

and

$$((m\mathrm{te})m^{-1}(m\mathrm{se}), m)\mathrm{c} = (m\mathrm{te})m^{-1}(m\mathrm{se})(m\mathrm{se})^{-1}m = m\mathrm{te},$$

that is, $(m\mathrm{te})m^{-1}(m\mathrm{se})$ is the inverse of m with respect to the composition c.

(c) We let $m \in \operatorname{Ker} \mathrm{t}$ and $n \in \operatorname{Ker} \mathrm{s}$ be given. Then we have $m\mathrm{t} = 1 = n\mathrm{s}$, that is, (m, n) is a pair of composable morphisms in C. According to (a), it follows that

$$mn = m(m\mathrm{te})^{-1}n = n(m\mathrm{te})^{-1}n = nm.$$

Thus $[m, n] = 1$ and since m and n were chosen arbitrary we get $[\operatorname{Ker} \mathrm{t}, \operatorname{Ker} \mathrm{s}] = \{1\}$. □

(5.13) Corollary. The underlying category of a categorical group is a groupoid.

Proof. This follows from proposition (5.12)(b). □

(5.14) Lemma. We let O, M be groups and $s\colon M \to O$, $t\colon M \to O$ be retractions with common coretraction $e\colon O \to M$. If $[\operatorname{Ker} s, \operatorname{Ker} t] = \{1\}$, then there exists a categorical group C with $\operatorname{Ob} C := O$, $\operatorname{Mor} C := M$, and categorical structure maps $\mathrm{s}^C = s$, $\mathrm{t}^C = t$, $\mathrm{e}^C = e$.

Proof. For elements $m, n \in M$ with $mt = ns$ we define their composite

$$(m, n)c := m(mte)^{-1}n = m(nse)^{-1}n.$$

Then $c\colon M\, {}_t\times_s^O\, M \to M$ is a group homomorphism since

$$\begin{aligned}((m, n)(m', n'))c &= (mm', nn')c = (mm')((mm')te)^{-1}(nn') = mm'(m'te)^{-1}(mte)^{-1}nn' \\ &= m(m'(m'te)^{-1})((nse)^{-1}n)n' = m((nse)^{-1}n)(m'(m'te)^{-1})n' = (m, n)c\,(m', n')c\end{aligned}$$

for all $m, n, m', n' \in M$ with $mt = ns$ and $m't = n's$. Now we have to verify that these data fulfill the category axioms given in definition (1.24):

(STC) We have

$$(m, n)cs = (m(nse)^{-1}n)s = (ms)(nses)^{-1}(ns) = (ms)(ns)^{-1}(ns) = ms$$

and

$$(m, n)ct = (m(mte)^{-1}n)t = (mt)(mtet)^{-1}(nt) = (mt)(mt)^{-1}(nt) = nt$$

for all $m, n \in M$ with $mt = ns$.

(STI) The identities $es = et = \mathrm{id}_{G_0}$ are given by assumption.

(AC) The composition is associative since

$$\begin{aligned}(k, (m, n)c)c &= (k, m(mte)^{-1}n)c = k(kte)^{-1}m(mte)^{-1}n = k(mse)^{-1}m(nse)^{-1}n = (k(mse)^{-1}m, n)c \\ &= ((k, m)c, n)c\end{aligned}$$

for all $k, m, n \in M$ with $kt = ms$ and $mt = ns$.

(CI) We have

$$(mse, m)c = (mse)(mse)^{-1}m = m$$

and

$$(m, mte)c = m(mte)^{-1}(mte) = m$$

for $m \in M$.

Thus C with $\operatorname{Ob} C := O$, $\operatorname{Mor} C := M$ and $\mathrm{s}^C := s$, $\mathrm{t}^C := t$, $\mathrm{e}^C := e$, $\mathrm{c}^C := c$ is a category object in **Grp**. □

(5.15) Lemma. We let C and D be categorical groups and we let $\varphi_0\colon \operatorname{Ob} C \to \operatorname{Ob} D$ and $\varphi_1\colon \operatorname{Mor} C \to \operatorname{Mor} D$ be group homomorphisms with $\varphi_1 \mathrm{s} = \mathrm{s}\varphi_0$, $\varphi_1 \mathrm{t} = \mathrm{t}\varphi_0$ and $\mathrm{e}\varphi_1 = \varphi_0 \mathrm{e}$. Then there exists a categorical group homomorphism $C \xrightarrow{\varphi} D$ with $\operatorname{Ob}\varphi = \varphi_0$ and $\operatorname{Mor}\varphi = \varphi_1$.

Proof. Since φ_0 and φ_1 interchange with s, t and e, it suffices to show the compatibility with the composition c. And indeed, proposition (5.12)(a) implies

$$\begin{aligned}(m, n)\mathrm{c}\varphi_1 &= (m(m\mathrm{te})^{-1}n)\varphi_1 = (m\varphi_1)((m\mathrm{te})^{-1}\varphi_1)(n\varphi_1) = (m\varphi_1)(m\mathrm{te}\varphi_1)^{-1}(n\varphi_1) \\ &= (m\varphi_1)(m\mathrm{t}\varphi_0\mathrm{e})^{-1}(n\varphi_1) = (m\varphi_1)(m\varphi_1\mathrm{te})^{-1}(n\varphi_1) = (m\varphi_1, n\varphi_1)\mathrm{c}\end{aligned}$$

for $m, n \in \operatorname{Mor} C$ with $m\mathrm{t} = n\mathrm{s}$. □

(5.16) Lemma. We let C and D be categorical groups and we let $C \xrightarrow{\varphi} D$ be a categorical group homomorphism such that $\operatorname{Ob}\varphi$ and $\operatorname{Mor}\varphi$ are group isomorphisms. Then φ is a categorical group isomorphism with $\operatorname{Ob}(\varphi^{-1}) = (\operatorname{Ob}\varphi)^{-1}$ and $\operatorname{Mor}(\varphi^{-1}) = (\operatorname{Mor}\varphi)^{-1}$.

Proof. Since $\operatorname{Ob}\varphi$ and $\operatorname{Mor}\varphi$ are group isomorphisms, their inverses $\psi_0 := (\operatorname{Ob}\varphi)^{-1}$ and $\psi_1 := (\operatorname{Mor}\varphi)^{-1}$ are group homomorphisms, too. Furthermore, the fact that φ is a categorical group homomorphism implies

$$(\operatorname{Mor}\varphi)\mathrm{s} = \mathrm{s}(\operatorname{Ob}\varphi), (\operatorname{Mor}\varphi)\mathrm{t} = \mathrm{t}(\operatorname{Ob}\varphi) \text{ and } \mathrm{e}(\operatorname{Mor}\varphi) = (\operatorname{Ob}\varphi)\mathrm{e}$$

and hence

$$\psi_1\mathrm{s} = \mathrm{s}\psi_0, \psi_1\mathrm{t} = \mathrm{t}\psi_0 \text{ and } \mathrm{e}\psi_1 = \psi_0\mathrm{e}.$$

Due to lemma (5.15), there exists a categorical group homomorphism $D \xrightarrow{\psi} C$ with $\operatorname{Ob}\psi = \psi_0$ and $\operatorname{Mor}\psi = \psi_1$. But then we have

$$(\operatorname{Ob}\varphi)(\operatorname{Ob}\psi) = (\operatorname{Ob}\varphi)\psi_0 = (\operatorname{Ob}\varphi)(\operatorname{Ob}\varphi)^{-1} = \mathrm{id}_{\operatorname{Ob} C}$$

and

$$(\operatorname{Mor}\varphi)(\operatorname{Mor}\psi) = (\operatorname{Mor}\varphi)\psi_0 = (\operatorname{Mor}\varphi)(\operatorname{Mor}\varphi)^{-1} = \mathrm{id}_{\operatorname{Mor} C},$$

that is, $\varphi\psi = \mathrm{id}_C$, and analogously $\psi\varphi = \mathrm{id}_D$. Thus φ is invertible with inverse $\varphi^{-1} = \psi$. □

§3 The equivalence of crossed modules and categorical groups

Our aim is to show that the categories **CrMod** and **cGrp** are equivalent (cf. [16]).

(5.17) Convention. Given a crossed module V, the semidirect product $\operatorname{Mp} V \rtimes \operatorname{Gp} V$ is formed using the ation of $\operatorname{Gp} V$ on $\operatorname{Mp} V$ the crossed module provides. Hence we have $(m, g)(m', g') = (m\,{}^g m', gg')$ and $(m, g)^{-1} = ({}^{g^{-1}}(m^{-1}), g^{-1})$ for $(m, g), (m', g') \in \operatorname{Mp} V \rtimes \operatorname{Gp} V$. The identity in $\operatorname{Mp} V \rtimes \operatorname{Gp} V$ is given by $(1, 1)$.

(5.18) Remark. For every crossed module V we have a categorical group C, in which the objects and morphisms are given by

$$\operatorname{Ob} C := \operatorname{Gp} V \text{ and } \operatorname{Mor} C := \operatorname{Mp} V \rtimes \operatorname{Gp} V,$$

source object, target object and identity morphisms are given by

$$(m,g)\mathrm{s} := mg \text{ and } (m,g)\mathrm{t} := g \text{ for } (m,g) \in \operatorname{Mor} C \text{ and } g\mathrm{e} := (1,g) \text{ for } g \in \operatorname{Ob} C,$$

the group of composable morphisms is $\{((m_2, m_1 g), (m_1, g)) \in \operatorname{Mor} C \times \operatorname{Mor} C \mid m_1, m_2 \in \operatorname{Mp} V, g \in \operatorname{Gp} V\}$ and the composition in C is given by

$$((m_2, m_1 g), (m_1, g))\mathrm{c} := (m_2 m_1, g) \text{ for } m_1, m_2 \in \operatorname{Mp} V, g \in \operatorname{Gp} V.$$

Proof. Since $\operatorname{Ob} C = \operatorname{Gp} V$ and $\operatorname{Mor} C = \operatorname{Mp} V \rtimes \operatorname{Gp} V$ are groups, we begin the proof by showing that s, t and e are group homomorphisms. We have

$$((m,g),(m',g'))\mathrm{s} = (m\,{}^g m', gg')\mathrm{s} = m\,{}^g m' gg' = mgm'g' = (m,g)\mathrm{s}(m',g')\mathrm{s}$$

and

$$((m,g),(m',g'))\mathrm{t} = (m\,{}^g m', gg')\mathrm{t} = gg' = (m,g)\mathrm{t}(m',g')\mathrm{t}$$

for $(m,g), (m',g') \in \operatorname{Mor} C$ as well as

$$(gg')\mathrm{e} = (1, gg') = (1,g)(1,g') = (g\mathrm{e})(g'\mathrm{e})$$

for $g, g' \in \operatorname{Ob} C$. The group of composable morphisms can be computed as follows.

$$\begin{aligned}\operatorname{Mor} C\ {}_{\mathrm{t}}\!\times_{\mathrm{s}} \operatorname{Mor} C &= \{((m_2,g_2),(m_1,g_1)) \in \operatorname{Mor} C \times \operatorname{Mor} C \mid (m_2,g_2)\mathrm{t} = (m_1,g_1)\mathrm{s}\} \\ &= \{((m_2,g_2),(m_1,g_1)) \in \operatorname{Mor} C \times \operatorname{Mor} C \mid g_2 = m_1 g_1\} \\ &= \{((m_2, m_1 g_1),(m_1,g_1)) \in \operatorname{Mor} C \times \operatorname{Mor} C \mid m_1, m_2 \in \operatorname{Mp} V, g_1 \in \operatorname{Gp} V\}.\end{aligned}$$

Thus c is a group homomorphism since

$$\begin{aligned}&(((m_2, m_1 g),(m_1,g))((m_2', m_1' g'),(m_1',g')))\mathrm{c} = (((m_2, m_1 g)(m_2', m_1' g'),(m_1,g)(m_1',g')))\mathrm{c} \\ &= ((m_2\,{}^{m_1 g} m_2', m_1 g m_1' g'),(m_1\,{}^g m_1', gg'))\mathrm{c} = ((m_2\,{}^{m_1 g} m_2', m_1\,{}^g m_1' gg'),(m_1\,{}^g m_1', gg'))\mathrm{c} \\ &= (m_2\,{}^{m_1 g} m_2' m_1\,{}^g m_1', gg') = (m_2 m_1\,{}^g m_2'\,{}^g m_1', gg') = (m_2 m_1\,{}^g(m_2' m_1'), gg') \\ &= (m_2 m_1, g)(m_2' m_1', g') = (((m_2, m_1 g),(m_1,g))\mathrm{c})(((m_2', m_1' g'),(m_1',g'))\mathrm{c})\end{aligned}$$

for all $m_1, m_2, m_1', m_2' \in \operatorname{Mp} V,\ g, g' \in \operatorname{Gp} V$.
At last, we have to show that C satisfies the axioms for a category object given in definition (1.24).

(STI) We have

$$g\mathrm{es} = (1,g)\mathrm{s} = g \text{ and } g\mathrm{et} = (1,g)\mathrm{t} = g \text{ for } g \in \operatorname{Ob} C,$$

that is, s and t are retractions with common coretraction e.

(STC) Given a pair of composable morphisms $((m_2, m_1 g),(m_1,g))$ in C, we have

$$((m_2, m_1 g),(m_1,g))\mathrm{cs} = (m_2 m_1, g)\mathrm{s} = m_2 m_1 g = (m_2, m_1 g)\mathrm{s}$$

and

$$((m_2, m_1 g),(m_1,g))\mathrm{ct} = (m_2 m_1, g)\mathrm{t} = g = (m_1, g)\mathrm{t}.$$

(AC) We have

$$\begin{aligned}&((m_3, m_2 m_1 g),((m_2, m_1 g),(m_1,g))\mathrm{c})\mathrm{c} = ((m_3, m_2 m_1 g),(m_2 m_1, g))\mathrm{c} = (m_3 m_2 m_1, g) \\ &= ((m_3 m_2, m_1 g),(m_1,g))\mathrm{c} = (((m_3, m_2 m_1 g),(m_2, m_1 g))\mathrm{c},(m_1,g))\mathrm{c}\end{aligned}$$

for $m_1, m_2, m_3 \in \operatorname{Mp} V,\ g \in \operatorname{Gp} V$, that is, the composition in C is associative.

(CI) We get

$$((m,g)\mathrm{se},(m,g))\mathrm{c} = (mg\mathrm{e},(m,g))\mathrm{c} = ((1,mg),(m,g))\mathrm{c} = (m,g),$$

and

$$((m,g),(m,g)\mathrm{te})\mathrm{c} = ((m,g),g\mathrm{e})\mathrm{c} = ((m,g),(1,g))\mathrm{c} = (m,g)$$

for $(m,g) \in \operatorname{Mor} C$.

Thus C is a category object in **Grp**. □

(5.19) Definition (associated categorical group)**.** We let V be a crossed module. The categorical group C given as in remark (5.18) by

$$\begin{aligned}
&\operatorname{Ob} C := \operatorname{Gp} V \text{ and } \operatorname{Mor} C := \operatorname{Mp} V \rtimes \operatorname{Gp} V,\\
&(m,g)\mathrm{s} := mg \text{ and } (m,g)\mathrm{t} := g \text{ for } (m,g) \in \operatorname{Mor} C,\\
&g\mathrm{e} := (1,g) \text{ for all } g \in \operatorname{Ob} C \text{ and}\\
&((m_2,m_1 g),(m_1,g))\mathrm{c} := (m_2 m_1, g) \text{ for } m_1,m_2 \in \operatorname{Mp} V, g \in \operatorname{Gp} V,
\end{aligned}$$

will be called the *associated categorical group* to V and will be denoted by $\mathtt{cGrp}(V) := C$.

(5.20) Proposition.

(a) We let V and W be crossed modules. Given a morphism of crossed modules $V \xrightarrow{\varphi} W$, we have an induced morphism

$$\mathtt{cGrp}(V) \xrightarrow{\mathtt{cGrp}(\varphi)} \mathtt{cGrp}(W)$$

given on the objects by $\operatorname{Ob} \mathtt{cGrp}(\varphi) := \operatorname{Gp} \varphi$ and on the morphisms by $\operatorname{Mor} \mathtt{cGrp}(\varphi) := (\operatorname{Mp} \varphi) \rtimes (\operatorname{Gp} \varphi)$, where $(m,g)((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)) := (m\varphi, g\varphi)$ for $(m,g) \in \operatorname{Mor} \mathtt{cGrp}(V)$.

(b) The construction in (a) yields a functor

$$\mathbf{CrMod} \xrightarrow{\mathtt{cGrp}} \mathbf{cGrp}.$$

Proof.

(a) We have

$$\begin{aligned}
((m,g)(m',g'))((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)) &= (m\,{}^{g}m', gg')((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)) = ((m\,{}^{g}m')\varphi, (gg')\varphi)\\
&= ((m\varphi)(({}^{g}m')\varphi), (g\varphi)(g'\varphi)) = ((m\varphi)\,{}^{g\varphi}(m'\varphi), (g\varphi)(g'\varphi))\\
&= (m\varphi, g\varphi)(m'\varphi, g'\varphi)\\
&= (m,g)((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))(m',g')((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))
\end{aligned}$$

for $(m,g),(m',g') \in \operatorname{Mor} \mathtt{cGrp}(V) = \operatorname{Mp} V \rtimes \operatorname{Gp} V$, that is, the map $(\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)\colon \operatorname{Mor} \mathtt{cGrp}(V) \to \operatorname{Mor} \mathtt{cGrp}(W)$ is a group homomorphism.

Thus we have to show that the group homomorphisms $\operatorname{Gp}\varphi$ and $(\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)$ are compatible with s, t and e. Indeed, we obtain

$$(m,g)((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))\mathrm{s} = (m\varphi, g\varphi)\mathrm{s} = (m\varphi)(g\varphi) = (mg)\varphi = (m,g)\mathrm{s}(\operatorname{Gp}\varphi)$$

and

$$(m,g)((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))\mathrm{t} = (m\varphi, g\varphi)\mathrm{t} = g\varphi = (m,g)\mathrm{t}(\operatorname{Gp}\varphi)$$

as well as

$$(g\mathrm{e})((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)) = (1,g)((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi)) = (1, g\varphi) = g(\operatorname{Gp}\varphi)\mathrm{e}$$

for $m \in \operatorname{Mp} V$, $g \in \operatorname{Gp} V$.

(b) We let V, W, X be crossed modules and we let $V \xrightarrow{\varphi} W$ and $W \xrightarrow{\psi} X$ be morphisms of crossed modules. Then we have

$$\begin{aligned}(m, g)((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))((\operatorname{Mp}\psi) \rtimes (\operatorname{Gp}\psi)) &= (m\varphi, g\varphi)((\operatorname{Mp}\psi) \rtimes (\operatorname{Gp}\psi)) = (m\varphi\psi, g\varphi\psi) \\ &= (m, g)((\operatorname{Mp}(\varphi\psi)) \rtimes (\operatorname{Gp}(\varphi\psi)))\end{aligned}$$

and

$$(m, g)((\operatorname{Mp}\operatorname{id}_V) \rtimes (\operatorname{Gp}\operatorname{id}_V)) = (m\operatorname{id}_V, g\operatorname{id}_V) = (m, g)$$

for $(m, g) \in \operatorname{Mor}\mathtt{cGrp}(V)$. Thus

$$((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))((\operatorname{Mp}\psi) \rtimes (\operatorname{Gp}\psi)) = (\operatorname{Mp}(\varphi\psi)) \rtimes (\operatorname{Gp}(\varphi\psi))$$

and

$$(\operatorname{Mp}\operatorname{id}_V) \rtimes (\operatorname{Gp}\operatorname{id}_V) = \operatorname{id}_{\operatorname{Mp}V \rtimes \operatorname{Gp}V}.$$

Because the validity of the functor axioms on the objects follows since $\operatorname{Gp}(\varphi\psi) = (\operatorname{Gp}\varphi)(\operatorname{Gp}\psi)$ and $\operatorname{Gp}\operatorname{id}_V = \operatorname{id}_{\operatorname{Gp}V}$, we have a functor

$$\mathbf{CrMod} \xrightarrow{\mathtt{cGrp}} \mathbf{cGrp}. \qquad \square$$

(5.21) Example. We consider the crossed module $V \cong \mathrm{C}_{4,4}^{2,-1}$ introduced in example (5.6). Recall that it has group part $\operatorname{Gp}V = \langle a \mid a^4 = 1\rangle$, module part $\operatorname{Mp}V = \langle b \mid b^4 = 1\rangle$, structure morphism $\mu^V \colon \operatorname{Mp}V \to \operatorname{Gp}V, b \mapsto a^2$ and action ${}^a b = b^{-1}$. Its associated categorical group has the group of objects $\langle a\rangle$ and the group of morphisms $\langle b\rangle \rtimes \langle a\rangle$. The source object morphism is given by

$$(b, 1)\mathrm{s} = a^2 \text{ and } (1, a)\mathrm{s} = a,$$

while the target object morphism is given by

$$(b, 1)\mathrm{t} = 1 \text{ and } (1, a)\mathrm{t} = a.$$

The identity morphism of a is given by

$$a\mathrm{e} = (1, a).$$

(5.22) Remark. For every categorical group C there is a crossed module V with group part and module part given by

$$\operatorname{Gp}V := \operatorname{Ob}C \text{ and } \operatorname{Mp}V := \operatorname{Ker}\mathrm{t},$$

structure morphism $\mu^V := \mathrm{s}|_{\operatorname{Ker}\mathrm{t}}$, where the action of the group part on the module part is given by ${}^o m := {}^{o\mathrm{e}} m$ for $o \in \operatorname{Gp}V$, $m \in \operatorname{Mp}V$.

Proof. Since $\operatorname{Mor}C$ is a group and s is a group homomorphism, the kernel $\operatorname{Ker}\mathrm{t}$ is a group and $\mathrm{s}|_{\operatorname{Ker}\mathrm{t}}$ is a group homomorphism, too, and since the conjugation turns M into an M-group and e is a group homomorphism, we have a well defined $(\operatorname{Ob}C)$-group action on $\operatorname{Ker}\mathrm{t}$. It remains to show (CM1) and (CM2).

(CM1) We have

$$({}^{o\mathrm{e}} m)\mathrm{s} = {}^{o\mathrm{es}}(m\mathrm{s}) = {}^{o}(m\mathrm{s})$$

for $o \in \operatorname{Ob}C$ and all $m \in \operatorname{Mor}C$, and hence in particular

$$({}^{o\mathrm{e}} m)(\mathrm{s}|_{\operatorname{Ker}\mathrm{t}}) = {}^{o}(m(\mathrm{s}|_{\operatorname{Ker}\mathrm{t}}))$$

for $o \in \operatorname{Ob}C$, $m \in \operatorname{Ker}\mathrm{t}$.

(CM2) Since $(n^{-1}(n\mathrm{se}))\mathrm{s} = (n^{-1}\mathrm{s})(n\mathrm{s}) = 1$, we have $(n^{-1}(n\mathrm{se})) \in \operatorname{Ker}\mathrm{s}$ for all $n \in \operatorname{Mor} C$. Hence proposition (5.12)(c) implies $(n^{-1}(n\mathrm{se}))m = m(n^{-1}(n\mathrm{se}))$ for all $m \in \operatorname{Ker}\mathrm{t}$, $n \in \operatorname{Mor} C$, whence $(n\mathrm{se})m(n\mathrm{se})^{-1} = nmn^{-1}$, and therefore

$$^{n(\mathrm{s}|_{\operatorname{Ker}\mathrm{t}})}m = {}^{n\mathrm{s}}m = {}^{n\mathrm{se}}m = (n\mathrm{se})m(n\mathrm{se})^{-1} = nmn^{-1} = {}^{n}m$$

for all $m, n \in \operatorname{Ker}\mathrm{t}$.

Altogether, there is a well-defined crossed module V with $\operatorname{Gp} V = \operatorname{Ob} C$, $\operatorname{Mp} V = \operatorname{Ker}\mathrm{t}$, $\mu^V = \mathrm{s}|_{\operatorname{Ker}\mathrm{t}}$ and operation ${}^{o}m = {}^{o\mathrm{e}}m$ for $o \in \operatorname{Gp} V$, $m \in \operatorname{Mp} V$. □

(5.23) Definition (associated crossed module)**.** We let C be a categorical group. The crossed module V with group part and module part given as in remark (5.22) by

$$\operatorname{Gp} V := \operatorname{Ob} C \text{ and } \operatorname{Mp} V := \operatorname{Ker}\mathrm{t},$$

structure morphism $\mu^V := \mathrm{s}|_{\operatorname{Ker}\mathrm{t}}$ and action ${}^{o}m := {}^{o\mathrm{e}}m$ for all $o \in \operatorname{Gp} V$, $m \in \operatorname{Mp} V$, is called the *associated crossed module* to C and will be denoted by $\mathtt{CrMod}(C) := V$.

(5.24) Proposition.

(a) We let C, D be categorical groups. If $C \xrightarrow{\varphi} D$ is a categorical group homomorphism, then we have an induced morphism of crossed modules

$$\mathtt{CrMod}(C) \xrightarrow{\mathtt{CrMod}(\varphi)} \mathtt{CrMod}(D)$$

given on the group part by $\operatorname{Gp}\mathtt{CrMod}(\varphi) := \operatorname{Ob}\varphi$ and given on the module part by $\operatorname{Mp}\mathtt{CrMod}(\varphi) := (\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}^{\operatorname{Mp}\mathtt{CrMod}(D)}$.

(b) The construction in (a) yields a functor

$$\mathbf{cGrp} \xrightarrow{\mathtt{CrMod}} \mathbf{CrMod}.$$

Proof.

(a) Since the categorical group homomorphism φ is in particular a functor, we have $\mathrm{s}^C(\operatorname{Ob}\varphi) = (\operatorname{Mor}\varphi)\mathrm{s}^D$ and $\mathrm{t}^C(\operatorname{Ob}\varphi) = (\operatorname{Mor}\varphi)\mathrm{t}^D$. Hence we have

$$m(\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}\mathrm{t}^D = m(\operatorname{Mor}\varphi)\mathrm{t}^D = m\mathrm{t}^C(\operatorname{Ob}\varphi) = 1(\operatorname{Ob}\varphi) = 1$$

for all $m \in \operatorname{Mp}\mathtt{CrMod}(C) = \operatorname{Ker}\mathrm{t}^C$ and therefore $\operatorname{Im}(\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)} \subseteq \operatorname{Ker}\mathrm{t}^D = \operatorname{Mp}\mathtt{CrMod}(D)$. Additionally, we get a commutative diagram

$$\begin{array}{ccc} \operatorname{Ker}\mathrm{t}^C & \xrightarrow{\mu^{\mathtt{CrMod}(C)}} & \operatorname{Ob} C \\ {\scriptstyle (\operatorname{Mor}\varphi)|_{\operatorname{Ker}\mathrm{t}^C}^{\operatorname{Ker}\mathrm{t}^D}}\downarrow & & \downarrow{\scriptstyle \operatorname{Ob}\varphi} \\ \operatorname{Ker}\mathrm{t}^D & \xrightarrow{\mu^{\mathtt{CrMod}(D)}} & \operatorname{Ob} D \end{array}$$

because

$$\mu^{\mathtt{CrMod}(C)}(\operatorname{Ob}\varphi) = \mathrm{s}^C|_{\operatorname{Ker}\mathrm{t}^C}(\operatorname{Ob}\varphi) = (\operatorname{Mor}\varphi)|_{\operatorname{Ker}\mathrm{t}^C}^{\operatorname{Ker}\mathrm{t}^D}\mathrm{s}^D|_{\operatorname{Ker}\mathrm{t}^D} = (\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}^{\operatorname{Mp}\mathtt{CrMod}(D)}\mu^{\mathtt{CrMod}(D)}.$$

Finally, we have

$$\begin{aligned} ({}^{o}m)(\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}^{\operatorname{Mp}\mathtt{CrMod}(D)} &= ({}^{o\mathrm{e}}m)(\operatorname{Mor}\varphi) = {}^{o\mathrm{e}(\operatorname{Mor}\varphi)}(m(\operatorname{Mor}\varphi)) = {}^{o(\operatorname{Ob}\varphi)\mathrm{e}}(m(\operatorname{Mor}\varphi)) \\ &= {}^{o(\operatorname{Ob}\varphi)}(m(\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}^{\operatorname{Mp}\mathtt{CrMod}(D)}) \end{aligned}$$

for all $o \in \operatorname{Gp}\mathtt{CrMod}(C)$, $m \in \operatorname{Mp}\mathtt{CrMod}(C)$. Since $\operatorname{Ob}\varphi$ and $(\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}^{\operatorname{Mp}\mathtt{CrMod}(D)}$ are group homomorphisms, we have a morphism of crossed modules $\mathtt{CrMod}(\varphi)$ with group part $\operatorname{Gp}\mathtt{CrMod}(\varphi) = \operatorname{Ob}\varphi$ and module part $\operatorname{Mp}\mathtt{CrMod}(\varphi) = (\operatorname{Mor}\varphi)|_{\operatorname{Mp}\mathtt{CrMod}(C)}^{\operatorname{Mp}\mathtt{CrMod}(D)}$.

(b) We let C, D, E be categorical groups and $C \xrightarrow{\varphi} D$, $D \xrightarrow{\psi} E$ categorical group homomorphisms. Then we have

$$\operatorname{Gp}\mathtt{CrMod}(\varphi\psi) = \operatorname{Ob}(\varphi\psi) = (\operatorname{Ob}\varphi)(\operatorname{Ob}\psi) = (\operatorname{Gp}\mathtt{CrMod}(\varphi))(\operatorname{Gp}\mathtt{CrMod}(\psi))$$

and

$$\begin{aligned}\operatorname{Mp}\mathtt{CrMod}(\varphi\psi) &= (\operatorname{Mor}(\varphi\psi))|^{\operatorname{Mp}\mathtt{CrMod}(E)}_{\operatorname{Mp}\mathtt{CrMod}(C)} = ((\operatorname{Mor}\varphi)(\operatorname{Mor}\psi))|^{\operatorname{Mp}\mathtt{CrMod}(E)}_{\operatorname{Mp}\mathtt{CrMod}(C)} \\ &= (\operatorname{Mor}\varphi)|^{\operatorname{Mp}\mathtt{CrMod}(D)}_{\operatorname{Mp}\mathtt{CrMod}(C)}(\operatorname{Mor}\psi)|^{\operatorname{Mp}\mathtt{CrMod}(E)}_{\operatorname{Mp}\mathtt{CrMod}(D)} = (\operatorname{Mp}\mathtt{CrMod}(\varphi))(\operatorname{Mp}\mathtt{CrMod}(\psi))\end{aligned}$$

as well as

$$\operatorname{Gp}\mathtt{CrMod}(\mathrm{id}_C) = \operatorname{Ob}\mathrm{id}_C = \mathrm{id}_{\operatorname{Ob}C} = \mathrm{id}_{\operatorname{Gp}\mathtt{CrMod}(C)}$$

and

$$\operatorname{Mp}\mathtt{CrMod}(\mathrm{id}_C) = (\operatorname{Mor}\mathrm{id}_C)|^{\operatorname{Mp}\mathtt{CrMod}(C)}_{\operatorname{Mp}\mathtt{CrMod}(C)} = (\mathrm{id}_{\operatorname{Mor}C})|^{\operatorname{Mp}\mathtt{CrMod}(C)}_{\operatorname{Mp}\mathtt{CrMod}(C)} = \mathrm{id}_{\operatorname{Mp}\mathtt{CrMod}(C)},$$

that is, $\mathtt{CrMod}(\varphi\psi) = \mathtt{CrMod}(\varphi)\mathtt{CrMod}(\psi)$ and $\mathtt{CrMod}(\mathrm{id}_C) = \mathrm{id}_{\mathtt{CrMod}(C)}$. □

(5.25) Theorem (Brown-Spencer theorem)**.** The category **CrMod** of crossed modules and the category **cGrp** of categorical groups are equivalent.

Proof. We show that the functors

$$\mathbf{CrMod} \xrightarrow{\mathtt{cGrp}} \mathbf{cGrp} \text{ and } \mathbf{cGrp} \xrightarrow{\mathtt{CrMod}} \mathbf{CrMod}$$

are equivalences of categories, mutually inverse up to isomorphy.
First, we let $V \in \operatorname{Ob}\mathbf{CrMod}$ be a crossed module. Then we obtain

$$\operatorname{Gp}\mathtt{CrMod}(\mathtt{cGrp}(V)) = \operatorname{Ob}\mathtt{cGrp}(V) = \operatorname{Gp}V$$

and

$$\begin{aligned}\operatorname{Mp}\mathtt{CrMod}(\mathtt{cGrp}(V)) &= \operatorname{Ker}\mathrm{t} = \{(m,g) \in \operatorname{Mor}\mathtt{cGrp}(V) \mid (m,g)\mathrm{t} = 1\} \\ &= \{(m,g) \in \operatorname{Mp}V \rtimes \operatorname{Gp}V \mid g = 1\} = \operatorname{Mp}V \rtimes \{1\}.\end{aligned}$$

Furthermore, the structure morphism of $\mathtt{CrMod}(\mathtt{cGrp}(V))$ is given by

$$(m,1)\mu^{\mathtt{CrMod}(\mathtt{cGrp}(V))} = (m,1)\mathrm{s}|_{\operatorname{Ker}\mathrm{t}} = (m,1)\mathrm{s} = m\mu^V$$

for $(m,1) \in \operatorname{Mp}\mathtt{CrMod}(\mathtt{cGrp}(V))$. The action of $\operatorname{Gp}\mathtt{CrMod}(\mathtt{cGrp}(V))$ is given by

$${}^{g}(m,1) = {}^{g\mathrm{e}}(m,1) = (1,g)(m,1)(1,g^{-1}) = ({}^{g}m,1)$$

for $g \in \operatorname{Gp}\mathtt{CrMod}(\mathtt{cGrp}(V))$, $(m,1) \in \operatorname{Mp}\mathtt{CrMod}(\mathtt{cGrp}(V))$. Additionally, given a crossed module W and a morphism of crossed modules $V \xrightarrow{\varphi} W$, we have

$$\operatorname{Gp}\mathtt{CrMod}(\mathtt{cGrp}(\varphi)) = \operatorname{Ob}\mathtt{cGrp}(\varphi) = \operatorname{Gp}\varphi$$

and

$$\operatorname{Mp}\mathtt{CrMod}(\mathtt{cGrp}(\varphi)) = (\operatorname{Mor}\mathtt{cGrp}(\varphi))|^{\operatorname{Mp}\mathtt{CrMod}(\mathtt{cGrp}(W))}_{\operatorname{Mp}\mathtt{CrMod}(\mathtt{cGrp}(V))} = ((\operatorname{Mp}\varphi) \rtimes (\operatorname{Gp}\varphi))|^{\operatorname{Mp}W \rtimes \{1\}}_{\operatorname{Mp}V \rtimes \{1\}} = (\operatorname{Mp}\varphi) \rtimes \{1\}.$$

Thus we have

$$\mathtt{CrMod} \circ \mathtt{cGrp} \cong \mathrm{id}_{\mathbf{CrMod}}.$$

Conversely, we let C be a categorical group. The categorical group associated to the crossed module that is associated to C has objects

$$\operatorname{Ob}\mathtt{cGrp}(\mathtt{CrMod}(C)) = \operatorname{Gp}\mathtt{CrMod}(C) = \operatorname{Ob}C$$

and morphisms

$$\mathrm{Mor}\,\mathtt{cGrp}(\mathtt{CrMod}(C)) = (\mathrm{Mp}\,\mathtt{CrMod}(C)) \rtimes (\mathrm{Gp}(\mathtt{CrMod}(C))) = (\mathrm{Ker}\,\mathrm{t}) \rtimes (\mathrm{Ob}\,C).$$

The source object and the target object of a morphism $(m, o) \in \mathrm{Mor}\,\mathtt{cGrp}(\mathtt{CrMod}(C))$ are given by

$$(m, o)\mathrm{s} = m\mu^{\mathtt{CrMod}(C)}o = m\mathrm{s}|_{\mathrm{Ker}\,\mathrm{t}}o = (m\mathrm{s})o$$

and

$$(m, o)\mathrm{t} = o,$$

while the identity of an object $o \in \mathrm{Ob}\,\mathtt{cGrp}(\mathtt{CrMod}(C))$ has the form

$$o\mathrm{e} = (1, o).$$

We define maps $\mathrm{Ob}\,\alpha_C \colon \mathrm{Ob}\,C \to \mathrm{Ob}\,\mathtt{cGrp}(\mathtt{CrMod}(C))$ and $\mathrm{Mor}\,\alpha_C \colon \mathrm{Mor}\,C \to \mathrm{Mor}\,\mathtt{cGrp}(\mathtt{CrMod}(C))$ by setting $\mathrm{Ob}\,\alpha_C := \mathrm{id}_{\mathrm{Ob}\,C}$ and $m(\mathrm{Mor}\,\alpha_C) := (m(m\mathrm{te})^{-1}, m\mathrm{t})$ for all $m \in \mathrm{Mor}\,C$, which is well-defined since

$$(m(m\mathrm{te})^{-1})\mathrm{t} = (m\mathrm{t})(m\mathrm{tet})^{-1} = (m\mathrm{t})(m\mathrm{t})^{-1} = 1.$$

Then $\mathrm{Ob}\,\alpha_C$ is a group homomorphism and we have

$$\begin{aligned}(m(\mathrm{Mor}\,\alpha_C))(n(\mathrm{Mor}\,\alpha_C)) &= (m(m\mathrm{te})^{-1}, m\mathrm{t})(n(n\mathrm{te})^{-1}, n\mathrm{t}) = (m(m\mathrm{te})^{-1}\,{}^{m\mathrm{t}}(n(n\mathrm{te})^{-1}), (m\mathrm{t})(n\mathrm{t}))\\ &= (m(m\mathrm{te})^{-1}\,{}^{m\mathrm{te}}(n(n\mathrm{te})^{-1}), (m\mathrm{t})(n\mathrm{t}))\\ &= (m(m\mathrm{te})^{-1}(m\mathrm{te})n(n\mathrm{te})^{-1}(m\mathrm{te})^{-1}, (m\mathrm{t})(n\mathrm{t})) = (mn((mn)\mathrm{te})^{-1}, (mn)\mathrm{t})\\ &= (mn)(\mathrm{Mor}\,\alpha_C)\end{aligned}$$

for $m, n \in \mathrm{Mor}\,C$, that is, $\mathrm{Mor}\,\alpha_C$ is a group homomorphism, too. To prove that $\mathrm{Ob}\,\alpha_C$ and $\mathrm{Mor}\,\alpha_C$ yield a categorical group homomorphism

$$C \xrightarrow{\alpha_C} \mathtt{cGrp}(\mathtt{CrMod}(C))$$

it remains to show the compatibility with s, t and e. Indeed, we have

$$\begin{aligned}m(\mathrm{Mor}\,\alpha_C)\mathrm{s} &= (m(m\mathrm{te})^{-1}, m\mathrm{t})\mathrm{s} = (m(m\mathrm{te})^{-1})\mathrm{s}(m\mathrm{t}) = (m\mathrm{s})(m\mathrm{tes})^{-1}(m\mathrm{t}) = (m\mathrm{s})(m\mathrm{t})^{-1}(m\mathrm{t}) = m\mathrm{s}\\ &= m\mathrm{s}(\mathrm{Ob}\,\alpha_C).\end{aligned}$$

and

$$m(\mathrm{Mor}\,\alpha_C)\mathrm{t} = (m(m\mathrm{te})^{-1}, m\mathrm{t})\mathrm{t} = m\mathrm{t} = m\mathrm{t}(\mathrm{Ob}\,\alpha_C)$$

for $m \in \mathrm{Mor}\,C$ as well as

$$o\mathrm{e}(\mathrm{Mor}\,\alpha_C) = ((o\mathrm{e})(o\mathrm{ete})^{-1}, o\mathrm{et}) = ((o\mathrm{e})(o\mathrm{e})^{-1}, o) = (1, o) = o\mathrm{e} = o(\mathrm{Ob}\,\alpha_C)\mathrm{e}$$

for $o \in \mathrm{Ob}\,C$, that is, $(\mathrm{Mor}\,\alpha_C)\mathrm{s} = \mathrm{s}(\mathrm{Ob}\,\alpha_C)$, $(\mathrm{Mor}\,\alpha_C)\mathrm{t} = \mathrm{t}(\mathrm{Ob}\,\alpha_C)$ and $\mathrm{e}(\mathrm{Mor}\,\alpha_C) = (\mathrm{Ob}\,\alpha_C)\mathrm{e}$. Given a categorical group D and a categorical group homomorphism $C \xrightarrow{\varphi} D$, we obtain

$$(\mathrm{Ob}\,\alpha_C)(\mathrm{Ob}\,\mathtt{cGrp}(\mathtt{CrMod}(\varphi))) = \mathrm{id}_{\mathrm{Ob}\,C}(\mathrm{Gp}\,\mathtt{CrMod}(\varphi)) = \mathrm{Ob}\,\varphi = (\mathrm{Ob}\,\varphi)\mathrm{id}_{\mathrm{Ob}\,D} = (\mathrm{Ob}\,\varphi)(\mathrm{Ob}\,\alpha_D)$$

and

$$\begin{aligned}m(\mathrm{Mor}\,\alpha_C)(\mathrm{Mor}\,\mathtt{cGrp}(\mathtt{CrMod}(\varphi))) &= (m(m\mathrm{te})^{-1}, m\mathrm{t})((\mathrm{Mp}\,\mathtt{CrMod}(\varphi)) \rtimes (\mathrm{Gp}\,\mathtt{CrMod}(\varphi)))\\ &= ((m(m\mathrm{te})^{-1})(\mathrm{Mp}\,\mathtt{CrMod}(\varphi)), (m\mathrm{t})(\mathrm{Gp}\,\mathtt{CrMod}(\varphi)))\\ &= ((m(m\mathrm{te})^{-1})(\mathrm{Mor}\,\varphi)|_{\mathrm{Mp}\,\mathtt{CrMod}(C)}^{\mathrm{Mp}\,\mathtt{CrMod}(D)}, m\mathrm{t}(\mathrm{Ob}\,\varphi))\\ &= ((m(m\mathrm{te})^{-1})(\mathrm{Mor}\,\varphi), m\mathrm{t}(\mathrm{Ob}\,\varphi))\\ &= ((m(\mathrm{Mor}\,\varphi))((m(\mathrm{Mor}\,\varphi))\mathrm{te})^{-1}, m(\mathrm{Mor}\,\varphi)\mathrm{t}) = m(\mathrm{Mor}\,\varphi)(\mathrm{Mor}\,\alpha_D)\end{aligned}$$

for $m \in \operatorname{Mor} C$. Hence the diagram

$$\begin{array}{ccc} C & \xrightarrow{\alpha_C} & \texttt{cGrp}(\texttt{CrMod}(C)) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\texttt{cGrp}(\texttt{CrMod}(\varphi))} \\ D & \xrightarrow{\alpha_D} & \texttt{cGrp}(\texttt{CrMod}(D)) \end{array}$$

commutes and we have a natural transformation

$$\mathrm{id}_{\mathbf{cGrp}} \xrightarrow{\alpha} \texttt{cGrp} \circ \texttt{CrMod}.$$

To show that $\texttt{cGrp} \circ \texttt{CrMod} \cong \mathrm{id}_{\mathbf{cGrp}}$, it remains to show that each categorical group homomorphism $C \xrightarrow{\alpha_C} \texttt{cGrp}(\texttt{CrMod}(C))$ is an isomorphism. Thereto, we define categorical group homomorphisms

$$\texttt{cGrp}(\texttt{CrMod}(C)) \xrightarrow{\beta_C} C$$

by setting $\operatorname{Ob} \beta_C := \mathrm{id}_{\operatorname{Ob} C}$ and $(m, o)(\operatorname{Mor} \beta_C) := m(o\mathrm{e})$ for all $(m, o) \in \operatorname{Mor} \texttt{cGrp}(\texttt{CrMod}(C))$, $C \in \operatorname{Ob} \mathbf{cGrp}$. Then $\operatorname{Ob} \beta_C = (\operatorname{Ob} \alpha_C)^{-1}$ and

$$m(\operatorname{Mor} \alpha_C)(\operatorname{Mor} \beta_C) = (m(m\mathrm{te})^{-1}, m\mathrm{t})(\operatorname{Mor} \beta_C) = m(m\mathrm{te})^{-1}(m\mathrm{te}) = m$$

for all $m \in \operatorname{Mor} C$ and

$$\begin{aligned}(m, o)(\operatorname{Mor} \beta_C)(\operatorname{Mor} \alpha_C) &= (m(o\mathrm{e}))(\operatorname{Mor} \alpha_C) = ((m(o\mathrm{e}))((m(o\mathrm{e}))\mathrm{te})^{-1}, (m(o\mathrm{e}))\mathrm{t}) \\ &= (m(o\mathrm{e})((m\mathrm{te})(o\mathrm{ete}))^{-1}, (m\mathrm{t})(o\mathrm{et})) = (m(o\mathrm{e})(o\mathrm{e})^{-1}, o) = (m, o)\end{aligned}$$

for all $(m, o) \in \operatorname{Mor} \texttt{cGrp}(\texttt{CrMod}(C))$, that is, $\operatorname{Mor} \beta_C = (\operatorname{Mor} \alpha_C)^{-1}$. Hence each α_C is invertible with inverse β_C, that is, α is a natural isotransformation and we obtain

$$\texttt{cGrp} \circ \texttt{CrMod} \cong \mathrm{id}_{\mathbf{cGrp}}.$$

Thus the functors cGrp and CrMod are category equivalences between **CrMod** and **cGrp**, mutually inverse up to isomorphy. □

Chapter VI

Homology of crossed modules

In this last chapter we associate to a crossed module a simplicial group, its coskeleton. This construction proceeds by associating to the given crossed module its category object in **Grp** via Brown-Spencer and then applying the nerve functor to that category object. We start by studying this nerve functor and construct its left adjoint, which will give us later a left adjoint for the coskeleton, again via Brown-Spencer. At the end, the (co)homology groups of a crossed module will be defined as the (co)homology groups for its coskeleton, using the definition for simplicial groups as in (4.34). Moreover, we consider the Jardine spectral sequences of a crossed module and compute some examples.

§1 Fundamental groupoid and categorical nerve

(6.1) Definition (categorical nerve of a categorical group)**.** The *categorical nerve functor*

$$\mathbf{cGrp} \xrightarrow{\mathrm{N}_{\mathbf{Cat}}} \mathbf{sGrp}$$

is defined as the composition of the isofunctor $\mathbf{cGrp} \xrightarrow{\mathtt{CatGrp}} \mathbf{CatGrp}$ with the nerve functor of **CatGrp**. ([1])

$$\begin{array}{ccc} \mathbf{cGrp} & \xrightarrow{\mathrm{N}_{\mathbf{Cat}}} & \mathbf{sGrp} \\ {\scriptstyle\mathtt{CatGrp}}\downarrow & \nearrow_{\mathrm{N}} & \\ \mathbf{CatGrp} & & \end{array}$$

(6.2) Definition. The full subcategory of **sGrp** with objects G that fulfill $\mathrm{M}_n G \cong 1$ for $n \geq 2$, will be denoted by $\mathbf{sGrp}_{\lfloor 1,0 \rfloor}$.

In the following, we use the morphisms $\mathrm{c}_{\lfloor j_1, j_0 \rfloor}$ and t_j as defined in (1.30).

(6.3) Proposition. We let C be a categorical group.

(a) The categorical nerve $\mathrm{N}_{\mathbf{Cat}} C$ of C is given by $(\mathrm{N}_{\mathbf{Cat}} C)_n = (\operatorname{Mor} C)^{{}^{\mathrm{t}}\times_{\mathrm{s}} n}$ for all $n \in \mathbb{N}_0$ and

$$(\mathrm{N}_{\mathbf{Cat}} C)_\theta = \begin{Bmatrix} \mathrm{t}_{0\theta} & \text{if } m = 0, \\ (\mathrm{c}_{\lfloor (i+1)\theta, i\theta \rfloor})_{i \in \lfloor m-1, 0 \rfloor} & \text{if } m > 0 \end{Bmatrix}$$

for a morphism $\theta \in \mathbf{\Delta}([m],[n])$. The faces $\mathrm{d}_k \colon (\mathrm{N}_{\mathbf{Cat}} C)_n \to (\mathrm{N}_{\mathbf{Cat}} C)_{n-1}$ are given by

$$\mathrm{d}_k = \begin{cases} (\mathrm{pr}_j)_{j \in \lfloor n-1, 1 \rfloor} & \text{if } k = 0, \\ (\mathrm{pr}_j)_{j \in \lfloor n-1, k+1 \rfloor} \cup (\mathrm{c}_{\lfloor k+1, k-1 \rfloor}) \cup (\mathrm{pr}_j)_{j \in \lfloor k-2, 0 \rfloor} & \text{if } k \in [1, n-1], \\ (\mathrm{pr}_j)_{j \in \lfloor n-2, 0 \rfloor} & \text{if } k = n \end{cases}$$

[1]Analogously, the *groupical nerve functor* $\mathbf{cGrp} \xrightarrow{\mathrm{N}_{\mathbf{Grp}}} \mathbf{sCat}$ can be defined as the composition of the isofunctor $\mathbf{cGrp} \xrightarrow{\mathtt{GrpCat}} \mathbf{GrpCat}$ with the nerve functor from **GrpCat** to **sCat**.

for all $k \in [0, n]$, $n \in \mathbb{N}$, $n \geq 2$, resp.

$$\mathrm{d}_k = \begin{cases} \mathrm{s} & \text{if } k = 0, \\ \mathrm{t} & \text{if } k = 1 \end{cases}$$

for $n = 1$. The degeneracies $\mathrm{s}_k \colon (\mathrm{N}_{\mathbf{Cat}} C)_n \to (\mathrm{N}_{\mathbf{Cat}} C)_{n+1}$ are given by

$$\mathrm{s}_k = (\mathrm{pr}_j)_{j \in \lfloor n-1, k \rfloor} \cup (\mathrm{t}_k \mathrm{e}) \cup (\mathrm{pr}_j)_{j \in \lfloor k-1, 0 \rfloor}$$

for all $k \in [0, n]$, $n \in \mathbb{N}_0$.

(b) The Moore complex of $\mathrm{N}_{\mathbf{Cat}} C$ is given by

$$\mathrm{M}_n \mathrm{N}_{\mathbf{Cat}} C = \begin{cases} \mathrm{Ob}\, C & \text{for } n = 0, \\ (\mathrm{Ker}\,\mathrm{t})^{\times 1} & \text{for } n = 1, \\ \{1\} & \text{for } n \geq 2, \end{cases}$$

while the differential morphism $\mathrm{M}_1 \mathrm{N}_{\mathbf{Cat}} C \xrightarrow{\partial} \mathrm{M}_0 \mathrm{N}_{\mathbf{Cat}} C$ is given by

$$(m)\partial = m\mathrm{s} \text{ for all } (m) \in \mathrm{M}_1 \mathrm{N}_{\mathbf{Cat}} C.$$

In particular, $\mathrm{N}_{\mathbf{Cat}}$ takes values in $\mathbf{sGrp}_{\lfloor 1,0 \rfloor}$.

Proof.

(a) This follows from definition (1.32) and proposition (1.33).

(b) We have

$$\mathrm{M}_0 \mathrm{N}_{\mathbf{Cat}} C = (\mathrm{N}_{\mathbf{Cat}} C)_0 = (\mathrm{Mor}\, C)^{{}_{\mathrm{t}}\times_{\mathrm{s}} 0} = \mathrm{Ob}\, C$$

and

$$\begin{aligned} \mathrm{M}_1 \mathrm{N}_{\mathbf{Cat}} C &= \mathrm{Ker}\, \mathrm{d}_1 = \{(m_0) \in (\mathrm{N}_{\mathbf{Cat}} C)_1 \mid (m_0)\mathrm{d}_1 = 1\} = \{(m_0) \in (\mathrm{Mor}\, C)^{{}_{\mathrm{t}}\times_{\mathrm{s}}^{\mathrm{Ob}\, C} 1} \mid (m_0 \mathrm{t}) = 1\} \\ &= (\mathrm{Ker}\, \mathrm{t})^{\times 1}. \end{aligned}$$

For $n \geq 2$, we suppose given an element $(m_i)_{i \in \lfloor n-1, 0 \rfloor} \in \mathrm{M}_n \mathrm{N}_{\mathbf{Cat}} C$. Then we have

$$1 = (m_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_n = (m_i)_{i \in \lfloor n-2, 0 \rfloor}$$

and

$$1 = (m_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_1 = (m_{n-1}, \dots, m_2, (m_1, m_0)\mathrm{c}) = (m_{n-1}, \dots, m_2, m_1 (m_0 \mathrm{se})^{-1} m_0),$$

cf. proposition (5.12)(a). For $n \geq 3$, we see directly from the second equation that $m_{n-1} = 1$; for $n = 2$ we have

$$1 = m_1 (m_0 \mathrm{se})^{-1} m_0 = m_1 (1\mathrm{se})^{-1} 1 = m_1 = m_{n-1}.$$

Thus we have $(m_i)_{i \in \lfloor n-1, 0 \rfloor} = 1$ in each case and hence $\mathrm{M}_n \mathrm{N}_{\mathbf{Cat}} C = \{1\}$ for $n \geq 2$.

The differential morphism $\mathrm{M}_1 \mathrm{N}_{\mathbf{Cat}} C \xrightarrow{\partial} \mathrm{M}_0 \mathrm{N}_{\mathbf{Cat}} C$ is given by

$$(m)\partial = (m)\mathrm{d}_0 = m\mathrm{s} \text{ for } (m) \in \mathrm{M}_1 \mathrm{N}_{\mathbf{Cat}} C.$$ □

(6.4) Example. We suppose given a categorical group C. An element of $(\mathrm{N}_{\mathbf{Cat}} C)_3 = (\mathrm{Mor}\, C) {}_{\mathrm{t}}\times_{\mathrm{s}} (\mathrm{Mor}\, C) {}_{\mathrm{t}}\times_{\mathrm{s}} (\mathrm{Mor}\, C)$ is a tuple $(m_2, m_1, m_0) \in (\mathrm{Mor}\, C) \times (\mathrm{Mor}\, C) \times (\mathrm{Mor}\, C)$ such that $m_2 \mathrm{t} = m_1 \mathrm{s}$ and $m_1 \mathrm{t} = m_0 \mathrm{s}$. We write $o_0 := m_0 \mathrm{t}$, $o_1 := m_1 \mathrm{t}$, $o_2 := m_2 \mathrm{t}$ and $o_3 := m_2 \mathrm{s}$.

$$o_3 \xrightarrow{m_2} o_2 \xrightarrow{m_1} o_1 \xrightarrow{m_0} o_0$$

Its images under the faces are given by

$$(m_2, m_1, m_0)\mathrm{d}_0 = (m_2, m_1),$$
$$(m_2, m_1, m_0)\mathrm{d}_1 = (m_2, (m_1, m_0)\mathrm{c}) = (m_2, m_1(o_1\mathrm{e})^{-1}m_0),$$
$$(m_2, m_1, m_0)\mathrm{d}_2 = ((m_2, m_1)\mathrm{c}, m_0) = (m_2(o_2\mathrm{e})^{-1}m_1, m_0),$$
$$(m_2, m_1, m_0)\mathrm{d}_3 = (m_1, m_0),$$

its images under the degeneracies by

$$(m_2, m_1, m_0)\mathrm{s}_0 = (m_2, m_1, m_0, o_0\mathrm{e}),$$
$$(m_2, m_1, m_0)\mathrm{s}_1 = (m_2, m_1, o_1\mathrm{e}, m_0),$$
$$(m_2, m_1, m_0)\mathrm{s}_2 = (m_2, o_2\mathrm{e}, m_1, m_0),$$
$$(m_2, m_1, m_0)\mathrm{s}_3 = (o_3\mathrm{e}, m_2, m_1, m_0).$$

We suppose $[4] \xrightarrow{\theta} [3]$ to be given by $0\theta := 0$, $1\theta := 0$, $2\theta := 2$, $3\theta := 2$, $4\theta := 3$, and $[0] \xrightarrow{\rho} [3]$ to be given by $0\rho := 2$. Then we have

$$(m_2, m_1, m_0)(\mathrm{N}_{\mathbf{Cat}}C)_\theta = (m_2, o_2\mathrm{e}, (m_1, m_0)\mathrm{c}, o_0\mathrm{e}) = (m_2, o_2\mathrm{e}, m_1(o_1\mathrm{e})^{-1}m_0, o_0\mathrm{e})$$

and

$$(m_2, m_1, m_0)(\mathrm{N}_{\mathbf{Cat}}C)_\rho = o_2.$$

We want to construct a left adjoint for the categorical nerve $\mathrm{N}_{\mathbf{Cat}}$ (cf. [4]).

(6.5) Remark. For every simplicial group G there exists a categorical group $\mathrm{F}G$ with group of objects $\operatorname{Ob}\mathrm{F}G = G_0$, group of morphisms $\operatorname{Mor}\mathrm{F}G = G_1/\mathrm{B}_1\mathrm{M}G$ and where the categorical structure maps s, t and e are induced by d_0, d_1 and s_1 respectively:

$$\mathrm{s}\colon \operatorname{Mor}\mathrm{F}G \to \operatorname{Ob}\mathrm{F}G, g_1\mathrm{B}_1\mathrm{M}G \mapsto g_1\mathrm{d}_0,$$
$$\mathrm{t}\colon \operatorname{Mor}\mathrm{F}G \to \operatorname{Ob}\mathrm{F}G, g_1\mathrm{B}_1\mathrm{M}G \mapsto g_1\mathrm{d}_1,$$
$$\mathrm{e}\colon \operatorname{Ob}\mathrm{F}G \to \operatorname{Mor}\mathrm{F}G, g_0 \mapsto (g_0\mathrm{s}_0)\mathrm{B}_1\mathrm{M}G.$$

Proof. According to lemma (4.4), $\mathrm{B}_1\mathrm{M}G$ is a normal subgroup of G_1 and so $G_1/\mathrm{B}_1\mathrm{M}G$ is a well-defined group. Since $\mathrm{B}_1\mathrm{M}G \subseteq \mathrm{Z}_1\mathrm{M}G = (\operatorname{Ker}\mathrm{d}_0) \cap (\operatorname{Ker}\mathrm{d}_1)$, we obtain induced group homomorphisms $\mathrm{s}\colon G_1/\mathrm{B}_1\mathrm{M}G \to G_0, g_1\mathrm{B}_1\mathrm{M}G \to g_1\mathrm{d}_0$ and $\mathrm{t}\colon G_1/\mathrm{B}_1\mathrm{M}G \to G_0, g_1\mathrm{B}_1\mathrm{M}G \to g_1\mathrm{d}_1$. We define $\mathrm{e}\colon G_0 \to G_1/\mathrm{B}_1\mathrm{M}G, g_0 \mapsto (g_0\mathrm{s}_0)\mathrm{B}_1\mathrm{M}G$. As composition of s_0 and the canonical epimorphism $G_1 \to G_1/\mathrm{B}_1\mathrm{M}G$, this is obviously a group homomorphism with

$$g_0\mathrm{es} = (g_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G)\mathrm{s} = g_0\mathrm{s}_0\mathrm{d}_0 = g_0$$

and

$$g_0\mathrm{et} = (g_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G)\mathrm{t} = g_0\mathrm{s}_0\mathrm{d}_1 = g_0$$

for every $g_0 \in G_0$. Thus s and t are retractions with common coretraction e. Since $[\operatorname{Ker}\mathrm{d}_0, \operatorname{Ker}\mathrm{d}_1] \subseteq \mathrm{B}_1\mathrm{M}G$ by lemma (4.7), we have $[\operatorname{Ker}\mathrm{s}, \operatorname{Ker}\mathrm{t}] = \{1\}$, and lemma (5.14) implies that C is a category object in **Grp** with $\operatorname{Ob}C = G_0$, $\operatorname{Mor}C = G_1/\mathrm{B}_1\mathrm{M}G$ and s, t, e defined as above. □

(6.6) Definition (fundamental groupoid)**.** We let G be a simplicial group. The categorical group $\mathrm{F}G$ given as in remark (6.5) with objects $\operatorname{Ob}\mathrm{F}G = G_0$, morphisms $\operatorname{Mor}\mathrm{F}G = G_1/\mathrm{B}_1\mathrm{M}G$ and categorical structure maps

$$\mathrm{s}\colon \operatorname{Mor}\mathrm{F}G \to \operatorname{Ob}\mathrm{F}G, g_1\mathrm{B}_1\mathrm{M}G \mapsto g_1\mathrm{d}_0,$$
$$\mathrm{t}\colon \operatorname{Mor}\mathrm{F}G \to \operatorname{Ob}\mathrm{F}G, g_1\mathrm{B}_1\mathrm{M}G \mapsto g_1\mathrm{d}_1,$$
$$\mathrm{e}\colon \operatorname{Ob}\mathrm{F}G \to \operatorname{Mor}\mathrm{F}G, g_0 \mapsto (g_0\mathrm{s}_0)\mathrm{B}_1\mathrm{M}G,$$

is called the *fundamental groupoid* of G.

(6.7) Proposition.

(a) If G and H are simplicial groups and $G \xrightarrow{\varphi} H$ is a simplicial group homomorphism, then we have an induced categorical group homomorphism

$$\mathrm{F}G \xrightarrow{\mathrm{F}\varphi} \mathrm{F}H$$

on the fundamental groupoids given on the objects by $\operatorname{Ob}\mathrm{F}\varphi = \varphi_0$ and on the morphisms by

$$\operatorname{Mor}\mathrm{F}\varphi\colon \operatorname{Mor}\mathrm{F}G \to \operatorname{Mor}\mathrm{F}H, g_1\mathrm{B}_1\mathrm{M}G \mapsto (g_1\varphi_1)\mathrm{B}_1\mathrm{M}H.$$

(b) The construction in (a) yields a functor

$$\mathbf{sGrp} \xrightarrow{\mathrm{F}} \mathbf{cGrp}.$$

Proof.

(a) For every $g_2 \in \mathrm{M}_2G$ we have

$$g_2\partial\varphi_1 = g_2\partial(\mathrm{M}_1\varphi) = g_2(\mathrm{M}_2\varphi)\partial \in \mathrm{B}_1\mathrm{M}H.$$

This implies $\mathrm{B}_1\mathrm{M}G \subseteq \operatorname{Ker}\varphi_1\nu$, where ν denotes the canonical epimorphism $H_1 \to H_1/\mathrm{B}_1\mathrm{M}H$, and thus we get a well-defined group homomorphism

$$\overline{\varphi_1}\colon \operatorname{Mor}\mathrm{F}G \to \operatorname{Mor}\mathrm{F}H, g_1\mathrm{B}_1\mathrm{M}G \mapsto (g_1\varphi_1)\mathrm{B}_1\mathrm{M}H.$$

Now we get

$$(g_1\mathrm{B}_1\mathrm{M}G)\mathrm{s}\varphi_0 = g_1\mathrm{d}_0\varphi_0 = g_1\varphi_1\mathrm{d}_0 = ((g_1\varphi_1)\mathrm{B}_1\mathrm{M}H)\mathrm{s} = (g_1\mathrm{B}_1\mathrm{M}G)\overline{\varphi_1}\mathrm{s}$$

and

$$(g_1\mathrm{B}_1\mathrm{M}G)\mathrm{t}\varphi_0 = g_1\mathrm{d}_1\varphi_0 = g_1\varphi_1\mathrm{d}_1 = ((g_1\varphi_1)\mathrm{B}_1\mathrm{M}H)\mathrm{t} = (g_1\mathrm{B}_1\mathrm{M}G)\overline{\varphi_1}\mathrm{t}$$

for $g_1 \in G_1$ as well as

$$g_0\mathrm{e}\overline{\varphi_1} = ((g_0\mathrm{s}_0)\mathrm{B}_1\mathrm{M}G)\overline{\varphi_1} = (g_0\mathrm{s}_0\varphi_1)\mathrm{B}_1\mathrm{M}H = (g_0\varphi_0\mathrm{s}_0)\mathrm{B}_1\mathrm{M}H = (g_0\varphi_0)\mathrm{e}$$

for $g_0 \in G_0$. Thus we get a categorical group homomorphism $\mathrm{F}\varphi$ with $\operatorname{Ob}\mathrm{F}\varphi = \varphi_0$ and $\operatorname{Mor}\mathrm{F}\varphi = \overline{\varphi_1}$.

(b) We let G, H, K be simplicial groups and $G \xrightarrow{\varphi} H$, $H \xrightarrow{\psi} K$ be simplicial group homomorphisms. Then we compute

$$\begin{aligned}(g_1\mathrm{B}_1\mathrm{M}G)(\operatorname{Mor}\mathrm{F}(\varphi\psi)) &= (g_1\varphi_1\psi_1)\mathrm{B}_1\mathrm{M}K = ((g_1\varphi_1)\mathrm{B}_1\mathrm{M}H)(\operatorname{Mor}\mathrm{F}\psi)\\ &= (g_1\mathrm{B}_1\mathrm{M}G)(\operatorname{Mor}\mathrm{F}\varphi)(\operatorname{Mor}\mathrm{F}\psi)\end{aligned}$$

and

$$(g_1\mathrm{B}_1\mathrm{M}G)(\operatorname{Mor}\mathrm{F}(\mathrm{id}_G)) = (g_1\mathrm{id}_G)\mathrm{B}_1\mathrm{M}G = g_1\mathrm{B}_1\mathrm{M}G$$

for all $g_1 \in G_1$. Hence we have $\operatorname{Mor}\mathrm{F}(\varphi\psi) = (\operatorname{Mor}\mathrm{F}\varphi)(\operatorname{Mor}\mathrm{F}\psi)$ and $\operatorname{Mor}\mathrm{F}(\mathrm{id}_G) = \mathrm{id}_{\operatorname{Mor}\mathrm{F}}$. Since

$$\operatorname{Ob}\mathrm{F}(\varphi\psi) = (\varphi\psi)_0 = \varphi_0\psi_0 = (\operatorname{Ob}\mathrm{F}\varphi)(\operatorname{Ob}\mathrm{F}\psi)$$

and

$$\operatorname{Ob}\mathrm{F}(\mathrm{id}_G) = (\mathrm{id}_G)_0 = \mathrm{id}_{G_0} = \mathrm{id}_{\operatorname{Ob}\mathrm{F}G},$$

this implies that F is a functor from the category of simplicial groups to the category of categorical groups. □

(6.8) Remark.

(a) We let C and D be categorical groups and $C \xrightarrow{\varphi} D$ be a categorical group homomorphism.

(i) The fundamental groupoid of the categorical nerve of C has objects $\operatorname{Ob} \mathrm{FN}_{\mathbf{Cat}} C = \operatorname{Ob} C$ and morphisms $\operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C = (\operatorname{Mor} C)^{\times 1}/\{1\}$. The categorical structure maps are given by

$$\begin{aligned}
&\mathrm{s}\colon \operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C \to \operatorname{Ob} \mathrm{FN}_{\mathbf{Cat}} C, (m)\{1\} \mapsto m\mathrm{s},\\
&\mathrm{t}\colon \operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C \to \operatorname{Ob} \mathrm{FN}_{\mathbf{Cat}} C, (m)\{1\} \mapsto m\mathrm{t},\\
&\mathrm{e}\colon \operatorname{Ob} \mathrm{FN}_{\mathbf{Cat}} C \to \operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C, o \mapsto (o\mathrm{e})\{1\},\\
&\mathrm{c}\colon (\operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C)\ {}_{\mathrm{t}}{\times}_{\mathrm{s}}\ (\operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C) \to \operatorname{Mor} \mathrm{FN}_{\mathbf{Cat}} C, ((m)\{1\}, (n)\{1\}) \mapsto ((m, n)\mathrm{c})\{1\}.
\end{aligned}$$

(ii) The categorical group homomorphism $\mathrm{FN}_{\mathbf{Cat}}\varphi$ induced by φ is given on the objects by $\operatorname{Ob} \mathrm{FN}_{\mathbf{Cat}}\varphi = \operatorname{Ob}\varphi$ and on the morphisms by $((m)\{1\})(\mathrm{FN}_{\mathbf{Cat}}\varphi) = (m\varphi)\{1\}$ for all $m \in \operatorname{Mor} C$.

(b) We let G and H be simplicial groups and $G \xrightarrow{\varphi} H$ be a simplicial group homomorphism.

(i) The group of n-simplices in the categorical nerve of the fundamental groupoid of G is

$$(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n = \{(g_{1,i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1, 0\rfloor} \mid g_{1,i+1}\mathrm{d}_1 = g_{1,i}\mathrm{d}_0 \text{ for all } i \in \lfloor n-2, 0\rfloor\}$$

for every $n \in \mathbb{N}_0$. The faces and degeneracies of $\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G$ are given by

$$((g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j \in \lfloor n-1,0\rfloor}\mathrm{d}_k)_i = \begin{cases} g_{1,i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2, k\rfloor, \\ (g_{1,k}(g_{1,k}^{-1}\mathrm{d}_1\mathrm{s}_0)g_{1,k-1})\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2, 0\rfloor \end{cases}$$

for $i \in \lfloor n-2, 0\rfloor$, $(g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j \in \lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n$, $k \in [0, n]$, $n \in \mathbb{N}$, $n \geq 2$, and

$$((g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j \in \lfloor n-1,0\rfloor}\mathrm{s}_k)_i = \begin{cases} g_{1,i-1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n, k+1\rfloor, \\ g_{1,k}\mathrm{d}_1\mathrm{s}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k, k \in [0, n-1], \\ g_{1,k-1}\mathrm{d}_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k, k \in [1, n], \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-1, 0\rfloor \end{cases}$$

for $i \in \lfloor n, 0\rfloor$, $(g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j \in \lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n$, $k \in [0, n]$, $n \in \mathbb{N}$. The faces $\mathrm{d}_0\colon (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_1 \to (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_0$ and $\mathrm{d}_1\colon (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_1 \to (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_0$ are given by

$$(g_1\mathrm{B}_1\mathrm{M}G)\mathrm{d}_0 = g_1\mathrm{d}_0 \text{ and } (g_1\mathrm{B}_1\mathrm{M}G)\mathrm{d}_1 = g_1\mathrm{d}_1$$

for $(g_1\mathrm{B}_1\mathrm{M}G) \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_1$, while the degeneracy $\mathrm{s}_0\colon (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_0 \to (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_1$ is given by

$$g_0\mathrm{s}_0 = g_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G$$

for $g_0 \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_0$.

(ii) The simplicial group homomorphism $\mathrm{N}_{\mathbf{Cat}}\mathrm{F}\varphi$ induced by φ is given by $(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}\varphi)_0 = \varphi_0$ and

$$(g_{1,i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1,0\rfloor}(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}\varphi)_n = ((g_{1,i}\varphi_1)\mathrm{B}_1\mathrm{M}H)_{i \in \lfloor n-1,0\rfloor}$$

for $(g_{1,i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n$, $n \in \mathbb{N}$.

Proof.

(a) (i) According to proposition (6.3), the Moore complex of the categorical nerve $\mathrm{N}_{\mathbf{Cat}}C$ is given by

$$\mathrm{M}(\mathrm{N}_{\mathbf{Cat}}C) = (\ldots \longrightarrow \{1\} \longrightarrow (\operatorname{Ker}\mathrm{t})^{\times 1} \xrightarrow{\partial} \operatorname{Ob} C),$$

with $(m)\partial = m\mathrm{s}|_{\operatorname{Ker}\mathrm{t}}$ for all $m \in \operatorname{Ker}\mathrm{t}$. Hence the fundamental groupoid $\mathrm{FN}_{\mathbf{Cat}}C$ has objects

$$\operatorname{Ob}\mathrm{FN}_{\mathbf{Cat}}C = (\mathrm{N}_{\mathbf{Cat}}C)_0 = \operatorname{Ob} C$$

and morphisms

$$\operatorname{Mor}\mathrm{FN}_{\mathbf{Cat}}C = (\mathrm{N}_{\mathbf{Cat}}C)_1/\mathrm{B}_1\mathrm{MN}_{\mathbf{Cat}}C = (\operatorname{Mor} C)^{\times 1}/\{1\}.$$

The categorical structure maps are given by

$$((m)\{1\})\mathrm{s} = (m)\mathrm{d}_0 = m\mathrm{s} \text{ and } (m\{1\})\mathrm{t} = (m)\mathrm{d}_1 = m\mathrm{t}$$

for $m \in \operatorname{Mor} C$ as well as

$$o\mathrm{e} = o\mathrm{s}_0 = (o\mathrm{e})\{1\}$$

for $o \in \operatorname{Ob} C$ and

$$\begin{aligned}((m)\{1\}, (n)\{1\})\mathrm{c} &= ((m)((m)\mathrm{d}_1\mathrm{s}_0)^{-1}(n))\{1\} = ((m)((m\mathrm{t})\mathrm{s}_0)^{-1}(n))\{1\} = ((m)(m\mathrm{te})^{-1}(n))\{1\} \\ &= (m(m\mathrm{te})^{-1}n)\{1\} = ((m,n)\mathrm{c})\{1\}\end{aligned}$$

for $m, n \in \operatorname{Mor} C$ with $m\mathrm{t} = n\mathrm{s}$.

(ii) We have $\operatorname{Ob}\mathrm{FN}_{\mathbf{Cat}}\varphi = (\mathrm{N}_{\mathbf{Cat}}\varphi)_0 = \operatorname{Ob}\varphi$ and

$$((m)\{1\})(\mathrm{FN}_{\mathbf{Cat}}\varphi) = ((m)(\mathrm{N}_{\mathbf{Cat}}\varphi))\{1\} = (m\varphi)\{1\} \text{ for all } m \in \operatorname{Mor} C.$$

(b) (i) The group of n-simplices of $\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G$ is given by

$$(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n = (\operatorname{Mor}\mathrm{F}G)^{{}^{\mathrm{t}}\times_{\mathrm{s}} n} = (G_1/\mathrm{B}_1\mathrm{M}G)^{{}^{\mathrm{t}}\times_{\mathrm{s}} n} \text{ for all } n \in \mathbb{N}_0.$$

The faces $\mathrm{d}_k\colon (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n \to (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_{n-1}$ are given by

$$\begin{aligned}((g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j\in\lfloor n-1,0\rfloor}\mathrm{d}_k)_i &= \begin{cases} g_{1,i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2,k\rfloor, \\ (g_{1,k}\mathrm{B}_1\mathrm{M}G, g_{1,k-1}\mathrm{B}_1\mathrm{M}G)\mathrm{c} & \text{if } i = k-1, \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2,0\rfloor \end{cases} \\ &= \begin{cases} g_{1,i+1}\mathrm{B}_1\mathrm{M}G, & \text{if } i \in \lfloor n-2,k\rfloor, \\ (g_{1,k}\mathrm{B}_1\mathrm{M}G)(g_{1,k}\mathrm{B}_1\mathrm{M}G)^{-1}\mathrm{te}(g_{1,k-1}\mathrm{B}_1\mathrm{M}G) & \text{if } i = k-1, \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2,0\rfloor \end{cases} \\ &= \begin{cases} g_{1,i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2,k\rfloor, \\ (g_{1,k}(g_{1,k}^{-1}\mathrm{d}_1\mathrm{s}_0)g_{1,k-1})\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2,0\rfloor \end{cases}\end{aligned}$$

for $i \in \lfloor n-2,0\rfloor$, $(g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j\in\lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n$, $k \in [0,n]$, $n \in \mathbb{N}$, $n \geq 2$, resp.

$$(g_1\mathrm{B}_1\mathrm{M}G)\mathrm{d}_0 = (g_1\mathrm{B}_1\mathrm{M}G)\mathrm{s} = g_1\mathrm{d}_0 \text{ and } (g_1\mathrm{B}_1\mathrm{M}G)\mathrm{d}_1 = (g_1\mathrm{B}_1\mathrm{M}G)\mathrm{t} = g_1\mathrm{d}_1$$

for $(g_1\mathrm{B}_1\mathrm{M}G) \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_1$. Similarly, the degeneracies $\mathrm{s}_k\colon (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n \to (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_{n+1}$ are given by

$$((g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j\in\lfloor n-1,0\rfloor}\mathrm{s}_k)_i = \begin{cases} g_{1,i-1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n,k+1\rfloor, \\ (g_{1,k}\mathrm{B}_1\mathrm{M}G)\mathrm{te} & \text{if } i = k, k \in [0,n-1], \\ (g_{1,k-1}\mathrm{B}_1\mathrm{M}G)\mathrm{se} & \text{if } i = k, k \in [1,n], \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-1,0\rfloor, \end{cases}$$

$$= \begin{cases} g_{1,i-1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n, k+1 \rfloor, \\ g_{1,k}\mathrm{d}_1\mathrm{s}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k, k \in [0, n-1], \\ g_{1,k-1}\mathrm{d}_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k, k \in [1, n], \\ g_{1,i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-1, 0 \rfloor \end{cases}$$

for $i \in \lfloor n, 0 \rfloor$, $(g_{1,j}\mathrm{B}_1\mathrm{M}G)_{j \in \lfloor n-1,0 \rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n$, $k \in [0, n]$, $n \in \mathbb{N}$, resp.

$$g_0\mathrm{s}_0 = g_0\mathrm{e} = g_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G$$

for $g_0 \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_0$.

(ii) For $(g_{1,i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1,0 \rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n$, $n \in \mathbb{N}$, we have

$$\begin{aligned}(g_{1,i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1,0 \rfloor}(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}\varphi)_n &= (g_{1,i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1,0 \rfloor}(\mathrm{Mor}\,\mathrm{F}\varphi)^{\times n} \\ &= ((g_{1,i}\mathrm{B}_1\mathrm{M}G)(\mathrm{F}\varphi))_{i \in \lfloor n-1,0 \rfloor} = ((g_{1,i}\varphi_1)\mathrm{B}_1\mathrm{M}H)_{i \in \lfloor n-1,0 \rfloor},\end{aligned}$$

and for $g_0 \in (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_0$, we have

$$g_0(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}\varphi)_0 = g_0(\mathrm{Mor}\,\mathrm{F}\varphi)^{{}_{\mathrm{t}}\times_{\mathrm{s}} 0} = g_0(\mathrm{F}\varphi) = g_0\varphi_0,$$

cf. definition (1.30). □

(6.9) Proposition. The functor $\mathbf{sGrp} \xrightarrow{\mathrm{F}} \mathbf{cGrp}$ is left adjoint to $\mathbf{cGrp} \xrightarrow{\mathrm{N}_{\mathbf{Cat}}} \mathbf{sGrp}$, and we have

$$\mathrm{F} \circ \mathrm{N}_{\mathbf{Cat}} \cong \mathrm{id}_{\mathbf{cGrp}}.$$

Proof. We let $C, D \in \mathrm{Ob}\,\mathbf{cGrp}$ be categorical groups and $C \xrightarrow{\varphi} D$ be a categorical group homomorphism. According to remark (6.8)(a) we have $\mathrm{Ob}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C = \mathrm{Ob}\,C$ and $\mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C = (\mathrm{Mor}\,C)^{\times 1}/\{1\}$, while the categorical structure maps are given by

$$\begin{aligned}&\mathrm{s}\colon \mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C \to \mathrm{Ob}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C, (m)\{1\} \mapsto m\mathrm{s}, \\ &\mathrm{t}\colon \mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C \to \mathrm{Ob}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C, (m)\{1\} \mapsto m\mathrm{t}, \\ &\mathrm{e}\colon \mathrm{Ob}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C \to \mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C, o \mapsto (o\mathrm{e})\{1\}, \\ &\mathrm{c}\colon (\mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C)\,{}_{\mathrm{t}}\times_{\mathrm{s}}\,(\mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C) \to \mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C, ((m)\{1\}, (n)\{1\}) \mapsto ((m, n)\mathrm{c})\{1\}.\end{aligned}$$

Further, the categorical group homomorphism $\mathrm{F}\mathrm{N}_{\mathbf{Cat}}\varphi$ induced by φ is given on the objects by $\mathrm{Ob}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}\varphi = \mathrm{Ob}\,\varphi$ and on the morphisms by $((m)\{1\})(\mathrm{F}\mathrm{N}_{\mathbf{Cat}}\varphi) = (m\varphi)\{1\}$ for all $m \in \mathrm{Mor}\,C$. Thus we obtain

$$\mathrm{F} \circ \mathrm{N}_{\mathbf{Cat}} \cong \mathrm{id}_{\mathbf{cGrp}}$$

by the natural isotransformation

$$\mathrm{F}\mathrm{N}_{\mathbf{Cat}} \xrightarrow{\eta} \mathrm{id}_{\mathbf{cGrp}},$$

which is defined by $\mathrm{Ob}\,\eta_C := \mathrm{id}_{\mathrm{Ob}\,C}$ and $\mathrm{Mor}\,\eta_C\colon \mathrm{Mor}\,\mathrm{F}\mathrm{N}_{\mathbf{Cat}}C \to \mathrm{Mor}\,C, (m)\{1\} \mapsto m$ at a categorical group $C \in \mathrm{Ob}\,\mathbf{cGrp}$.

To show $\mathrm{F} \dashv \mathrm{N}_{\mathbf{Cat}}$, we construct the unit $\mathrm{id}_{\mathbf{sGrp}} \xrightarrow{\varepsilon} \mathrm{N}_{\mathbf{Cat}} \circ \mathrm{F}$. Thereto, we let G be a simplicial group. Since

$$\begin{aligned}(g_n\mathrm{d}_{\lfloor n,0 \rfloor \wedge i+2 \wedge i+1}\mathrm{B}_1\mathrm{M}G)\mathrm{t} &= g_n\mathrm{d}_{\lfloor n,0 \rfloor \wedge i+2 \wedge i+1}\mathrm{d}_1 = g_n\mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1} = g_n\mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i}\mathrm{d}_0 \\ &= (g_n\mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i}\mathrm{B}_1\mathrm{M}G)\mathrm{s}\end{aligned}$$

for $i \in \lfloor n-1, 0 \rfloor$, $g_n \in G_n$, we have a well-defined map $(\varepsilon_G)_n\colon G_n \to (\mathrm{N}_{\mathbf{Cat}}FG)_n$ given by

$$g_n(\varepsilon_G)_n := \begin{cases} g_0 & \text{if } n = 0, \\ (g_n\mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i}\mathrm{B}_1\mathrm{M}G)_{i \in \lfloor n-1,0 \rfloor} & \text{if } n > 0 \end{cases}$$

for $g_n \in G_n$, which is a group homomorphism for every $n \in \mathbb{N}_0$ because all faces in G and the canonical epimorphism $G_1 \to G_1/\mathrm{B}_1\mathrm{M}G$ are group homomorphisms.

We want to show that $G \xrightarrow{\varepsilon_G} \mathrm{N}_{\mathbf{Cat}}\mathrm{F}G$ is even a simplicial group homomorphism. Thereto, we have to show the commutativity of the diagrams

$$\begin{array}{ccc} G_n & \xrightarrow{\mathrm{d}_k} & G_{n-1} \\ {\scriptstyle (\varepsilon_G)_n}\downarrow & & \downarrow{\scriptstyle (\varepsilon_G)_{n-1}} \\ (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n & \xrightarrow{\mathrm{d}_k} & (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_{n-1} \end{array}$$

for $k \in [0, n]$, $n \in \mathbb{N}$, and

$$\begin{array}{ccc} G_{n+1} & \xleftarrow{\mathrm{s}_k} & G_n \\ {\scriptstyle (\varepsilon_G)_{n+1}}\downarrow & & \downarrow{\scriptstyle (\varepsilon_G)_n} \\ (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_{n+1} & \xleftarrow{\mathrm{s}_k} & (\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)_n \end{array}$$

for $k \in [0, n]$, $n \in \mathbb{N}_0$.

First, we consider the faces. We let $n \in \mathbb{N}$ be a natural number, $k \in [0, n]$ and $g_n \in G_n$ a group element. If $n \geq 2$, then we obtain by remark (6.8)(b)(i)

$$\begin{aligned}
(g_n(\varepsilon_G)_n\mathrm{d}_k)_i &= ((g_n\mathrm{d}_{\lfloor n,0\rfloor \wedge j+1 \wedge j}\mathrm{B}_1\mathrm{M}G)_{j \in \lfloor n-1,0\rfloor}\mathrm{d}_k)_i \\
&= \left.\begin{cases} g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+2\wedge i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2, k\rfloor, \\ ((g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k+1\wedge k})(g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k+1\wedge k})^{-1}\mathrm{d}_1\mathrm{s}_0(g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k\wedge k-1}))\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2, 0\rfloor \end{cases}\right\} \\
&= \left.\begin{cases} g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+2\wedge i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2, k\rfloor, \\ ((g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k+1\wedge k})(g_n^{-1}\mathrm{d}_{\lfloor n,0\rfloor\wedge k}\mathrm{s}_0)(g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k\wedge k-1}))\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2, 0\rfloor \end{cases}\right\} \\
&= \left.\begin{cases} g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+2\wedge i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2, k\rfloor, \\ (g_n\mathrm{d}_{\lfloor n,1\rfloor\wedge 2}\mathrm{B}_1\mathrm{M}G)((g_n^{-1}\mathrm{d}_{\lfloor n,1\rfloor\wedge 2}\mathrm{s}_0) & \\ \quad\cdot(g_n\mathrm{d}_{\lfloor n,3\rfloor})(g_n^{-1}\mathrm{d}_{\lfloor n,2\rfloor}\mathrm{s}_1)(g_n\mathrm{d}_{\lfloor n,2\rfloor}\mathrm{s}_0))\mathrm{d}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, k = 1, \\ (g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k+1\wedge k-1}\mathrm{B}_1\mathrm{M}G)((g_n^{-1}\mathrm{d}_{\lfloor n,1\rfloor\wedge k+1\wedge k-1}) & \\ \quad\cdot(g_n\mathrm{d}_{\lfloor n,1\rfloor\wedge k+1\wedge k})(g_n^{-1}\mathrm{d}_{\lfloor n,1\rfloor\wedge k}\mathrm{s}_1)(g_n\mathrm{d}_{\lfloor n,1\rfloor\wedge k\wedge k-1}))\mathrm{d}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, k \in [2, n-1], \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2, 0\rfloor \end{cases}\right\} \\
&= \left.\begin{cases} g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+2\wedge i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2, k\rfloor, \\ (g_n\mathrm{d}_{\lfloor n,1\rfloor\wedge 2})\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, k = 1, \\ (g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k+1\wedge k-1})\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, k \in [2, n-1], \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2, 0\rfloor \end{cases}\right\} \\
&= \left.\begin{cases} g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+2\wedge i+1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n-2, k\rfloor, \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+2\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i = k-1, \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-2, 0\rfloor \end{cases}\right\} = g_n\mathrm{d}_k\mathrm{d}_{\lfloor n-1,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G = (g_n\mathrm{d}_k(\varepsilon_G)_{n-1})_i
\end{aligned}$$

for all $i \in \lfloor n-2, 0\rfloor$, that is, $g_n(\varepsilon_G)_n\mathrm{d}_k = g_n\mathrm{d}_k(\varepsilon_G)_{n-1}$. If $n = 1$, we have

$$g_1(\varepsilon_G)_1\mathrm{d}_k = (g_1\mathrm{B}_1\mathrm{M}G)\mathrm{d}_k = g_1\mathrm{d}_k = g_1\mathrm{d}_k(\varepsilon_G)_0.$$

Thus we have $(\varepsilon_G)_n\mathrm{d}_k = \mathrm{d}_k(\varepsilon_G)_{n-1}$ for all $n \in \mathbb{N}$, $k \in [0, n]$.

Next, we show the compability with the degeneracies. Thereto, we let $n \in \mathbb{N}_0$, $k \in [0, n]$ and $g_n \in G_n$. For $n \geq 1$ we get, by remark (6.8)(b)(i)

$$\begin{aligned}
(g_n(\varepsilon_G)_n\mathrm{s}_k)_i &= ((g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge j+1\wedge j}\mathrm{B}_1\mathrm{M}G)_{j\in\lfloor n-1,0\rfloor}\mathrm{s}_k)_i \\
&= \left.\begin{cases} g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i\wedge i-1}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor n, k+1\rfloor, \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k+1\wedge k}\mathrm{d}_1\mathrm{s}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k, k \in [0, n-1], \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge k\wedge k-1}\mathrm{d}_0\mathrm{s}_0\mathrm{B}_1\mathrm{M}G & \text{if } i = k, k \in [1, n], \\ g_n\mathrm{d}_{\lfloor n,0\rfloor\wedge i+1\wedge i}\mathrm{B}_1\mathrm{M}G & \text{if } i \in \lfloor k-1, 0\rfloor \end{cases}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \begin{array}{ll} g_n \mathrm{d}_{\lfloor n,0 \rfloor \wedge i \wedge i-1} \mathrm{B}_1 \mathrm{M} G & \text{if } i \in \lfloor n, k+1 \rfloor, \\ g_n \mathrm{d}_{\lfloor n,0 \rfloor \wedge k} \mathrm{s}_0 \mathrm{B}_1 \mathrm{M} G & \text{if } i = k, \\ g_n \mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i} \mathrm{B}_1 \mathrm{M} G & \text{if } i \in \lfloor k-1, 0 \rfloor \end{array} \right\} \\
&= \left\{ \begin{array}{ll} g_n \mathrm{s}_k \mathrm{d}_{\lfloor n+1,0 \rfloor \wedge i+1 \wedge i} \mathrm{B}_1 \mathrm{M} G & \text{if } i \in \lfloor n, k+1 \rfloor, \\ g_n \mathrm{s}_k \mathrm{d}_{\lfloor n+1,0 \rfloor \wedge k+1 \wedge k} \mathrm{B}_1 \mathrm{M} G & \text{if } i = k, \\ g_n \mathrm{s}_k \mathrm{d}_{\lfloor n+1,0 \rfloor \wedge i+1 \wedge i} \mathrm{B}_1 \mathrm{M} G & \text{if } i \in \lfloor k-1, 0 \rfloor \end{array} \right\} = g_n \mathrm{s}_k \mathrm{d}_{\lfloor n+1,0 \rfloor \wedge i+1 \wedge i} \mathrm{B}_1 \mathrm{M} G \\
&= (g_n \mathrm{s}_k (\varepsilon_G)_{n+1})_i
\end{aligned}
$$

for all $i \in \lfloor n, 0 \rfloor$, that is, $g_n (\varepsilon_G)_n \mathrm{s}_k = g_n \mathrm{s}_k (\varepsilon_G)_{n+1}$. If $n = 0$, we have

$$g_0 (\varepsilon_G)_0 \mathrm{s}_0 = g_0 \mathrm{s}_0 = (g_0 \mathrm{s}_0 \mathrm{B}_1 \mathrm{M} G) = g_0 \mathrm{s}_0 (\varepsilon_G)_1.$$

Hence $(\varepsilon_G)_n \mathrm{s}_k = \mathrm{s}_k (\varepsilon_G)_{n+1}$ for all $n \in \mathbb{N}$, $k \in [0, n]$, and we have shown that we have a simplicial group homomorphism

$$G \xrightarrow{\varepsilon_G} \mathrm{N}_{\mathbf{Cat}} \mathrm{F} G.$$

Next, we show the naturality of $(\varepsilon_G)_{G \in \operatorname{Ob} \mathbf{sGrp}}$. We let $G \xrightarrow{\varphi} H$ be a simplicial group homomorphism for $G, H \in \operatorname{Ob} \mathbf{sGrp}$. Then, by remark (6.8)(b)(ii), we have

$$
\begin{aligned}
g_n (\varepsilon_G)_n (\mathrm{N}_{\mathbf{Cat}} \mathrm{F} \varphi)_n &= (g_n \mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i} \mathrm{B}_1 \mathrm{M} G)_{i \in \lfloor n-1, 0 \rfloor} (\mathrm{N}_{\mathbf{Cat}} \mathrm{F} \varphi)_n = ((g_n \mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i} \varphi_1) \mathrm{B}_1 \mathrm{M} H)_{i \in \lfloor n-1, 0 \rfloor} \\
&= ((g_n \varphi_n \mathrm{d}_{\lfloor n,0 \rfloor \wedge i+1 \wedge i}) \mathrm{B}_1 \mathrm{M} H)_{i \in \lfloor n-1, 0 \rfloor} = g_n \varphi_n (\varepsilon_H)_n
\end{aligned}
$$

for $g_n \in G_n$, $n \in \mathbb{N}$, and

$$(\varepsilon_G)_0 (\mathrm{N}_{\mathbf{Cat}} \mathrm{F} \varphi)_0 = \mathrm{id}_{G_0} \varphi_0 = \varphi_0 = \varphi_0 \mathrm{id}_{H_0} = \varphi_0 (\varepsilon_H)_0.$$

Hence the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\varepsilon_G} & \mathrm{N}_{\mathbf{Cat}} \mathrm{F} G \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{N}_{\mathbf{Cat}} \mathrm{F} \varphi} \\
H & \xrightarrow{\varepsilon_H} & \mathrm{N}_{\mathbf{Cat}} \mathrm{F} H
\end{array}
$$

commutes and the morphisms ε_G for $G \in \operatorname{Ob} \mathbf{sGrp}$ yield a natural transformation

$$\mathrm{id}_{\mathbf{sGrp}} \xrightarrow{\varepsilon} \mathrm{N}_{\mathbf{Cat}} \circ \mathrm{F}.$$

It remains to show that ε resp. η yield a unit resp. a counit of an adjunction. But indeed we have

$$
\begin{aligned}
(m_i)_{i \in \lfloor n-1, 0 \rfloor} (\varepsilon_{\mathrm{N}_{\mathbf{Cat}} C})_n (\mathrm{N}_{\mathbf{Cat}} \eta_C)_n &= ((m_j)_{j \in \lfloor n-1, 0 \rfloor} \mathrm{d}_{\lfloor n-1,0 \rfloor \wedge i+1 \wedge i} \mathrm{B}_1 \mathrm{M} \mathrm{N}_{\mathbf{Cat}} C)_{i \in \lfloor n-1, 0 \rfloor} (\mathrm{N}_{\mathbf{Cat}} \eta_C)_n \\
&= ((m_i)\{1\})_{i \in \lfloor n-1, 0 \rfloor} (\mathrm{N}_{\mathbf{Cat}} \eta_C)_n = ((m_i)\{1\} \eta_C)_{i \in \lfloor n-1, 0 \rfloor} \\
&= (m_i)_{i \in \lfloor n-1, 0 \rfloor}
\end{aligned}
$$

for all $(m_i)_{i \in \lfloor n-1, 0 \rfloor} \in (\mathrm{N}_{\mathbf{Cat}} C)_n$, $n \in \mathbb{N}$, and

$$(\varepsilon_{\mathrm{N}_{\mathbf{Cat}} C})_0 (\mathrm{N}_{\mathbf{Cat}} \eta_C)_0 = \mathrm{id}_{(\mathrm{N}_{\mathbf{Cat}} C)_0} (\operatorname{Ob} \eta_C) = \mathrm{id}_{\operatorname{Ob} C} \mathrm{id}_{\operatorname{Ob} C} = \mathrm{id}_{\operatorname{Ob} C} = (\mathrm{id}_{\mathrm{N}_{\mathbf{Cat}} C})_0,$$

that is,

$$\varepsilon_{\mathrm{N}_{\mathbf{Cat}} C} (\mathrm{N}_{\mathbf{Cat}} \eta_C) = \mathrm{id}_{\mathrm{N}_{\mathbf{Cat}} C} \text{ for every } C \in \operatorname{Ob} \mathbf{cGrp}.$$

Furthermore, we have

$$(\operatorname{Ob} \mathrm{F} \varepsilon_G)(\operatorname{Ob} \eta_{\mathrm{F} G}) = (\varepsilon_G)_0 \mathrm{id}_{\operatorname{Ob} \mathrm{F} G} = \mathrm{id}_{\operatorname{Ob} \mathrm{F} G} = \operatorname{Ob} \mathrm{id}_{\mathrm{F} G}$$

and

$$(g_1 \mathrm{B}_1 \mathrm{M} G)(\mathrm{F} \varepsilon_G) \eta_{\mathrm{F} G} = ((g_1 (\varepsilon_G)_1)\{1\}) \eta_{\mathrm{F} G} = ((g_1 \mathrm{B}_1 \mathrm{M} G)\{1\}) \eta_{\mathrm{F} G} = g_1 \mathrm{B}_1 \mathrm{M} G$$

for $g_1 \mathrm{B}_1 \mathrm{M} G \in \operatorname{Mor} \mathrm{F} G$, and thus

$$(\mathrm{F} \varepsilon_G) \eta_{\mathrm{F} G} = \mathrm{id}_{\mathrm{F} G} \text{ for every } G \in \operatorname{Ob} \mathbf{sGrp}.$$ □

(6.10) Corollary. The categories **CrMod**, **cGrp** and $\mathbf{sGrp}_{\lfloor 1,0 \rfloor}$ are equivalent.

Proof. The equivalence of **CrMod** and **cGrp** is the assertion of the Brown-Spencer theorem (5.25).
We let $\mathrm{id}_{\mathbf{sGrp}} \xrightarrow{\varepsilon} \mathrm{N}_{\mathbf{Cat}} \circ \mathrm{F}$ denote the unit from the proof of proposition (6.9) and we let $G \in \mathrm{Ob}\, \mathbf{sGrp}_{\lfloor 1,0 \rfloor}$ be a simplicial group with $\mathrm{M}_n G \cong 1$ for $n \geq 2$. Then we have $(\varepsilon_G)_0 = \mathrm{id}_{G_0}$ and $g_1(\varepsilon_G)_1 = (g_1 \mathrm{B}_1 \mathrm{M}G) = (g_1\{1\})$ for all $g_1 \in G_1$, whence $(\varepsilon_G)_0$ and $(\varepsilon_G)_1$ are group isomorphisms. Hence $\mathrm{M}_0 \varepsilon_G$ and $\mathrm{M}_1 \varepsilon_G$ are group isomorphisms, and since $\mathrm{M}_n G \cong \{1\}$ for all $n \geq 2$, the morphisms $\mathrm{M}_n \varepsilon_G$ for $n \geq 2$ are necessarily group isomorphisms as well. This is sufficient for $\mathrm{M}\varepsilon_G$ being an isomorphism in $\mathbf{C}(\mathbf{Grp})$, and with lemma (4.14), this implies that ε_G is an isomorphism of simplicial groups. By abuse of notation, we have

$$\mathrm{N}_{\mathbf{Cat}} \circ \mathrm{F}|_{\mathbf{sGrp}_{\lfloor 1,0 \rfloor}} \cong \mathrm{id}_{\mathbf{sGrp}_{\lfloor 1,0 \rfloor}},$$

that is, the fundamental groupoid functor restricts to a category equivalence $\mathbf{sGrp}_{\lfloor 1,0 \rfloor} \longrightarrow \mathbf{cGrp}$. □

§2 Truncation and coskeleton

In this section, we want to transfer our results of §1 on categorical groups to get equivalent facts for crossed modules.

(6.11) Definition (truncation)**.** The *truncation functor*

$$\mathbf{sGrp} \xrightarrow{\mathrm{Trunc}} \mathbf{CrMod}$$

is defined to be the composition $\mathrm{Trunc} := \mathtt{CrMod} \circ \mathrm{F}$.

$$\begin{array}{ccc} \mathbf{sGrp} & \xrightarrow{\mathrm{F}} & \mathbf{cGrp} \\ & {\scriptstyle \mathrm{Trunc}} \searrow & \downarrow {\scriptstyle \mathtt{CrMod}} \\ & & \mathbf{CrMod} \end{array}$$

Given a simplicial group $G \in \mathrm{Ob}\, \mathbf{sGrp}$, we call $\mathrm{Trunc}\, G$ the *crossed module truncated from* G.

(6.12) Proposition. We let $G, H \in \mathrm{Ob}\, \mathbf{sGrp}$ be simplicial groups and $G \xrightarrow{\varphi} H$ be a simplicial group homomorphism.

(a) The crossed module $\mathrm{Trunc}\, G$ is given by $\mathrm{Gp}\, \mathrm{Trunc}\, G = G_0$, $\mathrm{Mp}\, \mathrm{Trunc}\, G = \mathrm{M}_1 G / \mathrm{B}_1 \mathrm{M}G$ and

$$(g_1 \mathrm{B}_1 \mathrm{M}G)\mu^{\mathrm{Trunc}\, G} = g_1 \partial = g_1 \mathrm{d}_0 \text{ for } g_1 \mathrm{B}_1 \mathrm{M}G \in \mathrm{Mp}\, \mathrm{Trunc}\, G,$$

where $\mathrm{Gp}\, \mathrm{Trunc}\, G$ acts on $\mathrm{Mp}\, \mathrm{Trunc}\, G$ by

$${}^{g_0}(g_1 \mathrm{B}_1 \mathrm{M}G) = {}^{g_0 \mathrm{s}_0} g_1 \mathrm{B}_1 \mathrm{M}G \text{ for } g_0 \in \mathrm{Gp}\, \mathrm{Trunc}\, G, g_1 \mathrm{B}_1 \mathrm{M}G \in \mathrm{Mp}\, \mathrm{Trunc}\, G.$$

(b) We have $\mathrm{Gp}\, \mathrm{Trunc}\, \varphi = \varphi_0$ and $(g_1 \mathrm{B}_1 \mathrm{M}G)(\mathrm{Trunc}\, \varphi) = (g_1 \varphi_1)\mathrm{B}_1 \mathrm{M}H$ for $g_1 \mathrm{B}_1 \mathrm{M}G \in \mathrm{Mp}\, \mathrm{Trunc}\, G$.

Proof.

(a) We have

$$\mathrm{Gp}\, \mathrm{Trunc}\, G = \mathrm{Gp}\, \mathtt{CrMod}(\mathrm{F}G) = \mathrm{Ob}\, \mathrm{F}G = G_0$$

and

$$\begin{aligned} \mathrm{Mp}\, \mathrm{Trunc}\, G &= \mathrm{Mp}\, \mathtt{CrMod}(\mathrm{F}G) = \mathrm{Ker}\, \mathrm{t} = \{g_1 \mathrm{B}_1 \mathrm{M}G \in \mathrm{Mor}\, \mathrm{F}G \mid (g_1 \mathrm{B}_1 \mathrm{M}G)\mathrm{t} = 1\} \\ &= \{g_1 \mathrm{B}_1 \mathrm{M}G \in \mathrm{Mor}\, \mathrm{F}G \mid g_1 \mathrm{d}_1 = 1\} = \{g_1 \mathrm{B}_1 \mathrm{M}G \in G_1/\mathrm{B}_1 \mathrm{M}G \mid g_1 \in \mathrm{M}_1 G\} = \mathrm{M}_1 G / \mathrm{B}_1 \mathrm{M}G. \end{aligned}$$

Furthermore, we have

$$(g_1 \mathrm{B}_1 \mathrm{M}G)\mu^{\mathrm{Trunc}\, G} = (g_1 \mathrm{B}_1 \mathrm{M}G)\mu^{\mathtt{CrMod}(\mathrm{F}G)} = (g_1 \mathrm{B}_1 \mathrm{M}G)\mathrm{s}|_{\mathrm{Ker}\, \mathrm{t}} = g_1 \mathrm{d}_0 = g_1 \partial$$

and

$${}^{g_0}(g_1 \mathrm{B}_1 \mathrm{M}G) = {}^{g_0 \mathrm{e}}(g_1 \mathrm{B}_1 \mathrm{M}G) = {}^{g_0 \mathrm{s}_0 \mathrm{B}_1 \mathrm{M}G}(g_1 \mathrm{B}_1 \mathrm{M}G) = {}^{g_0 \mathrm{s}_0} g_1 \mathrm{B}_1 \mathrm{M}G$$

for $g_0 \in \mathrm{Gp}\, \mathrm{Trunc}\, G$, $g_1 \mathrm{B}_1 \mathrm{M}G \in \mathrm{Mp}\, \mathrm{Trunc}\, G$.

(b) We have

$$\mathrm{Gp}\,\mathrm{Trunc}\,\varphi = \mathrm{Gp}\,\mathtt{CrMod}(\mathrm{F}\varphi) = \mathrm{Ob}\,\mathrm{F}\varphi = \varphi_0$$

and

$$\begin{aligned}(g_1\mathrm{B}_1\mathrm{M}G)(\mathrm{Trunc}\,\varphi) &= (g_1\mathrm{B}_1\mathrm{M}G)(\mathtt{CrMod}(\mathrm{F}\varphi)) = (g_1\mathrm{B}_1\mathrm{M}G)(\mathrm{Mor}\,\mathrm{F}\varphi)|_{\mathrm{Mp}\,\mathtt{CrMod}(\mathrm{F}G)} \\ &= (g_1\mathrm{B}_1\mathrm{M}G)(\mathrm{F}\varphi) = g_1\varphi_1\mathrm{B}_1\mathrm{M}H\end{aligned}$$

for $g_1\mathrm{B}_1\mathrm{M}G \in \mathrm{Mp}\,\mathrm{Trunc}\,G$. □

(6.13) Definition. We let V be a crossed module. For every $n \in \mathbb{N}_0$, we define the *n-fold semidirect product*

$$\mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V := (\mathrm{Mp}\,V)^{\times n} \times \mathrm{Gp}\,V.$$

The elements in $\mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V$ are denoted by

$$(m_i, g)_{i\in\lfloor n-1,0\rfloor} := (m_i)_{i\in\lfloor n-1,0\rfloor} \cup (g) = (m_{n-1}, \ldots, m_0, g).$$

We equip $\mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V$ with the multiplication that is given by

$$(m_i, g)_{i\in\lfloor n-1,0\rfloor}(m'_i, g')_{i\in\lfloor n-1,0\rfloor} := (m_i \,{}^{(\prod_{k\in\lfloor i-1,0\rfloor} m_k)g} m'_i, gg')_{i\in\lfloor n-1,0\rfloor}$$

for $(m_i, g)_{i\in\lfloor n-1,0\rfloor}, (m'_i, g')_{i\in\lfloor n-1,0\rfloor} \in \mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V$.

(6.14) Remark. There exists a functor

$$\mathbf{CrMod} \xrightarrow{(\mathrm{Mp}\,-)_*\rtimes(\mathrm{Gp}\,-)} \mathbf{sGrp}$$

isomorphic to $\mathrm{N}_{\mathbf{Cat}} \circ \mathtt{cGrp}$ such that $((\mathrm{Mp}\,-)\,{}_*\!\rtimes(\mathrm{Gp}\,-))_n V = \mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V$, equipped with the multiplication in definition (6.13), and such that

$$\begin{aligned}(m_j, g)_{j\in\lfloor n-1,0\rfloor}(\mathrm{Mp}\,V \,{}_\theta\!\rtimes \mathrm{Gp}\,V) &:= (m_j, g)_{j\in\lfloor n-1,0\rfloor}((\mathrm{Mp}\,-\,{}_*\!\rtimes \mathrm{Gp}\,-)_\theta V) \\ &= (\prod_{k\in\lfloor (i+1)\theta-1, i\theta\rfloor} m_k, (\prod_{k\in\lfloor 0\theta-1,0\rfloor} m_k)g)_{i\in\lfloor m-1,0\rfloor}\end{aligned}$$

for $V \in \mathrm{Ob}\,\mathbf{CrMod}$, $\theta \in {}_{\mathbf{\Delta}}([m],[n])$, $m, n \in \mathbb{N}_0$. In particular, the n-fold semidirect product $\mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V$ with the multiplication given in definition (6.13) is a well-defined group for all $n \in \mathbb{N}_0$. Furthermore, given a morphism $V \xrightarrow{\varphi} W$ between crossed modules V and W, we have

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}(\mathrm{Mp}\,\varphi \,{}_n\!\rtimes \mathrm{Gp}\,\varphi) := (m_j, g)_{j\in\lfloor n-1,0\rfloor}((\mathrm{Mp}\,-\,{}_*\!\rtimes \mathrm{Gp}\,-)_n\varphi) = (m_j\varphi, g\varphi)_{j\in\lfloor n-1,0\rfloor}$$

for all $(m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \mathrm{Mp}\,V \,{}_n\!\rtimes \mathrm{Gp}\,V$, $n \in \mathbb{N}_0$.

Proof. We let $V \in \mathrm{Ob}\,\mathbf{CrMod}$ be a crossed module. Then we have

$$\begin{aligned}&(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_n = (\mathrm{Mor}\,\mathtt{cGrp}(V))^{{}_\mathrm{t}\times_\mathrm{s} n} \\ &= \begin{cases}\mathrm{Gp}\,V & \text{if } n = 0, \\ \{((m_i, (\prod_{k\in\lfloor i-1,0\rfloor} m_k)g))_{i\in\lfloor n-1,0\rfloor} \mid m_i \in \mathrm{Mp}\,V \text{ for } i \in \lfloor n-1,0\rfloor, g \in \mathrm{Gp}\,V\} & \text{if } n \geq 1.\end{cases}\end{aligned}$$

The multiplication in $(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_n$ is given by

$$\begin{aligned}&((m_i, (\prod_{k\in\lfloor i-1,0\rfloor} m_k)g))_{i\in\lfloor n-1,0\rfloor}((m'_i, (\prod_{k\in\lfloor i-1,0\rfloor} m'_k)g'))_{i\in\lfloor n-1,0\rfloor} \\ &= ((m_i, (\prod_{k\in\lfloor i-1,0\rfloor} m_k)g)(m'_i, (\prod_{k\in\lfloor i-1,0\rfloor} m'_k)g'))_{i\in\lfloor n-1,0\rfloor} \\ &= ((m_i \,{}^{(\prod_{k\in\lfloor i-1,0\rfloor} m_k)g} m'_i, (\prod_{k\in\lfloor i-1,0\rfloor} m_k)g(\prod_{k\in\lfloor i-1,0\rfloor} m'_k)g'))_{i\in\lfloor n-1,0\rfloor}\end{aligned}$$

for $((m_i, (\prod_{k\in\lfloor i-1,0\rfloor} m_k)g))_{i\in\lfloor n-1,0\rfloor}, ((m'_i, (\prod_{k\in\lfloor i-1,0\rfloor} m'_k)g'))_{i\in\lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_n$, $n \in \mathbb{N}_0$.
We wish to compute the morphism $(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_\theta$ for a morphism $\theta \in \mathbf{\Delta}([m],[n])$, where $m, n \in \mathbb{N}_0$. If $m \neq 0$, then

$$((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor}(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_\theta = (((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor}\mathrm{c}_{\lfloor (i+1)\theta, i\theta\rfloor})_{i\in\lfloor m-1,0\rfloor}$$
$$= ((\prod_{k\in\lfloor (i+1)\theta-1, i\theta\rfloor} m_k, (\prod_{k\in\lfloor i\theta-1,0\rfloor} m_k)g))_{i\in\lfloor m-1,0\rfloor}$$

for $((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_n$, $n \in \mathbb{N}_0$. If $m = 0$, then

$$((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor}(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_\theta = ((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor}\mathrm{t}_{0\theta}$$
$$= \begin{cases} (m_{0\theta}, (\prod_{k\in\lfloor 0\theta-1,0\rfloor} m_k)g)\mathrm{t} & \text{if } 0\theta \in [0, n-1], \\ (m_{n-1}, (\prod_{k\in\lfloor n-2,0\rfloor} m_k)g)\mathrm{s} & \text{if } 0\theta = n \end{cases} = \begin{cases} (\prod_{k\in\lfloor 0\theta-1,0\rfloor} m_k)g & \text{if } 0\theta \in [0, n-1], \\ m_{n-1}(\prod_{k\in\lfloor n-2,0\rfloor} m_k)g & \text{if } 0\theta = n \end{cases}$$
$$= (\prod_{k\in\lfloor 0\theta-1,0\rfloor} m_k)g$$

for $((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor} \in (\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_n$, $n \in \mathbb{N}_0$.
By transport of structure, the sets $\mathrm{Mp}\, V \;{}_n\!\rtimes\, \mathrm{Gp}\, V$ for $n \in \mathbb{N}_0$ fit into a simplicial group that is isomorphic to $\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V)$ via the natural isotransformation

$$\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V) \xrightarrow{\psi_V} \mathrm{Mp}\, V \;{}_n\!\rtimes\, \mathrm{Gp}\, V$$

given by

$$g(\psi_V)_0 := (g) \text{ and } ((m_i, (\prod_{k\in\lfloor i-1,0\rfloor} m_k)g))_{i\in\lfloor n-1,0\rfloor}(\psi_V)_n := (m_i, g)_{i\in\lfloor n-1,0\rfloor} \text{ for } n \geq 1.$$

Note that the second entry of the image of $(\psi_V)_n$ for $n \geq 1$ is obtained by reading off the second entry of the argument at $i = 0$. Thus ψ_V is compatible with the multiplication on $(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_n$ and the multiplication on $\mathrm{Mp}\, V \;{}_n\!\rtimes\, \mathrm{Gp}\, V$ defined in definition (6.13), as one can take from the computation of the product above.
Given a morphism $\theta \in \mathbf{\Delta}([m],[n])$, we have

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}(\mathrm{Mp}\, V \;{}_\theta\!\rtimes\, \mathrm{Gp}\, V) = (m_j, g)_{j\in\lfloor n-1,0\rfloor}(\psi_V)_n^{-1}(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_\theta(\psi_V)_m$$
$$= ((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor}(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(V))_\theta(\psi_V)_m$$
$$= \begin{cases} ((\prod_{k\in\lfloor 0\theta-1,0\rfloor} m_k)g)(\psi_V)_0 & \text{if } m = 0, \\ ((\prod_{k\in\lfloor (i+1)\theta-1, i\theta\rfloor} m_k, (\prod_{k\in\lfloor i\theta-1,0\rfloor} m_k)g))_{i\in\lfloor m-1,0\rfloor}(\psi_V)_m & \text{if } m \geq 1, \end{cases}$$
$$= (\prod_{k\in\lfloor (i+1)\theta-1, i\theta\rfloor} m_k, (\prod_{k\in\lfloor 0\theta-1,0\rfloor} m_k)g)_{i\in\lfloor m-1,0\rfloor}$$

for $(m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \mathrm{Mp}\, V \;{}_n\!\rtimes\, \mathrm{Gp}\, V$.
Furthermore, given a morphism $V \xrightarrow{\varphi} W$ between crossed modules V and W, we have

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}((\mathrm{Mp}\,\varphi) \;{}_n\!\rtimes\, (\mathrm{Gp}\,\varphi)) = (m_j, g)_{j\in\lfloor n-1,0\rfloor}(\psi_V)_n^{-1}(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(\varphi))_n(\psi_W)_n$$
$$= \begin{cases} g(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(\varphi))_0(\psi_W)_0 & \text{if } n = 0, \\ ((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g))_{j\in\lfloor n-1,0\rfloor}(\mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(\varphi))_n(\psi_W)_n & \text{if } n \geq 1 \end{cases}$$
$$= \begin{cases} (g\mathtt{cGrp}(\varphi))(\psi_W)_0 & \text{if } n = 0, \\ ((m_j, (\prod_{l\in\lfloor j-1,0\rfloor} m_l)g)\mathtt{cGrp}(\varphi))_{j\in\lfloor n-1,0\rfloor}(\psi_W)_n & \text{if } n \geq 1 \end{cases}$$
$$= \begin{cases} g\varphi(\psi_W)_0 & \text{if } n = 0, \\ ((m_j\varphi, ((\prod_{l\in\lfloor j-1,0\rfloor} m_l)g)\varphi))_{j\in\lfloor n-1,0\rfloor}(\psi_W)_n & \text{if } n \geq 1 \end{cases}$$

$$= \begin{cases} g\varphi(\psi_W)_0 & \text{if } n = 0, \\ ((m_j\varphi, (\prod_{l\in\lfloor j-1,0\rfloor}(m_l\varphi))(g\varphi)))_{j\in\lfloor n-1,0\rfloor}(\psi_W)_n & \text{if } n \geq 1 \end{cases} \Bigg\} = (m_j\varphi, g\varphi)_{j\in\lfloor n-1,0\rfloor}.$$

□

(6.15) Definition (1st coskeleton)**.** For a crossed module V, we call $\operatorname{Cosk} V := \operatorname{Mp} V \;{}_*\!\rtimes \operatorname{Gp} V$ the *coskeleton* of V. The functor

$$\mathbf{CrMod} \xrightarrow{\operatorname{Cosk}} \mathbf{sGrp}$$

given by $\operatorname{Cosk} := (\operatorname{Mp} -) \;{}_*\!\rtimes (\operatorname{Gp} -)$ is called the (1*st*) *coskeleton.*

(6.16) Proposition. We let V be a crossed module. The faces and degeneracies of $\operatorname{Cosk} V$ are given by

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}\mathrm{d}_k = \begin{cases} (m_j, m_0 g)_{j\in\lfloor n-1,1\rfloor} & \text{if } k = 0, \\ (m_j)_{j\in\lfloor n-1,k+1\rfloor} \cup (m_k m_{k-1}) \cup (m_j, g)_{j\in\lfloor k-2,0\rfloor} & \text{if } k \in [1, n-1], \\ (m_j, g)_{j\in\lfloor n-2,0\rfloor} & \text{if } k = n \end{cases}$$

for $(m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Cosk}_n V$, $k \in [1, n]$, $n \in \mathbb{N}$, and

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}\mathrm{s}_k = (m_j)_{j\in\lfloor n-1,k\rfloor} \cup (1) \cup (m_j, g)_{j\in\lfloor k-1,0\rfloor}$$

for $(m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Cosk}_n V$, $k \in [0, n]$, $n \in \mathbb{N}_0$.

Proof. We compute

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}\mathrm{d}_k = (m_j, g)_{j\in\lfloor n-1,0\rfloor}(\operatorname{Mp} V \;{}_{\delta^k}\!\rtimes \operatorname{Gp} V) = (\prod_{r\in\lfloor(i+1)\delta^k-1,i\delta^k\rfloor} m_r, (\prod_{r\in\lfloor 0\delta^k-1,0\rfloor} m_r)g)_{i\in\lfloor n-2,0\rfloor}$$

$$= \left\{\begin{matrix} (\prod_{r\in\lfloor i+2-1,i+1\rfloor} m_r, (\prod_{r\in\lfloor 1-1,0\rfloor} m_r)g)_{i\in\lfloor n-2,0\rfloor} & \text{if } k = 0, \\ (\prod_{r\in\lfloor i+2-1,i+1\rfloor} m_r)_{i\in\lfloor n-2,k\rfloor} \cup (\prod_{r\in\lfloor k+1-1,k-1\rfloor} m_r) & \\ \quad \cup(\prod_{r\in\lfloor i+1-1,i\rfloor} m_r, (\prod_{r\in\lfloor 0-1,0\rfloor} m_r)g)_{i\in\lfloor k-2,0\rfloor} & \text{if } k \in [1, n-1], \\ (\prod_{r\in\lfloor i+1-1,i\rfloor} m_r, (\prod_{r\in\lfloor 0-1,0\rfloor} m_r)g)_{i\in\lfloor n-2,0\rfloor} & \text{if } k = n \end{matrix}\right\}$$

$$= \left\{\begin{matrix} (m_{i+1}, m_0 g)_{i\in\lfloor n-2,0\rfloor} & \text{if } k = 0, \\ (m_{i+1})_{i\in\lfloor n-2,k\rfloor} \cup (m_k m_{k-1}) \cup (m_i, g)_{i\in\lfloor k-2,0\rfloor} & \text{if } k \in [1, n-1], \\ (m_i, g)_{i\in\lfloor n-2,0\rfloor} & \text{if } k = n \end{matrix}\right\}$$

$$= \begin{cases} (m_j, m_0 g)_{j\in\lfloor n-1,1\rfloor} & \text{if } k = 0, \\ (m_j)_{j\in\lfloor n-1,k+1\rfloor} \cup (m_k m_{k-1}) \cup (m_j, g)_{j\in\lfloor k-2,0\rfloor} & \text{if } k \in [1, n-1], \\ (m_j, g)_{j\in\lfloor n-2,0\rfloor} & \text{if } k = n \end{cases}$$

for $(m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Cosk}_n V$, $k \in [0, n]$, $n \in \mathbb{N}$, and

$$(m_j, g)_{j\in\lfloor n-1,0\rfloor}\mathrm{s}_k = (m_j, g)_{j\in\lfloor n-1,0\rfloor}(\operatorname{Mp} V \;{}_{\sigma^k}\!\rtimes \operatorname{Gp} V)$$
$$= (\prod_{r\in\lfloor(i+1)\sigma^k-1,i\sigma^k\rfloor} m_r, (\prod_{r\in\lfloor 0\sigma^k-1,0\rfloor} m_r)g)_{i\in\lfloor n-2,0\rfloor}$$
$$= (\prod_{r\in\lfloor i+1-1-1,i-1\rfloor} m_r)_{i\in\lfloor n,k+1\rfloor} \cup (\prod_{r\in\lfloor k-1,k\rfloor} m_r) \cup (\prod_{r\in\lfloor i+1-1,i\rfloor} m_r, (\prod_{r\in\lfloor 0-1,0\rfloor} m_r)g)_{i\in\lfloor k-1,0\rfloor}$$
$$= (m_{i-1})_{i\in\lfloor n,k+1\rfloor} \cup (1) \cup (m_i, g)_{i\in\lfloor k-1,0\rfloor} = (m_j)_{j\in\lfloor n-1,k\rfloor} \cup (1) \cup (m_j, g)_{j\in\lfloor k-1,0\rfloor}$$

for $(m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Cosk}_n V$, $k \in [0, n]$, $n \in \mathbb{N}_0$. □

(6.17) Proposition. The functor $\mathbf{sGrp} \xrightarrow{\operatorname{Trunc}} \mathbf{CrMod}$ is left adjoint to $\mathbf{CrMod} \xrightarrow{\operatorname{Cosk}} \mathbf{sGrp}$, and we have

$$\operatorname{Trunc} \circ \operatorname{Cosk} \cong \mathrm{id}_{\mathbf{CrMod}}.$$

Proof. Since CrMod and cGrp are mutually inverse category equivalences, they fulfill in particular $\mathtt{CrMod} \dashv \mathtt{cGrp}$ and $\mathtt{CrMod} \circ \mathtt{cGrp} \cong \mathrm{id}_{\mathbf{CrMod}}$. Hence $\mathrm{F} \dashv \mathrm{N}_{\mathbf{Cat}}$ implies

$$\mathrm{Trunc} = \mathtt{CrMod} \circ \mathrm{F} \dashv \mathrm{N}_{\mathbf{Cat}} \circ \mathtt{cGrp} \cong \mathrm{Cosk},$$

and from $\mathrm{F} \circ \mathrm{N}_{\mathbf{Cat}} \cong \mathrm{id}_{\mathbf{cGrp}}$ we can conclude that

$$\mathrm{Trunc} \circ \mathrm{Cosk} \cong \mathtt{CrMod} \circ \mathrm{F} \circ \mathrm{N}_{\mathbf{Cat}} \circ \mathtt{cGrp} \cong \mathtt{CrMod} \circ \mathrm{id}_{\mathbf{cGrp}} \circ \mathtt{cGrp} \cong \mathrm{id}_{\mathbf{CrMod}}.$$ □

We collect some examples and further properties of the n-fold semidirect product and the coskeleton functor.

(6.18) Example. If V is a crossed module with $\mathrm{Mp}\, V \cong 1$, then $\mathrm{Cosk}\, V \cong \mathrm{Cosk}(\mathrm{Gp}\, V)$; cf. definition (6.15) and definition (4.16).

(6.19) Example. We consider the crossed module $V \cong \mathrm{C}_{4,4}^{2,-1}$ introduced in example (5.6). Recall that it has group part $\mathrm{Gp}\, V = \langle a \mid a^4 = 1\rangle$, module part $\mathrm{Mp}\, V = \langle b \mid b^4 = 1\rangle$, structure morphism $\mu^V \colon \mathrm{Mp}\, V \to \mathrm{Gp}\, V, b \mapsto a^2$ and action ${}^a b = b^{-1}$. Its coskeleton is given by

$$\mathrm{Cosk}_n\, V = \langle b\rangle \,{}_n\!\rtimes \langle a\rangle.$$

Hence it consists of elements $(b^{f_j}, a^e)_{j \in \lfloor n-1,0\rfloor}$, where $e, f_j \in \{0,1,2,3\}$ for $j \in \lfloor n-1, 0\rfloor$. The multiplication of two elements $(b^{f_j}, a^e)_{j\in\lfloor n-1,0\rfloor}, (b^{f'_j}, a^{e'})_{j\in\lfloor n-1,0\rfloor} \in \mathrm{Cosk}_n\, V$ is given by

$$(b^{f_j}, a^e)_{j\in\lfloor n-1,0\rfloor}(b^{f'_j}, a^{e'})_{j\in\lfloor n-1,0\rfloor} = (b^{f_j+(-1)^e f'_j}, a^{e+e'})$$

for $e, f_j, e', f'_j \in \{0,1,2,3\}$, $j \in \lfloor n-1, 0\rfloor$.

(6.20) Remark. We let G be a group and M be a G-group. Then we have

$$(\prod_{j\in\lfloor n-1,0\rfloor} m_j)\ {}^g(\prod_{j\in\lfloor n-1,0\rfloor} m'_j) = \prod_{j\in\lfloor n-1,0\rfloor} (m_j\, {}^{(\prod_{l\in\lfloor j-1,0\rfloor} m_l)}({}^g m'_j))$$

for all $m_j, m'_j \in M$, $i \in \lfloor n-1, 0\rfloor$, $g \in G$, $n \in \mathbb{N}_0$.

Proof. We proceed by induction on $n \in \mathbb{N}_0$. For $n = 0$, there is nothing to show. We let $n \in \mathbb{N}$ be given and we assume that the asserted formula holds for $n-1$. This implies

$$\begin{aligned}
(\prod_{i\in\lfloor n-1,0\rfloor} m_j)\ {}^g(\prod_{i\in\lfloor n-1,0\rfloor} m'_j) &= m_{n-1}(\prod_{i\in\lfloor n-2,0\rfloor} m_j)({}^g m'_{n-1})\ {}^g(\prod_{i\in\lfloor n-2,0\rfloor} m'_j) \\
&= m_{n-1}\, {}^{(\prod_{i\in\lfloor n-2,0\rfloor} m_j)}({}^g m'_{n-1})(\prod_{i\in\lfloor n-2,0\rfloor} m_j)\ {}^g(\prod_{i\in\lfloor n-2,0\rfloor} m'_j) \\
&= m_{n-1}\, {}^{(\prod_{i\in\lfloor n-2,0\rfloor} m_j)}({}^g m'_{n-1}) \prod_{i\in\lfloor n-2,0\rfloor} (m_j\, {}^{(\prod_{j\in\lfloor i-1,0\rfloor} m_l)}({}^g m'_j)) \\
&= \prod_{i\in\lfloor n-1,0\rfloor} (m_j\, {}^{(\prod_{j\in\lfloor i-1,0\rfloor} m_l)}({}^g m'_j)).
\end{aligned}$$ □

(6.21) Proposition. We let V be a crossed module. Then $\mathrm{Mp}\, V$ becomes a $(\mathrm{Mp}\, V \,{}_n\!\rtimes \mathrm{Gp}\, V)$-group by

$${}^{(m_j,g)_{j\in\lfloor n-1,0\rfloor}} m := {}^{\prod_{j\in\lfloor n-1,0\rfloor} m_g}({}^g m) \text{ for all } m \in \mathrm{Mp}\, V, (m_j, g)_{j\in\lfloor n-1,0\rfloor} \in \mathrm{Mp}\, V \,{}_n\!\rtimes \mathrm{Gp}\, V$$

and we have

$$\mathrm{Mp}\, V \,{}_{n+1}\!\rtimes \mathrm{Gp}\, V \cong \mathrm{Mp}\, V \rtimes (\mathrm{Mp}\, V \,{}_n\!\rtimes \mathrm{Gp}\, V)$$

for all $n \in \mathbb{N}_0$.

Proof. We fix some $n \in \mathbb{N}_0$. Remark (6.20) shows that

$$\begin{aligned}{}^{(m_j,g)_{j\in\lfloor n-1,0\rfloor}}\big({}^{(m'_j,g')_{j\in\lfloor n-1,0\rfloor}}m\big) &= {}^{(\prod_{j\in\lfloor n-1,0\rfloor} m_j)g}\big({}^{(\prod_{j\in\lfloor n-1,0\rfloor} m'_j)g'}m\big) = {}^{(\prod_{j\in\lfloor n-1,0\rfloor} m_j)\,{}^{g}(\prod_{j\in\lfloor n-1,0\rfloor} m'_j)gg'}m \\ &= {}^{\prod_{j\in\lfloor n-1,0\rfloor}(m_j\,{}^{(\prod_{l\in\lfloor j-1,0\rfloor} m_l)}({}^{g}m'_j))}\big({}^{gg'}m\big) \\ &= {}^{(m_j\,{}^{(\prod_{l\in\lfloor j-1,0\rfloor} m_l)g}m'_j,\,gg')_{j\in\lfloor n-1,0\rfloor}}m = {}^{((m_j,g)_{j\in\lfloor n-1,0\rfloor}(m'_j,g')_{j\in\lfloor n-1,0\rfloor})}m\end{aligned}$$

for $(m_j,g)_{j\in\lfloor n-1,0\rfloor}, (m'_j,g')_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V$, $m \in \operatorname{Mp} V$, and

$${}^{(1,1)_{j\in\lfloor n-1,0\rfloor}}m = m$$

for $m \in \operatorname{Mp} V$. Furthermore, we have

$$\begin{aligned}{}^{(m_j,g)_{j\in\lfloor n-1,0\rfloor}}(mm') &= {}^{(\prod_{j\in\lfloor n-1,0\rfloor} m_j)g}(mm') = {}^{\prod_{j\in\lfloor n-1,0\rfloor} m_j}\big({}^{g}(mm')\big) = {}^{\prod_{j\in\lfloor n-1,0\rfloor} m_j}\big({}^{g}m\,{}^{g}m'\big) \\ &= {}^{\prod_{j\in\lfloor n-1,0\rfloor} m_j}\big({}^{g}m\big)\,{}^{\prod_{j\in\lfloor n-1,0\rfloor} m_j}\big({}^{g}m'\big) = {}^{(\prod_{j\in\lfloor n-1,0\rfloor} m_j)g}m\;{}^{(\prod_{j\in\lfloor n-1,0\rfloor} m_j)g}m' \\ &= {}^{(m_j,g)_{j\in\lfloor n-1,0\rfloor}}m\,{}^{(m_j,g)_{j\in\lfloor n-1,0\rfloor}}m'\end{aligned}$$

for $(m_j,g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V$, $m, m' \in \operatorname{Mp} V$. Thus $\operatorname{Mp} V$ becomes a $(\operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V)$-group and we can form the semidirect product

$$\operatorname{Mp} V \rtimes (\operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V).$$

Since the multiplication in $\operatorname{Mp} V \rtimes (\operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V)$ is given by

$$\begin{aligned}&(m_n, (m_j,g)_{j\in\lfloor n-1,0\rfloor})(m'_n, (m'_j,g')_{j\in\lfloor n-1,0\rfloor}) \\ &= (m_n\,{}^{(m_j,g)_{j\in\lfloor n-1,0\rfloor}}m'_n, (m_j,g)_{j\in\lfloor n-1,0\rfloor}(m'_j,g')_{j\in\lfloor n-1,0\rfloor}) \\ &= (m_n\,{}^{(\prod_{l\in\lfloor n-1,0\rfloor} m_l)g}m'_n, (m_j\,{}^{(\prod_{l\in\lfloor j-1,0\rfloor} m_l)g}m'_j, gg'))_{j\in\lfloor n-1,0\rfloor},\end{aligned}$$

for $(m_n, (m_j,g)_{j\in\lfloor n-1,0\rfloor}), (m'_n, (m'_j,g')_{j\in\lfloor n-1,0\rfloor}) \in \operatorname{Mp} V \rtimes (\operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V)$, we have a group isomorphism given by

$$\operatorname{Mp} V \rtimes (\operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V) \to \operatorname{Mp} V\,{}_{n+1}{\rtimes}\operatorname{Gp} V, (m_n, (m_j,g)_{j\in\lfloor n-1,0\rfloor}) \mapsto (m_j,g)_{j\in\lfloor n,0\rfloor}. \qquad \square$$

§3 Homotopy groups of a crossed module

(6.22) Proposition. We suppose given a crossed module V. Then $\operatorname{Cosk} V \in \mathbf{sGrp}_{\lfloor 1,0\rfloor}$ and we have

$$\mathrm{M} \circ \operatorname{Cosk} \cong \mathrm{U},$$

where U denotes the forgetful functor $\mathbf{CrMod} \xrightarrow{\mathrm{U}} \mathbf{C}(\mathbf{Grp})$ that sends the crossed module V to the complex of groups $\operatorname{Mp} V \xrightarrow{\mu^V} \operatorname{Gp} V$, concentrated in dimensions 1 and 0, and forgets the action of $\operatorname{Gp} V$ on $\operatorname{Mp} V$.

Proof. We let V be a crossed module. For $n \geq 2$, we obtain

$$\begin{aligned}\mathrm{M}_n \operatorname{Cosk} V &= \bigcap_{k\in[1,n]} \operatorname{Ker} \mathrm{d}_k \subseteq \operatorname{Ker} \mathrm{d}_1 \cap \operatorname{Ker} \mathrm{d}_n \\ &= \{(m_j,g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V \mid (m_j,g)_{j\in\lfloor n-1,0\rfloor}\mathrm{d}_1 = 1 \text{ and } (m_j,g)_{j\in\lfloor n-1,0\rfloor}\mathrm{d}_n = 1\} \\ &= \{(m_j,g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V \mid (m_{n-1},\ldots,m_2,m_1m_0,g) = 1 \text{ and } (m_{n-2},\ldots,m_0,g) = 1\} \\ &= \{(m_j,g)_{j\in\lfloor n-1,0\rfloor} \in \operatorname{Mp} V\,{}_{n}{\rtimes}\operatorname{Gp} V \mid m_j = 1 \text{ for all } j \in \lfloor n-1,0\rfloor \text{ and } g = 1\} = \{1\}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\mathrm{M}_1 \operatorname{Cosk} V &= \operatorname{Ker} \mathrm{d}_1 = \{(m_0,g) \in \operatorname{Mp} V\,{}_{1}{\rtimes}\operatorname{Gp} V \mid (m_0,g)\mathrm{d}_1 = 1\} = \{(m_0,g) \in \operatorname{Mp} V\,{}_{1}{\rtimes}\operatorname{Gp} V \mid (g) = 1\} \\ &= \{(m,1) \in \operatorname{Mp} V \rtimes \operatorname{Gp} V \mid m \in \operatorname{Mp} V\} = \operatorname{Mp} V \rtimes \{1\}.\end{aligned}$$

and $\mathrm{M}_0 \operatorname{Cosk} V = \operatorname{Mp} V \,{}_0\!\rtimes \operatorname{Gp} V = (\operatorname{Gp} V)^{\times 1}$. Moreover, we have

$$(m,1)\partial = (m,1)\mathrm{d}_0 = (m1) = (m\mu^V)$$

for all $m \in \operatorname{Mp} V$. Hence

$$\mathrm{M}(\operatorname{Cosk} V) = (\ldots \longrightarrow \{1\} \longrightarrow \operatorname{Mp} V \rtimes \{1\} \xrightarrow{\partial} (\operatorname{Gp} V)^{\times 1}),$$

where $(m,1)\partial = (m,1)\mathrm{d}_0 = (m\mu^V)$ for $m \in M$.
Now, we let W also be a crossed module and $V \xrightarrow{\varphi} W$ be a morphism of crossed modules. Then we have

$$(m,1)(\mathrm{M}_1 \operatorname{Cosk} \varphi) = (m,1)(\operatorname{Mp} \varphi \,{}_1\!\rtimes \operatorname{Gp} \varphi) = (m\varphi, 1) \text{ for } m \in \operatorname{Mp} \varphi$$

and

$$(g)(\mathrm{M}_0 \operatorname{Cosk} \varphi) = (g)(\operatorname{Mp} \varphi \,{}_0\!\rtimes \operatorname{Gp} \varphi) = (g\varphi) \text{ for } g \in \operatorname{Gp} V.$$

Consequently, we obtain the isomorphy

$$\mathrm{M} \circ \operatorname{Cosk} \cong \mathrm{U}$$

via

$$\mathrm{M} \circ \operatorname{Cosk} \xrightarrow{\alpha} \mathrm{U},$$

where $\mathrm{M} \operatorname{Cosk} V \xrightarrow{\alpha_V} V$ is given by $(m,1)\alpha_V := m$ for $m \in \operatorname{Mp} V$ and by $(g)\alpha_V := g$ for $g \in \operatorname{Gp} V$, $V \in \operatorname{Ob} \mathbf{CrMod}$. □

(6.23) Corollary. We let $V, W \in \operatorname{Ob} \mathbf{CrMod}$ be crossed modules and $V \xrightarrow{\varphi} W$ be a morphism of crossed modules.

(a) We have

$$\pi_n(\operatorname{Cosk} V) \cong \begin{cases} \operatorname{Coker} \mu^V & \text{for } n = 0, \\ \operatorname{Ker} \mu^V & \text{for } n = 1, \\ \{1\} & \text{for } n \geq 2. \end{cases}$$

(b) The induced morphisms $\pi_1(\operatorname{Cosk} \varphi)$ resp. $\pi_0(\operatorname{Cosk} \varphi)$ are the induced morphisms on the kernels resp. cokernels of μ^V and μ^W.

Proof. By proposition (6.22), we have

$$\pi_n \circ \operatorname{Cosk} = \mathrm{H}_n \circ \mathrm{M} \circ \operatorname{Cosk} \cong \mathrm{H}_n \circ \mathrm{U} \text{ for all } n \in \mathbb{N}_0.$$

This implies all assertions. □

(6.24) Definition (homotopy groups of a crossed module)**.** The *homotopy groups* of a crossed module V are defined by

$$\pi_n(V) := \begin{cases} \operatorname{Coker} \mu^V & \text{for } n = 0, \\ \operatorname{Ker} \mu^V & \text{for } n = 1, \\ \{1\} & \text{for } n \geq 2. \end{cases}$$

Moreover, if $V \xrightarrow{\varphi} W$ is a morphism of crossed module, then we define $\pi_0(\varphi)$ resp. $\pi_1(\varphi)$ to be the induced morphisms on the cokernels resp. kernels of μ^V and μ^W; the morphisms $\pi_n(\varphi)$ for $n \geq 2$ are defined to be trivial.

$$\begin{array}{ccccccc} \pi_1(V) & \longrightarrow & \operatorname{Mp} V & \xrightarrow{\mu^V} & \operatorname{Gp} V & \longrightarrow & \pi_0(V) \\ \downarrow{\scriptstyle \pi_1(\varphi)} & & \downarrow{\scriptstyle \operatorname{Mp} \varphi} & & \downarrow{\scriptstyle \operatorname{Gp} \varphi} & & \downarrow{\scriptstyle \pi_0(\varphi)} \\ \pi_1(W) & \longrightarrow & \operatorname{Mp} W & \xrightarrow{\mu^W} & \operatorname{Gp} W & \longrightarrow & \pi_0(W) \end{array}$$

(6.25) Proposition. The homotopy group functors π_0 and π_1 for simplicial groups fulfill

$$\pi_n = \pi_n \circ \mathrm{Trunc} \text{ for } n \in \{0, 1\}.$$

Proof. We let G be a simplicial group. Due to proposition (6.12), we have

$$\begin{aligned}\pi_1(\mathrm{Trunc}\,G) &= \mathrm{Ker}\,\mu^{\mathrm{Trunc}\,G} = \{g_1\mathrm{B}_1\mathrm{M}G \in \mathrm{Mp}\,\mathrm{Trunc}\,G \mid (g_1\mathrm{B}_1\mathrm{M}G)\mu^{\mathrm{Trunc}\,G} = 1\} \\ &= \{g_1\mathrm{B}_1\mathrm{M}G \in \mathrm{M}_1G/\mathrm{B}_1\mathrm{M}G \mid g_1\partial = 1\} = \{g_1\mathrm{B}_1\mathrm{M}G \in \mathrm{M}_1G/\mathrm{B}_1\mathrm{M}G \mid g_1 \in \mathrm{Z}_1\mathrm{M}G\} \\ &= \mathrm{Z}_1\mathrm{M}G/\mathrm{B}_1\mathrm{M}G = \mathrm{H}_1\mathrm{M}G = \pi_1(G),\end{aligned}$$

and

$$\begin{aligned}\pi_0(\mathrm{Trunc}\,G) &= \mathrm{Coker}\,\mu^{\mathrm{Trunc}\,G} = (\mathrm{Gp}\,\mathrm{Trunc}\,G)/\,\mathrm{Im}\,\mu^{\mathrm{Trunc}\,G} = G_0/\,\mathrm{Im}\,\partial = \mathrm{Z}_0\mathrm{M}G/\mathrm{B}_0\mathrm{M}G = \mathrm{H}_0\mathrm{M}G \\ &= \pi_0(G).\end{aligned}$$

Furthermore, given a simplicial group homomorphism $G \xrightarrow{\varphi} H$, where $H \in \mathrm{Ob}\,\mathbf{sGrp}$, then we have

$$\begin{aligned}(g_1\mathrm{B}_1\mathrm{M}G)(\pi_1(\mathrm{Trunc}\,\varphi)) &= (g_1\mathrm{B}_1\mathrm{M}G)(\mathrm{Trunc}\,\varphi) = (g_1\varphi_1)\mathrm{B}_1\mathrm{M}H = (g_1(\mathrm{M}_1\varphi))\mathrm{B}_1\mathrm{M}H \\ &= (g_1\mathrm{B}_1\mathrm{M}G)(\mathrm{H}_1\mathrm{M}\varphi) = (g_1\mathrm{B}_1\mathrm{M}G)\pi_1(\varphi)\end{aligned}$$

for $g_1\mathrm{B}_1\mathrm{M}G \in \pi_1(\mathrm{Trunc}\,G)$ and

$$\begin{aligned}(g_0\,\mathrm{Im}\,\mu^{\mathrm{Trunc}\,G})\pi_0(\mathrm{Trunc}\,\varphi) &= (g_0(\mathrm{Trunc}_0\,\varphi))\,\mathrm{Im}\,\mu^{\mathrm{Trunc}\,H} = (g_0\varphi_0)\mathrm{B}_0\mathrm{M}H = (g_0(\mathrm{M}_0\varphi))\mathrm{B}_0\mathrm{M}H \\ &= (g_0\mathrm{B}_0\mathrm{M}G)(\mathrm{H}_0\mathrm{M}\varphi) = (g_0\,\mathrm{Im}\,\mu^{\mathrm{Trunc}\,G})\pi_0(\varphi)\end{aligned}$$

for $g_0\,\mathrm{Im}\,\mu^{\mathrm{Trunc}\,G} \in \pi_0(\mathrm{Trunc}\,G)$. □

(6.26) Theorem. If G is a simplicial group with $\pi_n(G) \cong 1$ for all $n \geq 2$, then there exists a simplicial group homomorphism $G \longrightarrow \mathrm{Cosk}\,\mathrm{Trunc}\,G$ that induces an isomorphism $\pi_n(G) \longrightarrow \pi_n(\mathrm{Cosk}\,\mathrm{Trunc}\,G)$ for each $n \in \mathbb{N}_0$.

Proof. First, we remark that

$$\mathrm{Cosk}\,\mathrm{Trunc}\,G \cong \mathrm{N}_{\mathbf{Cat}}\mathtt{cGrp}(\mathtt{CrMod}(\mathrm{F}G)) \cong \mathrm{N}_{\mathbf{Cat}}\mathrm{F}G$$

by remark (6.14) and definition (6.15).
We let $\mathrm{id}_{\mathbf{sGrp}} \xrightarrow{\varepsilon} \mathrm{N}_{\mathbf{Cat}} \circ \mathrm{F}$ denote the unit and $\mathrm{F} \circ \mathrm{N}_{\mathbf{Cat}} \xrightarrow{\eta} \mathrm{id}_{\mathbf{cGrp}}$ denote the counit from the proof of proposition (6.9) and we let $G \in \mathrm{Ob}\,\mathbf{sGrp}$ be a simplicial group with $\pi_n(G) \cong 1$ for $n \geq 2$. Then we have

$$(\mathrm{F}\varepsilon_G)\eta_{\mathrm{F}G} = \mathrm{id}_{\mathrm{F}G}$$

and $\eta_{\mathrm{F}G}$ is an isomorphism. Therefore, $\mathrm{F}\varepsilon_G = \eta_{\mathrm{F}G}^{-1}$ is an isomorphism as well. For $n \in \{0, 1\}$, this implies by proposition (6.25) that

$$\pi_n(\varepsilon_G) = \pi_n(\mathrm{Trunc}\,\varepsilon_G) = \pi_n(\mathtt{CrMod}(\mathrm{F}\varepsilon_G)) = \pi_n(\mathtt{CrMod}(\eta_{\mathrm{F}G}^{-1})).$$

Thus $\pi_n(\varepsilon_G)$ is an isomorphism for $n \in \{0, 1\}$. Hence $\pi_n(G) \cong 1 \cong \pi_n(\mathrm{N}_{\mathbf{Cat}}\mathrm{F}G)$ for $n \geq 2$. Thus, the induced homomorphisms $\pi_n(\varepsilon_G)$ for $n \geq 2$ have to be trivial and, in particular, they have to be isomorphisms, too. □

§4 The classifying simplicial set of a crossed module: an example

In this last section, we compute some homology and cohomology groups in low dimensions for the crossed module $\mathrm{C}_{4,4}^{2,-1}$.

(6.27) Definition (classifying (bi)simplicial set of a crossed module)**.** We define $\mathrm{B}V := \mathrm{B}\,\mathrm{Cosk}\,V$ resp. $\mathrm{B}^{(2)}V := \mathrm{B}^{(2)}\,\mathrm{Cosk}\,V$ to be the *classifying simplicial set* resp. the *classifying bisimplicial set* of a crossed module $V \in \mathrm{Ob}\,\mathbf{CrMod}$.

(6.28) Example. The classifying bisimplicial set of the crossed module $V \cong \mathrm{C}_{4,4}^{2,-1}$, given as in example (5.6) by $\mathrm{Gp}\, V = \langle a \mid a^4 = 1 \rangle$, $\mathrm{Mp}\, V = \langle b \mid b^4 = 1 \rangle$, $\mu^V \colon \mathrm{Mp} \to \mathrm{Gp}\, V, b \mapsto a^2$ and action ${}^a b = b^{-1}$, is given by

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\ldots & \longrightarrow & \langle b \rangle \,{}_{3}{\rtimes}\, \langle a \rangle & \longrightarrow & \langle b \rangle \,{}_{2}{\rtimes}\, \langle a \rangle & \longrightarrow & \langle b \rangle \,{}_{1}{\rtimes}\, \langle a \rangle & \longrightarrow & \langle b \rangle \,{}_{0}{\rtimes}\, \langle a \rangle \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\ldots & \longrightarrow & (\langle b \rangle \,{}_{3}{\rtimes}\, \langle a \rangle)^{\times 2} & \longrightarrow & (\langle b \rangle \,{}_{2}{\rtimes}\, \langle a \rangle)^{\times 2} & \longrightarrow & (\langle b \rangle \,{}_{1}{\rtimes}\, \langle a \rangle)^{\times 2} & \longrightarrow & (\langle b \rangle \,{}_{0}{\rtimes}\, \langle a \rangle)^{\times 2} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\ldots & \longrightarrow & (\langle b \rangle \,{}_{3}{\rtimes}\, \langle a \rangle)^{\times 3} & \longrightarrow & (\langle b \rangle \,{}_{2}{\rtimes}\, \langle a \rangle)^{\times 3} & \longrightarrow & (\langle b \rangle \,{}_{1}{\rtimes}\, \langle a \rangle)^{\times 3} & \longrightarrow & (\langle b \rangle \,{}_{0}{\rtimes}\, \langle a \rangle)^{\times 3} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\ldots & \longrightarrow & (\langle b \rangle \,{}_{3}{\rtimes}\, \langle a \rangle)^{\times 4} & \longrightarrow & (\langle b \rangle \,{}_{2}{\rtimes}\, \langle a \rangle)^{\times 4} & \longrightarrow & (\langle b \rangle \,{}_{1}{\rtimes}\, \langle a \rangle)^{\times 4} & \longrightarrow & (\langle b \rangle \,{}_{0}{\rtimes}\, \langle a \rangle)^{\times 4} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Here, the arrows denote the direction of the faces (for better readability we have omitted the degeneracies) and $*$ denotes a set with a single element (we have $(\langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle)^{\times 0} \cong *$ for all $p \in \mathbb{N}_0$). In the p-th column, where $p \in \mathbb{N}_0$, one can see the classifying simplicial set of $\mathrm{Cosk}_p\, V = \langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle$, that is,

$$\ldots \longrightarrow (\langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle)^{\times 4} \longrightarrow (\langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle)^{\times 3} \longrightarrow (\langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle)^{\times 2} \longrightarrow (\langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle)^{\times 1} \longrightarrow (\langle b \rangle \,{}_{p}{\rtimes}\, \langle a \rangle)^{\times 0}.$$

(6.29) Definition (homology and cohomology of crossed modules)**.** We let V be a crossed module, R be a commutative ring, M be an R-module and $n \in \mathbb{N}_0$ be a non-negative integer. The *n-th homology group* of V with coefficients in M over R is defined to be

$$\mathrm{H}_n(V, M; R) := \mathrm{H}_n(\mathrm{B}V, M; R).$$

The *nth cohomology group* of V with coefficients in M over R is defined to be

$$\mathrm{H}^n(V, M; R) := \mathrm{H}^n(\mathrm{B}V, M; R).$$

As in definition (2.18), we abbreviate

$$\begin{aligned}
&\mathrm{H}_n(V; R) := \mathrm{H}_n(V, R; R), \\
&\mathrm{H}_n(V, M) := \mathrm{H}_n(V, M; \mathbb{Z}), \\
&\mathrm{H}_n(V) := \mathrm{H}_n(V, \mathbb{Z}; \mathbb{Z}),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{H}^n(V; R) := \mathrm{H}^n(V, R; R), \\
&\mathrm{H}^n(V, M) := \mathrm{H}^n(V, M; \mathbb{Z}), \\
&\mathrm{H}^n(V) := \mathrm{H}^n(V, \mathbb{Z}; \mathbb{Z}).
\end{aligned}$$

(6.30) Remark. We let V be a crossed module, R be a commutative ring, M be an R-module. Then

$$\mathrm{H}_n(V, M; R) = \mathrm{H}_n(\mathrm{Cosk}\, V, M; R) \text{ and } \mathrm{H}^n(V, M; R) = \mathrm{H}^n(\mathrm{Cosk}\, V, M; R) \text{ for all } n \in \mathbb{N}_0.$$

Proof. We have

$$\mathrm{H}_n(V, M; R) = \mathrm{H}_n(\mathrm{B}V, M; R) = \mathrm{H}_n(\mathrm{B}\,\mathrm{Cosk}\, V, M; R) = \mathrm{H}_n(\mathrm{Cosk}\, V, M; R)$$

and, analogously,

$$\mathrm{H}^n(V, M; R) = \mathrm{H}^n(\mathrm{B}V, M; R) = \mathrm{H}^n(\mathrm{B}\,\mathrm{Cosk}\, V, M; R) = \mathrm{H}^n(\mathrm{Cosk}\, V, M; R)$$

for all $n \in \mathbb{N}_0$. □

(6.31) Proposition. Suppose given a crossed module V, a commutative ring R and an R-module M.

(a) There exists a spectral sequence E with $E^1_{p,n-p} \cong \mathrm{H}_{n-p}(\mathrm{Mp}\, V \;{}_p\!\rtimes\, \mathrm{Gp}\, V, M; R)$ that converges to the homology group $\mathrm{H}_n(V, M; R)$, where $p \in [0, n]$, $n \in \mathbb{N}_0$.

(b) There exists a spectral sequence E with $E_1^{p,n-p} \cong \mathrm{H}^{n-p}(\mathrm{Mp}\, V \;{}_p\!\rtimes\, \mathrm{Gp}\, V, M; R)$ that converges to the cohomology group $\mathrm{H}^n(V, M; R)$, where $p \in [0, n]$, $n \in \mathbb{N}_0$.

Proof. Follows from remark (6.30) and theorem (4.36); concerning the coskeleton cf. definition (6.15). □

(6.32) Definition (Jardine spectral sequences of crossed modules)**.** We let V be a crossed module. The spectral sequences in proposition (6.31) will be called *Jardine spectral sequences* of V (in the case of homology resp. in the case of cohomology).

(6.33) Example. We want to compute some homology and cohomology groups of the crossed module $V \cong \mathrm{C}^{2,-1}_{4,4}$ introduced in example (5.6). For the notation and the classifying bisimplicial set $\mathrm{B}^{(2)}V$, see example (6.28).

(a) The associated double complex over the ring of integers $\mathbb{Z}$ is isomorphic to

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathbb{Z}^{1\times 1} & \longrightarrow & \mathbb{Z}^{1\times 1} & \longrightarrow & \mathbb{Z}^{1\times 1} & \longrightarrow & \mathbb{Z}^{1\times 1} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \mathbb{Z}^{1\times 4^4} & \longrightarrow & \mathbb{Z}^{1\times 4^3} & \longrightarrow & \mathbb{Z}^{1\times 4^2} & \longrightarrow & \mathbb{Z}^{1\times 4} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \mathbb{Z}^{1\times 4^8} & \longrightarrow & \mathbb{Z}^{1\times 4^6} & \longrightarrow & \mathbb{Z}^{1\times 4^4} & \longrightarrow & \mathbb{Z}^{1\times 4^2} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \mathbb{Z}^{1\times 4^{12}} & \longrightarrow & \mathbb{Z}^{1\times 4^9} & \longrightarrow & \mathbb{Z}^{1\times 4^6} & \longrightarrow & \mathbb{Z}^{1\times 4^3} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \mathbb{Z}^{1\times 4^{16}} & \longrightarrow & \mathbb{Z}^{1\times 4^{12}} & \longrightarrow & \mathbb{Z}^{1\times 4^8} & \longrightarrow & \mathbb{Z}^{1\times 4^4} \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

where the morphisms are given by multiplication with suitable matrices from the right. Thus linear algebra over $\mathbb{Z}$ allows us to compute homology with respect to the vertical differentials and the associated morphisms. We obtain, using Maple,

$$\begin{array}{l}
\cdots \longrightarrow (\mathbb{Z}) \xrightarrow{\cdot(0)} (\mathbb{Z}) \xrightarrow{\cdot(1)} (\mathbb{Z}) \xrightarrow{\cdot(0)} (\mathbb{Z}) \\[2ex]
\cdots \longrightarrow \begin{pmatrix}\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/4\end{pmatrix} \xrightarrow{\cdot\left(\begin{smallmatrix}0&0\\0&0\\0&1\end{smallmatrix}\right)} \begin{pmatrix}\mathbb{Z}/2 & \mathbb{Z}/4\end{pmatrix} \xrightarrow{\cdot\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)} (\mathbb{Z}/4) \\[2ex]
\cdots \longrightarrow (\mathbb{Z}/2) \xrightarrow{\cdot(0)} (0) \\[2ex]
\cdots \longrightarrow (\mathbb{Z}/4)
\end{array}$$

where, for example, the elements of $\begin{pmatrix}\mathbb{Z}/2 & \mathbb{Z}/2 & \mathbb{Z}/4\end{pmatrix}$ are row vectors $\begin{pmatrix}a_2 & a_1 & a_0\end{pmatrix}$ with $a_2, a_1 \in \mathbb{Z}/2$ and $a_0 \in \mathbb{Z}/4$.

Here, the dots at the left should, except in the first row, not indicate any regularity in the appearing complexes - here the entries of $\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)$ were just too large to compute with.

Taking homology (in horizontal direction) yields the following homology groups:

$$
\begin{array}{lll}
 & & \mathrm{H}_0\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong \mathbb{Z}, \\
 & \mathrm{H}_0\mathrm{H}_1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong \mathbb{Z}/2, & \mathrm{H}_1\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0, \\
\mathrm{H}_0\mathrm{H}_2(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0, & \mathrm{H}_1\mathrm{H}_1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0, & \mathrm{H}_2\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0.
\end{array}
$$

By the Jardine spectral sequence, we know that $\mathrm{H}_0(V)$ is isomorphic to a subquotient of $\mathbb{Z}$, $\mathrm{H}_1(V)$ is isomorphic to a subquotient of $\mathbb{Z}/2$ and $\mathrm{H}_2(V) \cong 0$.

To compute $\mathrm{H}_n(V)$ for $n \in \{0, 1, 2\}$ directly, it is possible to use the Kan classifying simplicial set according to corollary (4.35):

$$\mathrm{H}_n(V) = \mathrm{H}_n(\operatorname{Cosk} V) \cong \mathrm{H}_n(\overline{\mathrm{W}} \operatorname{Cosk} V).$$

The Kan classifying simplicial set of $\operatorname{Cosk} V$ is given by

$$\ldots \longrightarrow \mathop{\Large\times}_{j\in\lfloor 2,0\rfloor} (\langle b\rangle \,{}_j\!\rtimes \langle a\rangle) \longrightarrow \mathop{\Large\times}_{j\in\lfloor 1,0\rfloor} (\langle b\rangle \,{}_j\!\rtimes \langle a\rangle) \longrightarrow \mathop{\Large\times}_{j\in\lfloor 0,0\rfloor} (\langle b\rangle \,{}_j\!\rtimes \langle a\rangle) \longrightarrow *.$$

(For the arrows, cf. example (6.28).) Its associated complex $\mathrm{C}(\overline{\mathrm{W}} \operatorname{Cosk} V)$ is isomorphic to

$$\ldots \longrightarrow \mathbb{Z}^{1\times 4^6} \longrightarrow \mathbb{Z}^{1\times 4^3} \longrightarrow \mathbb{Z}^{1\times 4} \longrightarrow \mathbb{Z}^{1\times 1}.$$

Computing homology via Maple yields

$$
\begin{aligned}
&\mathrm{H}_0(V) \cong \mathbb{Z}, \\
&\mathrm{H}_1(V) \cong \mathbb{Z}/2, \\
&\mathrm{H}_2(V) \cong 0.
\end{aligned}
$$

(b) To compute cohomology groups over $\mathbb{Z}$, one has to dualise the double complex $\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)$ resp. the complex $\mathrm{C}(\overline{\mathrm{W}} \operatorname{Cosk} V)$ by applying ${}_{\mathbb{Z}}(-, \mathbb{Z})$ pointwise and taking homology after that. This means to deal with column vectors resp. matrix multiplications from the left instead of row vectors resp. matrix multiplications from the right.

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^{4^4\times 1} & \longrightarrow & \mathbb{Z}^{4^8\times 1} & \longrightarrow & \mathbb{Z}^{4^{12}\times 1} & \longrightarrow & \mathbb{Z}^{4^{16}\times 1} & \longrightarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^{4^3\times 1} & \longrightarrow & \mathbb{Z}^{4^6\times 1} & \longrightarrow & \mathbb{Z}^{4^9\times 1} & \longrightarrow & \mathbb{Z}^{4^{12}\times 1} & \longrightarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^{4^2\times 1} & \longrightarrow & \mathbb{Z}^{4^4\times 1} & \longrightarrow & \mathbb{Z}^{4^6\times 1} & \longrightarrow & \mathbb{Z}^{4^8\times 1} & \longrightarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^{4\times 1} & \longrightarrow & \mathbb{Z}^{4^2\times 1} & \longrightarrow & \mathbb{Z}^{4^3\times 1} & \longrightarrow & \mathbb{Z}^{4^4\times 1} & \longrightarrow & \ldots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathbb{Z}^{1\times 1} & \longrightarrow & \mathbb{Z}^{1\times 1} & \longrightarrow & \mathbb{Z}^{1\times 1} & \longrightarrow & \mathbb{Z}^{1\times 1} & \longrightarrow & \ldots
\end{array}
$$

We obtain

$$
\begin{array}{ll}
\mathrm{H}^0\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong \mathbb{Z}, & \\
\mathrm{H}^1\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0, & \mathrm{H}^0\mathrm{H}^1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0,
\end{array}
$$

$$\mathrm{H}^2\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0, \qquad \mathrm{H}^1\mathrm{H}^1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong 0, \qquad \mathrm{H}^0\mathrm{H}^2(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V)) \cong \mathbb{Z}/2$$

and, using the Kan classifying simplicial set,

$$\begin{aligned} &\mathrm{H}^0(V) \cong \mathbb{Z}, \\ &\mathrm{H}^1(V) \cong 0, \\ &\mathrm{H}^2(V) \cong \mathbb{Z}/2. \end{aligned}$$

(c) Taking the field with two elements $\mathbb{F}_2$ as ground ring for homology resp. cohomology yields

$$\begin{aligned} &\mathrm{H}_0\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2, \\ &\mathrm{H}_0\mathrm{H}_1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2, \qquad \mathrm{H}_1\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong 0, \\ &\mathrm{H}_0\mathrm{H}_2(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2, \qquad \mathrm{H}_1\mathrm{H}_1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2, \qquad \mathrm{H}_2\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong 0 \end{aligned}$$

and

$$\begin{aligned} &\mathrm{H}_0(V;\mathbb{F}_2) \cong \mathbb{F}_2, \\ &\mathrm{H}_1(V;\mathbb{F}_2) \cong \mathbb{F}_2, \\ &\mathrm{H}_2(V;\mathbb{F}_2) \cong \mathbb{F}_2 \end{aligned}$$

resp.

$$\begin{aligned} &\mathrm{H}^0\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2, \\ &\mathrm{H}^1\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong 0, \qquad \mathrm{H}^0\mathrm{H}^1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2, \\ &\mathrm{H}^2\mathrm{H}^0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong 0, \qquad \mathrm{H}^0\mathrm{H}^2(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong 0, \qquad \mathrm{H}^1\mathrm{H}^1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \cong \mathbb{F}_2 \end{aligned}$$

and

$$\begin{aligned} &\mathrm{H}^0(V;\mathbb{F}_2) \cong \mathbb{F}_2, \\ &\mathrm{H}^1(V;\mathbb{F}_2) \cong \mathbb{F}_2, \\ &\mathrm{H}^2(V;\mathbb{F}_2) \cong \mathbb{F}_2. \end{aligned}$$

(6.34) Remark. Since, in example (6.33)(c),

$$\begin{aligned} \mathrm{H}_2(V;\mathbb{F}_2) &\cong \mathbb{F}_2 \\ &\ncong \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus 0 \\ &\cong \mathrm{H}_0\mathrm{H}_2(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \oplus \mathrm{H}_1\mathrm{H}_1(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)) \oplus \mathrm{H}_2\mathrm{H}_0(\mathrm{C}^{(2)}(\mathrm{B}^{(2)}V;\mathbb{F}_2)), \end{aligned}$$

we can conclude that the Jardine spectral sequence in the case of homology (cf. definition (6.32)) does not degenerate in general.

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Sebastian Thomas
Lehrstuhl D für Mathematik
RWTH Aachen
Templergraben 64
D-52062 Aachen
sebastian.thomas@math.rwth-aachen.de
`http://www.math.rwth-aachen.de/~Sebastian.Thomas/`

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I hereby ensure that the thesis at hand is entirely my own work, employing only the referenced media and sources.
Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig und nur unter Benutzung der angegebenen Hilfsmittel und Quellen angefertigt habe.