On the 3-arrow calculus for homotopy categories

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Abstract

We develop a localisation theory for certain categories, yielding a 3-arrow calculus: Every morphism in the localisation is represented by a diagram of length 3, and two such diagrams represent the same morphism if and only if they can be embedded in a 3-by-3 diagram in an appropriate way. The method we use to construct this localisation is similar to the Ore localisation for a 2-arrow calculus; in particular, we do not have to use zigzags of arbitrary length. Applications include the localisation of an arbitrary Quillen model category with respect to its weak equivalences as well as the localisation of its full subcategories of cofibrant, fibrant and bifibrant objects, giving the homotopy category in all four cases. In contrast to the approach of Dwyer, Hirschhorn, Kan and Smith, the Quillen model category under consideration does not need to admit functorial factorisations. Moreover, it follows that the derived category of any abelian (or idempotent splitting exact) category admits a 3-arrow calculus if we localise the category of complexes instead of its homotopy category.

1 Introduction

Localisations of categories occur in homological and homotopical algebra. Prominent examples are the construction of the derived category of an abelian category [34, ch. II, §1, not. 1.1] (as localisation of the homotopy category of complexes) and – more generally – localisations of Verdier triangulated categories with respect to thick subcategories [34, ch. I, §2, déf. 3-3], localisations of abelian categories with respect to thick subcategories [33, ch. I, sec. 2] [15, sec. 1.11] and the definition of the homotopy category of a Quillen model category [29, ch. I, sec. 1, def. 6]. Usually, the construction of the first three examples is done by a procedure known under the name of Ore localisation, which can only be applied in the special case where the denominator set, that is, the subset of morphisms to be formally inverted, fulfills some additional properties. Let us call such a special denominator set a classical denominator set for the moment. The basic ideas of this method have their historical origin in ring theory, in particular in the works of Ore [28, sec. 2] and Asano [2, Satz 1], while the categorical version comes from the Grothendieck school, see Verdier [34, ch. I, §2, sec. 3.2] and Grothendieck and Hartshorne [18, ch. I, §3, prop. 3.1], based on the work of Serre [33, ch. I, sec. 2]. In contrast, the construction of the homotopy category of a Quillen model category is usually done by a formal construction working for arbitrary denominator sets, which is commonly called Gabriel-Zisman localisation ([1]). Of course, if the denominator set under consideration is classical, then the Ore localisation and the Gabriel-Zisman localisation are isomorphic since localisation of categories is defined by a universal property.

The advantage of Ore localisation is in the manageability of morphisms in the localisation: We suppose given a category $\mathcal{C}$ and a denominator set $D \subseteq \operatorname{Mor} \mathcal{C}$. The morphisms in the Gabriel-Zisman localisation are represented by zigzags

$$\longrightarrow \;\overset{\approx}{\longleftarrow}\; \longrightarrow \dots \overset{\approx}{\longleftarrow}\; \longrightarrow$$

of finite but arbitrary length, where the "backward" arrows are in D. In contrast, if D is a classical denominator set, then every morphism in the Ore localisation is represented by a diagram

$$\longrightarrow \;\overset{\approx}{\longleftarrow}\; .$$

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[1] To the authors knowledge, it first explicitly appeared in the monograph of Gabriel and Zisman [12, sec. 1.1]. One can find earlier mentions, for example in [18, ch. I, §3, rem., p. 29] and in [34, ch. I, §2, n. 3, p. 17]. In the latter source, one finds moreover a citation "[C.G.G.]", which might be the unpublished manuscript *Catégories et foncteurs* of Chevalley, Gabriel and Grothendieck occurring in the bibliography of [32].

Furthermore, in the Gabriel-Zisman localisation one has, in general, no convenient criterion to decide whether two zigzags represent the same morphism in the localisation, while already from the construction of the Ore localisation it follows that two of these diagrams represent the same morphism if and only if they can be embedded as the top and the bottom row in a commutative diagram of the following form.

$$\begin{tikzcd}
{} \arrow[r] \arrow[d, equal] & {} \arrow[d] & {} \arrow[l, "\approx"'] \arrow[d, equal] \\
{} \arrow[r] & {} & {} \arrow[l, "\approx"'] \\
{} \arrow[r] \arrow[u, equal] & {} \arrow[u, "\approx"] & {} \arrow[l, "\approx"'] \arrow[u, equal]
\end{tikzcd}$$

Unfortunately, the set of weak equivalences in a Quillen model category $\mathcal{M}$ is not a classical denominator set in general, and the homotopy category $\operatorname{Ho}\mathcal{M}$, that is, the localisation of $\mathcal{M}$ with respect to its set of weak equivalences, does in general not fulfill a 2-arrow calculus in the above sense. Instead, Dwyer, Hirschhorn, Kan and Smith developed in [9, sec. 10, sec. 36] a 3-arrow calculus for the homotopy category of $\mathcal{M}$, provided $\mathcal{M}$ admits functorial factorisations (cf. [9, sec. 9.1, ax. MC5]). That is, they showed that each morphism in $\operatorname{Ho}\mathcal{M}$ is represented by a diagram

$$\begin{tikzcd}
{} & {} \arrow[l, "\approx"'] \arrow[r] & {} & {} \arrow[l, "\approx"']
\end{tikzcd},$$

and, moreover, that two of these diagrams represent the same morphism if and only if they can be embedded as the top and the bottom row in a commutative diagram of the following form.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "\approx"'] \arrow[r] & {} & {} \arrow[l, "\approx"'] \arrow[d, equal] \\
{} & {} \arrow[l, "\approx"'] \arrow[r] \arrow[u, "\approx"] \arrow[d, "\approx"] & {} \arrow[u, "\approx"] \arrow[d, "\approx"] & {} \arrow[l, "\approx"'] \\
{} & {} \arrow[l, "\approx"'] \arrow[r] & {} & {} \arrow[l, "\approx"'] \\
{} \arrow[u, equal] \arrow[uu, equal] \arrow[uuu, equal] & {} \arrow[l, "\approx"'] \arrow[r] \arrow[u, "\approx"] & {} \arrow[u, "\approx"] & {} \arrow[l, "\approx"'] \arrow[u, equal] \arrow[uu, equal]
\end{tikzcd}$$

To do this, they introduced the notion of a *homotopical category admitting a 3-arrow calculus* [9, sec. 33.1, 36.1] and developed a 3-arrow calculus in this context [9, sec. 36.3].

In this article, we introduce the concept of a uni-fractionable category, see definition (3.1)(a). Our main result is the construction of a localisation of a uni-fractionable category (with respect to its set of denominators) that satisfies a 3-arrow calculus in the sense described above, see theorem (5.18). In contrast to [9], we will not make use of the Gabriel-Zisman localisation. Instead, we will give an elementary ad hoc construction of a localisation of a uni-fractionable category, in the spirit of the Ore localisation for a 2-arrow calculus. ([2])

Both in the approach of [9, sec. 36.1] and in our uni-fractionable categories, one has three distinguished kinds of morphisms, which, in our terminology, are called denominators, S-denominators and T-denominators. The denominators are the morphisms to be formally inverted, while the S- and T-denominators are particular denominators. The essential stipulations in [9, sec. 36.1] are that every denominator factors functorially into an S-denominator followed by a T-denominator ([3]) and that one has functorial Ore completions along S-denominators resp. T-denominators. For uni-fractionable categories, we omit the stipulations of functoriality; instead, we require the existence of weakly universal Ore completions along S-denominators resp. T-denominators.

The advantage of uni-fractionable categories is that functoriality of factorisations is not needed. On the one hand, this is convenient for applications. On the other hand, the theory developed here can be applied to arbitrary Quillen model categories. Moreover, it can also be applied to the full subcategories of the cofibrant, fibrant resp. bifibrant objects of a Quillen model category. As a consequence, all of them admit a 3-arrow calculus.

[2] It is easy to show that every morphism in the Gabriel-Zisman localisation of a uni-fractionable category can be represented by a diagram of length 3 (cf. the definition of the composition in proposition (5.4)). However, the author does not know how to prove in that context that two of these diagrams represent the same morphism if and only if they can be embedded in a 3-by-3 diagram as above.

[3] The S resp. the T should remind us of the fact that the S-denominator resp. the T-denominator in a factorisation has the same source resp. the same target as the factorised morphism.

Furthermore, a derivable category in the sense of CISINSKI [7, sec. 2.25] ([4]), which is a self-dual generalisation of a category of fibrant objects in the sense of K. BROWN [4, sec. 1], admits a 3-arrow calculus, provided stronger variants of the factorisation axioms and the axioms which ensure stability of acyclic cofibrations under pushouts resp. of acyclic fibrations under pullbacks hold. For the relationship of CISINSKI's approach with other axiom systems, see [31, sec. 2].
A further example of a uni-fractionable category structure is provided by the category of complexes in an arbitrary abelian category, where the denominators are given by the quasi-isomorphisms, that is, by those morphisms inducing isomorphisms on the homology objects. To obtain the derived category, instead of localising the homotopy category of complexes, we may directly localise the category of complexes itself. The price to pay is that instead of a 2-arrow calculus, we obtain a 3-arrow calculus. Similarly for the derived category of an idempotent splitting exact category.
One feature of this 3-arrow approach is its self-duality. This might be a reason why 3-arrows occurred implicitly in GROTHENDIECK's construction of a localisation of an abelian category with respect to a thick subcategory [15, sec. 1.11, p. 138], cf. example (7.7)(b), although a 2-arrow approach is of course sufficient.

Outline We recall in section 2 some notions of localisation theory and indicate how quotients of (ordered) graphs with respect to so-called graph congruences can be constructed. In section 3, uni-fractionable categories are introduced. Recall that the aim of this article is to construct a localisation of a uni-fractionable category with respect to its set of denominators. To this end, we proceed in two steps: In section 4, we assign to a uni-fractionable category a certain graph, its 3-arrow graph, and introduce a graph congruence on this graph. Then, in section 5, it turns out that the quotient graph has a canonically given category structure, and we will show that this category is a localisation of the uni-fractionable category we started with. Our main theorem (5.18) then gives a criterion on when two 3-arrows represent the same morphism in the localisation. In section 6, we give a sufficient criterion for the localisation and the localisation functor being additive. Finally, in section 7, we show how Quillen model categories, derivable categories (under additional conditions), complexes and some further classical examples fit into this framework. The example of complexes with entries in an idempotent splitting exact category is best understood when generalised; this requires a little theory of formal cones as provided in appendix A.

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Conventions and notations

We use the following conventions and notations.

- The composite of morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ is usually denoted by $fg\colon X \to Z$. The composite of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ is usually denoted by $G \circ F\colon \mathcal{C} \to \mathcal{E}$.
- Isomorphy of objects X and Y is denoted by $X \cong Y$. Equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is denoted by $\mathcal{C} \simeq \mathcal{D}$.
- Given a category $\mathcal{C}$ and objects X and Y in $\mathcal{C}$, we write ${}_{\mathcal{C}}(X, Y)$ for the set of morphisms from X to Y.
- Given a subobject U of an object X, we denote by $\mathrm{inc} = \mathrm{inc}^U\colon U \to X$ the inclusion. Dually, given a quotient object Q of an object X, we denote by $\mathrm{quo} = \mathrm{quo}^Q\colon X \to Q$ the quotient morphism.
- Given a coproduct C of X_1 and X_2, the embedding $X_k \to C$ is denoted by $\mathrm{emb}_k = \mathrm{emb}_k^C$ for $k \in \{1, 2\}$. Given morphisms $f_k\colon X_k \to Y$ for $k \in \{1, 2\}$, the induced morphism $C \to Y$ is denoted by $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^C$. Dually for products: We write $\mathrm{pr}_k = \mathrm{pr}_k^P$ for the projections of a product of X_1 and X_2 and $\begin{pmatrix} f_1 & f_2 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^P$ for the induced morphism $Y \to P$ of morphisms $f_k\colon Y \to X_k$, $k \in \{1, 2\}$.

[4] Also called an Anderson-Brown-Cisinski premodel category by RĂDULESCU-BANU [31, def. 1.1.3].

- By a sum of objects X_1 and X_2, we understand an object S such that S carries the structure of a coproduct and a product of X_1 and X_2, and such that $\mathrm{emb}_k \mathrm{pr}_l = \delta_{k,l}$ for $k, l \in \{1, 2\}$, where δ denotes the Kronecker delta.

- Given an initial object I, the unique morphism $I \to X$ to an object X will be denoted by $\mathrm{ini} = \mathrm{ini}_X = \mathrm{ini}_X^I$. Dually, given a terminal object T, the unique morphism $X \to T$ from an object T will be denoted by $\mathrm{ter} = \mathrm{ter}_X = \mathrm{ter}_X^T$. Given a zero object N, the unique morphism $X \to Y$ that factors over N will be denoted by 0.

- Given a category admitting finite coproducts and objects X_1, X_2, we denote by $X_1 \amalg X_2$ a chosen coproduct and by ¡ a chosen initial object. Analogously, given morphisms $f_k \colon X_k \to Y_k$ for $k \in \{1, 2\}$, the coproduct of f_1 and f_2 is denoted by $f_1 \amalg f_2$. Analogously for finite products resp. finite sums, where we write $X_1 \Pi X_2$ and $f_1 \Pi f_2$ for a chosen product and ! for a chosen terminal object resp. $X_1 \oplus X_2$ and $f_1 \oplus f_2$ for a chosen sum and 0 for a chosen zero object.

- Given a category admitting finite coproducts $\mathcal{C}$ and a category $\mathcal{D}$, we say that a functor $F \colon \mathcal{C} \to \mathcal{D}$ preserves finite coproducts if $F¡$ is an initial object in $\mathcal{D}$, and if, given $X_1, X_2 \in \mathrm{Ob}\,\mathcal{C}$, the object $F(X_1 \amalg X_2)$ is a coproduct of FX_1 and FX_2, where the embeddings are given by $\mathrm{emb}_1^{F(X_1 \amalg X_2)} = F(\mathrm{emb}_1^{X_1 \amalg X_2})$ and $\mathrm{emb}_2^{F(X_1 \amalg X_2)} = F(\mathrm{emb}_2^{X_1 \amalg X_2})$. Dually for finite products and analogously for finite sums.

- Given a category admitting kernels, we denote by $\mathrm{Ker}\, f$ a chosen kernel of a morphism f. Given a category admitting cokernels, we denote by $\mathrm{Coker}\, f$ a chosen cokernel of a morphism f.

- The opposite category of a category $\mathcal{C}$ is denoted by $\mathcal{C}^{\mathrm{op}}$.

- By a weak pushout rectangle (resp. weak pullback rectangle) we understand a quadrangle having the universal property of a pushout rectangle (resp. pullback rectangle) except for the uniqueness of the induced morphism.

- The category of complexes in an additive category $\mathcal{A}$ is denoted by $\mathrm{C}(\mathcal{A})$, its homotopy category by $\mathrm{K}(\mathcal{A})$. The derived category of an exact category $\mathcal{E}$ is denoted by $\mathrm{D}(\mathcal{E})$.

- Arrows a and b in an (oriented) graph are called parallel if $\mathrm{Source}\, a = \mathrm{Source}\, b$ and $\mathrm{Target}\, a = \mathrm{Target}\, b$.

- In an exact category $\mathcal{E}$, the distinguished short exact sequences in $\mathcal{E}$ will be called *pure short exact sequences*. Likewise, the monomorphisms occurring in a pure short exact sequence are called *pure monomorphisms*, and the epimorphisms occurring in a pure short exact sequence are called *pure epimorphisms*.

- We use the notations $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

- Given integers $a, b \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ for the set of integers lying between a and b.

A remark on Grothendieck universes To avoid set-theoretical difficulties, we work with Grothendieck universes [1, exp. I, sec. 0] in this article. In particular, every category has an object *set* and a morphism *set*. Given a Grothendieck universe $\mathfrak{U}$, we say that a category $\mathcal{C}$ is a $\mathfrak{U}$-*category* if $\mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Mor}\,\mathcal{C}$ are elements of $\mathfrak{U}$. The *category of* $\mathfrak{U}$-*categories*, whose object set consists of all $\mathfrak{U}$-categories and whose morphism set consists of all functors between $\mathfrak{U}$-categories (and source, target, composition and identities given by ordinary source, target, composition of functors and the identity functors, respectively), will be denoted $\mathbf{Cat} = \mathbf{Cat}_{(\mathfrak{U})}$.

2 Preliminaries

In this section, we give some preliminaries on localisations of categories and quotient graphs with respect to graph congruences.

Localisations of categories

We suppose given a category $\mathcal{C}$. A *denominator set* in $\mathcal{C}$ is a subset $D \subseteq \operatorname{Mor} \mathcal{C}$. We will consider denominator sets with special properties later in this article, but at the moment, a denominator set D is just an arbitrary subset of $\operatorname{Mor} \mathcal{C}$. Informally, it is a subset singled out with the "intention of localising with respect to it", in the following sense.
A *localisation* of $\mathcal{C}$ with respect to a denominator set D in $\mathcal{C}$ consists of a category $\mathcal{L}$ and a functor $L\colon \mathcal{C} \to \mathcal{L}$ such that the following axioms hold.

(Inv) *Invertibility.* For all $d \in D$, the morphism Ld is invertible.

(1-uni) *1-universality.* Given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible for all $d \in D$, there exists a unique functor $\hat{F}\colon \mathcal{L} \to \mathcal{D}$ with $F = \hat{F} \circ L$.

(2-uni) *2-universality.* We suppose given a category $\mathcal{D}$ and functors $F, G\colon \mathcal{C} \to \mathcal{D}$ such that Fd and Gd are invertible for all $d \in D$, and we denote by $\hat{F}\colon \mathcal{L} \to \mathcal{D}$ resp. $\hat{G}\colon \mathcal{L} \to \mathcal{D}$ the unique functor with $F = \hat{F} \circ L$ resp. $G = \hat{G} \circ L$. Given a transformation $\alpha\colon F \to G$, there exists a unique transformation $\hat{\alpha}\colon \hat{F} \to \hat{G}$ such that $\hat{\alpha}_{LX} = \alpha_X$ for all $X \in \operatorname{Ob} \mathcal{C}$.

$$
\begin{tikzcd}
\mathcal{L} \arrow[r, dotted, bend left, "\hat{F}"{name=A}] \arrow[r, dotted, bend right, "\hat{G}"'{name=B}] \arrow[from=A, to=B, dotted, Rightarrow, "\hat{\alpha}"] & \mathcal{D} \\
\mathcal{C} \arrow[u, "L"] \arrow[ur, bend left, "F"{name=C}] \arrow[ur, bend right, "G"'{name=E}] \arrow[from=C, to=E, Rightarrow, "\alpha"] &
\end{tikzcd}
$$

By abuse of notation, we refer to the localisation as well as to its underlying category just by $\mathcal{L}$. The functor L is said to be the *localisation functor* of the localisation $\mathcal{L}$. Given a localisation $\mathcal{L}$ of $\mathcal{C}$ with respect to D with localisation functor $L\colon \mathcal{C} \to \mathcal{L}$, we write $\operatorname{loc} = \operatorname{loc}^{\mathcal{L}} := L$.
Gabriel and Zisman have shown in [12, sec. 1.1] that there exists a localisation of every category $\mathcal{C}$ with respect to an arbitrary denominator set D in $\mathcal{C}$. We will not make use of this result. Rather, given a uni-fractionable category, see definition (3.1), we construct a localisation directly, cf. propositions (5.4) and (5.7).

Saturatedness

We suppose given a category $\mathcal{C}$, a denominator set D in $\mathcal{C}$, and a localisation $\mathcal{L}$ of $\mathcal{C}$ with respect to D. By definition of a localisation, $\operatorname{loc}(d)$ is invertible for every $d \in D$. But in general, not every morphism f in $\mathcal{C}$ for which $\operatorname{loc}(f)$ is invertible in $\mathcal{L}$ has to be an element of D. The denominator set D is said to be *saturated* if $f \in D$ for all $f \in \operatorname{Mor} \mathcal{C}$ with $\operatorname{loc}(f)$ invertible in $\mathcal{L}$. We use the following notions to indicate how far D is away from this property.
The denominator set D is said to be *multiplicative* if it fulfills:

(Cat) *Multiplicativity.* For all $d, e \in D$ with $\operatorname{Target} d = \operatorname{Source} e$, their composite de is in D, and for every object X in $\mathcal{C}$, the identity 1_X is in D.

The denominator set D is said to be *semi-saturated* if it is multiplicative and fulfills:

(2 of 3) *2 out of 3 axiom.* We suppose given morphisms f and g in $\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Source} g$. If two out of the morphisms f, g, fg are in D, then so is the third.

Finally, the denominator set D is said to be *weakly saturated* if it is multiplicative and fulfills:

(2 of 6) *2 out of 6 axiom.* We suppose given morphisms f, g, h in $\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Source} g$ and $\operatorname{Target} g = \operatorname{Source} h$. If $fg, gh \in D$, then $f, g, h, fgh \in D$.

Saturatedness implies weak saturatedness, weak saturatedness implies semi-saturatedness, and semi-saturatedness implies multiplicativity (the last impliciation holds by definition).

Categories with denominators

A *category with denominators* consists of a category $\mathcal{C}$ together with a denominator set D in $\mathcal{C}$. By abuse of notation, we refer to the category with denominators as well as to its underlying category just by $\mathcal{C}$. The elements of D are called *denominators* in $\mathcal{C}$.
Given a category with denominators $\mathcal{C}$ with set of denominators D, we write $\operatorname{Den}\mathcal{C} := D$. In diagrams, a denominator d in $\mathcal{C}$ will usually be depicted as

$$\xrightarrow[\approx]{d}\ .$$

We suppose given categories with denominators $\mathcal{C}$ and $\mathcal{D}$. A *morphism of categories with denominators* from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F\colon \mathcal{C} \to \mathcal{D}$ that *preserves denominators*, that is, such that Fd is a denominator in $\mathcal{D}$ for every denominator d in $\mathcal{C}$.
We suppose given a Grothendieck universe $\mathfrak{U}$. A category with denominators is said to be a $\mathfrak{U}$-*category with denominators* if its underlying category is in $\mathfrak{U}$. The category $\mathbf{CatD} = \mathbf{CatD}_{(\mathfrak{U})}$ consisting of the set of $\mathfrak{U}$-categories with denominators as set of objects and the set of morphisms of categories with denominators between $\mathfrak{U}$-categories with denominators as set of morphisms (and categorical structure maps induced from $\mathbf{Cat}_{(\mathfrak{U})}$) is called the *category of categories with denominators* (more precisely, the *category of* $\mathfrak{U}$*-categories with denominators*).
Given a category with denominators $\mathcal{C}$, a *localisation* of $\mathcal{C}$ is defined to be a localisation of (the underlying category of) $\mathcal{C}$ with respect to its set of denominators $\operatorname{Den}\mathcal{C}$.
A category with denominators $\mathcal{C}$ is said to be *multiplicative* resp. *semi-saturated* resp. *weakly saturated* resp. *saturated* if its set of denominators $\operatorname{Den}\mathcal{C}$ is multiplicative resp. semi-saturated resp. weakly saturated resp. saturated denominator set in the category $\mathcal{C}$.

Graph congruences and quotient graphs

We suppose given an (oriented) graph $\mathcal{G}$. An equivalence relation $\equiv$ on $\operatorname{Arr}\mathcal{G}$ is said to be a *graph congruence* on $\mathcal{G}$ if $\operatorname{Source} a = \operatorname{Source}\tilde{a}$ and $\operatorname{Target} a = \operatorname{Target}\tilde{a}$ for all $a, \tilde{a} \in \operatorname{Arr}\mathcal{G}$ with $a \equiv \tilde{a}$. Given a graph congruence $\equiv$ on $\mathcal{G}$, the *quotient graph* of $\mathcal{G}$ with respect to $\equiv$ is the graph $\mathcal{G}/{\equiv}$ with $\operatorname{Ob}\mathcal{G}/{\equiv} := \operatorname{Ob}\mathcal{G}$, $\operatorname{Arr}\mathcal{G}/{\equiv} := (\operatorname{Arr}\mathcal{G})/{\equiv}$ and $\operatorname{Source}[a]_{\equiv} := \operatorname{Source} a$, $\operatorname{Target}[a]_{\equiv} := \operatorname{Target} a$ for $a \in \operatorname{Arr}\mathcal{G}$. The graph morphism $\operatorname{quo} = \operatorname{quo}^{\mathcal{G}/\equiv}\colon \mathcal{G} \to \mathcal{G}/{\equiv}$ given by $\operatorname{quo}(X) := X$ and $\operatorname{quo}(a) := [a]_{\equiv}$ is called the *quotient graph morphism*.
The quotient graph of $\mathcal{G}$ with respect to a graph congrunce $\equiv$ fulfills the following universal property. Given $a, \tilde{a} \in \operatorname{Arr}\mathcal{G}$ with $a \equiv \tilde{a}$, we have $\operatorname{quo}(a) = \operatorname{quo}(\tilde{a})$. For every graph $\mathcal{H}$ and every graph morphism $F\colon \mathcal{G} \to \mathcal{H}$ with $Fa = F\tilde{a}$ for $a, \tilde{a} \in \operatorname{Arr}\mathcal{G}$ with $a \equiv \tilde{a}$, there exists a unique graph morphism $\overline{F}\colon \mathcal{G}/{\equiv} \to \mathcal{H}$ with $F = \overline{F} \circ \operatorname{quo}$.

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "F"] \arrow[d, "\operatorname{quo}"'] & \mathcal{H} \\
\mathcal{G}/{\equiv} \arrow[ur, dotted, "\overline{F}"']
\end{tikzcd}$$

3 Uni-fractionable categories

(3.1) Definition (uni-fractionable categories, morphisms of uni-fractionable categories)**.**

(a) A *uni-fractionable category* ([5]) consists of a semi-saturated category with denominators $\mathcal{C}$ together with multiplicative subsets $S, T \subseteq \operatorname{Den}\mathcal{C}$ such that the following axioms hold.

(WU) *Weakly universal Ore completions.* Given morphisms i and f in $\mathcal{C}$ with $i \in S$ and $\operatorname{Source} i = \operatorname{Source} f$, there exists a weak pushout rectangle

$$\begin{tikzcd}
\cdot \arrow[r, "f'"] & \cdot \\
\cdot \arrow[u, "i"] \arrow[r, "f"] & \cdot \arrow[u, "i'"']
\end{tikzcd}$$

[5] There exists also the notion of a fractionable category, cf. the author's forthcoming doctoral thesis.

in $\mathcal{C}$ such that $i' \in S$. Dually, given morphisms p and f in $\mathcal{C}$ with $p \in T$ and $\operatorname{Target} p = \operatorname{Target} f$, there exists a weak pullback rectangle

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "p'"'] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

in $\mathcal{C}$ such that $p' \in T$.

(Fac) *Factorisations.* For every denominator d in $\mathcal{C}$, there exist $i \in S$ and $p \in T$ with $d = ip$.

$$\begin{tikzcd} & \cdot \arrow[dr, dotted, "p"] & \\ \cdot \arrow[ur, dotted, "i"] \arrow[rr, "d"'] & & \cdot \end{tikzcd}$$

By abuse of notation, we refer to the uni-fractionable category as well as to its underlying category with denominators just by $\mathcal{C}$. The elements of S are called *S-denominators* in $\mathcal{C}$, and the elements of T are called *T-denominators* in $\mathcal{C}$.

Given a uni-fractionable category $\mathcal{C}$ with set of S-denominators S and set of T-denominators T, we write $\operatorname{SDen} \mathcal{C} := S$ and $\operatorname{TDen} \mathcal{C} := T$. In diagrams, an S-denominator i resp. a T-denominator p in $\mathcal{C}$ will usually be depicted as

$$\xrightarrow{i} \quad \text{resp.} \quad \xrightarrow{p} \quad .$$

(b) We suppose given uni-fractionable categories $\mathcal{C}$ and $\mathcal{D}$. A *morphism of uni-fractionable categories* from $\mathcal{C}$ to $\mathcal{D}$ is a morphism of categories with denominators $F \colon \mathcal{C} \to \mathcal{D}$ that *preserves S-denominators* and *T-denominators*, that is, such that Fi is an S-denominator in $\mathcal{D}$ for every S-denominator i in $\mathcal{C}$ and such that Fp is a T-denominator in $\mathcal{D}$ for every T-denominator p in $\mathcal{C}$.

Some examples of uni-fractionable categories can be found in section 7.
Since the composite of composable morphisms of uni-fractionable categories is again a morphism of uni-fractionable categories and the identity functor on a uni-fractionable category is a morphism of uni-fractionable categories, we get a category of uni-fractionable categories:

(3.2) Definition (uni-fractionable category in a Grothendieck universe)**.** We suppose given a Grothendieck universe $\mathfrak{U}$. A uni-fractionable category $\mathcal{C}$ is said to be a *$\mathfrak{U}$-uni-fractionable category* if its underlying category with denominators is a category with denominators in $\mathfrak{U}$.

(3.3) Remark.

(a) We suppose given a Grothendieck universe $\mathfrak{U}$. A uni-fractionable category $\mathcal{C}$ is a $\mathfrak{U}$-uni-fractionable category if and only if it is an element of $\mathfrak{U}$.

(b) For every uni-fractionable category $\mathcal{C}$ there exists a Grothendieck universe $\mathfrak{U}$ such that $\mathcal{C}$ is in $\mathfrak{U}$.

(3.4) Definition (category of uni-fractionable categories)**.** We suppose given a Grothendieck universe $\mathfrak{U}$.

(a) The category $\mathbf{UFrCat} = \mathbf{UFrCat}_{(\mathfrak{U})}$ consisting of the set of $\mathfrak{U}$-uni-fractionable categories as set of objects and the set of morphisms of uni-fractionable categories between $\mathfrak{U}$-uni-fractionable categories as set of morphisms (and categorical structure maps induced from $\mathbf{CatD}_{(\mathfrak{U})}$) is called the *category of uni-fractionable categories* (more precisely, the *category of $\mathfrak{U}$-uni-fractionable categories*).

(b) We denote by $\mathbf{UFr}(\mathbf{CatD}_{(\mathfrak{U})})$ the full subcategory of $\mathbf{CatD}_{(\mathfrak{U})}$ with

$$\begin{aligned} \operatorname{Ob} \mathbf{UFr}(\mathbf{CatD}_{(\mathfrak{U})}) = \{\mathcal{C} \in \operatorname{Ob} \mathbf{CatD}_{(\mathfrak{U})} \mid {} & \text{there exist } S, T \subseteq \operatorname{Den} \mathcal{C} \text{ such that } \mathcal{C} \text{ becomes a} \\ & \text{uni-fractionable category with } \operatorname{SDen} \mathcal{C} = S \text{ and } \operatorname{TDen} \mathcal{C} = T\}, \end{aligned}$$

the *category of categories with denominators admitting the structure of a uni-fractionable category* (more precisely, the *category of $\mathfrak{U}$-categories with denominators admitting the structure of a uni-fractionable category*).

4 The 3-arrow graph

We want to construct a localisation $\operatorname{Frac} \mathcal{C}$ of a uni-fractionable category $\mathcal{C}$ (with respect to its set of denominators $\operatorname{Den} \mathcal{C}$). To this end, we begin in this section by introducing its 3-arrow graph $\operatorname{AG} \mathcal{C}$ and a graph congruence $\equiv$ on $\operatorname{AG} \mathcal{C}$.
In this section, we suppose given a uni-fractionable category $\mathcal{C}$.

(4.1) Definition (3-arrow shape)**.** The graph

$$0 \xleftarrow{\tau} 1 \xrightarrow{\nu} 2 \xleftarrow{\sigma} 3$$

is said to be the 3-*arrow shape* and will be denoted by $\mathbf{\Theta}$.

Recall that a *diagram* of *shape* $\mathbf{\Theta}$ in $\mathcal{C}$ is just a graph morphism $D \colon \mathbf{\Theta} \to \mathcal{C}$. Given a diagram D of shape $\mathbf{\Theta}$ in $\mathcal{C}$, we write $D_i := D(i)$ for $i \in \operatorname{Ob} \mathbf{\Theta}$ and $D_a := D(a)$ for $a \in \operatorname{Arr} \mathbf{\Theta}$. Given diagrams D and E, a *diagram morphism* from D to E is a family $f = (f_i)_{i \in \operatorname{Ob} \mathbf{\Theta}}$ in $\operatorname{Mor} \mathcal{C}$ with $D_a f_j = f_i E_a$ for all arrows $a \colon i \to j$ in $\mathbf{\Theta}$. The category consisting of diagrams of shape $\mathbf{\Theta}$ in $\mathcal{C}$ as objects and diagram morphisms between those diagrams as morphisms will be denoted by $\mathcal{C}^{\mathbf{\Theta}}$. ([6])

(4.2) Definition (3-arrow graph)**.** The 3-*arrow graph* of $\mathcal{C}$ is defined to be the graph $\operatorname{AG} \mathcal{C}$ with object set

$$\operatorname{Ob} \operatorname{AG} \mathcal{C} := \operatorname{Ob} \mathcal{C}$$

and arrow set

$$\operatorname{Arr} \operatorname{AG} \mathcal{C} := \{A \in \operatorname{Ob} \mathcal{C}^{\mathbf{\Theta}} \mid A_\sigma, A_\tau \in \operatorname{Den} \mathcal{C}\}.$$

The source resp. the target of $A \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ are defined by $\operatorname{Source} A := A_0$ resp. $\operatorname{Target} A := A_3$.
An arrow A in $\operatorname{AG} \mathcal{C}$ is called a 3-*arrow* in $\mathcal{C}$. Given a denominator $b \colon \tilde{X} \to X$, a morphism $f \colon \tilde{X} \to \tilde{Y}$ and a denominator $a \colon Y \to \tilde{Y}$ in $\mathcal{C}$, we abuse notation and denote the unique 3-arrow A with $A_\tau = b$, $A_\nu = f$, $A_\sigma = a$ by $(b, f, a) := A$. Moreover, we use the notation $(b, f, a) \colon X \leftarrow \tilde{X} \to \tilde{Y} \leftarrow Y$.

$$X \xleftarrow[\approx]{b} \tilde{X} \xrightarrow{f} \tilde{Y} \xleftarrow[\approx]{a} Y$$

(4.3) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\mathbf{\Theta}$ is in $\mathfrak{U}$. If $\mathcal{C}$ is in $\mathfrak{U}$, then its 3-arrow graph $\operatorname{AG} \mathcal{C}$ is in $\mathfrak{U}$.

Proof. We suppose that $\mathcal{C}$ is in $\mathfrak{U}$. Then $\operatorname{Ob} \mathcal{C}$ and $\operatorname{Mor} \mathcal{C}$ are in $\mathfrak{U}$ and hence $\operatorname{Map}(\operatorname{Arr} \mathbf{\Theta}, \operatorname{Mor} \mathcal{C})$ is in $\mathfrak{U}$. But then $\operatorname{Ob} \operatorname{AG} \mathcal{C}$ and $\operatorname{Arr} \operatorname{AG} \mathcal{C}$ are in $\mathfrak{U}$, that is, $\operatorname{AG} \mathcal{C}$ is in $\mathfrak{U}$. □

Our next step will be the introduction of an equivalence relation on the arrow set of the 3-arrow graph.

(4.4) Definition (fraction equality)**.** The equivalence relation $\equiv$ on $\operatorname{Arr} \operatorname{AG} \mathcal{C}$ is defined to be generated by the following relation on $\operatorname{Arr} \operatorname{AG} \mathcal{C}$: Given $(b, f, a) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ and $c \in \operatorname{Mor} \mathcal{C}$ with $ac \in \operatorname{Den} \mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow (b, fc, ac); and given $(b, f, a) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ and $c \in \operatorname{Mor} \mathcal{C}$ with $cb \in \operatorname{Den} \mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow (cb, cf, a).

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "f"] \arrow[d, equal] & {} \arrow[d, "c"] & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "b", "\approx"'] \arrow[r, "fc"] & {} & {} \arrow[l, "ac", "\approx"']
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "f"] & {} \arrow[d, equal] & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "cb", "\approx"'] \arrow[r, "cf"] \arrow[u, "c"] & {} & {} \arrow[l, "a", "\approx"']
\end{tikzcd}$$

Given $(b, f, a), (\tilde{b}, \tilde{f}, \tilde{a}) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ with $(b, f, a) \equiv (\tilde{b}, \tilde{f}, \tilde{a})$, we say that (b, f, a) and $(\tilde{b}, \tilde{f}, \tilde{a})$ are *fraction equal*.

In practice, it is sometimes convenient to work with different generating sets for fraction equality. These are stated in the following remark.

[6] By the adjunction "free category on a graph – underlying graph of a category", diagrams of shape $\mathbf{\Theta}$ in $\mathcal{C}$ correspond in a unique way to functors from the free category on $\mathbf{\Theta}$ to $\mathcal{C}$, and diagram morphisms correspond to transformations.

(4.5) Remark.

(a) The fraction equality relation $\equiv$ on $\operatorname{Arr} \operatorname{AG} \mathcal{C}$ is generated by the following relation: Given $(b, f, a) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ and $c, c' \in \operatorname{Mor} \mathcal{C}$ with $ac, c'b \in \operatorname{Den} \mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow $(c'b, c'fc, ac)$.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "f"] & {} \arrow[d, "c"] & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "c'b", "\approx"'] \arrow[r, "c'fc"] \arrow[u, "c'"'] & {} & {} \arrow[l, "ac", "\approx"']
\end{tikzcd}$$

(b) The fraction equality relation $\equiv$ on $\operatorname{Arr} \operatorname{AG} \mathcal{C}$ is generated by the following relation: Given $(b, f, a), (\tilde{b}, \tilde{f}, \tilde{a}) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$, the 3-arrow (b, f, a) is in relation to the 3-arrow $(\tilde{b}, \tilde{f}, \tilde{a})$ if there exist $c, c' \in \operatorname{Mor} \mathcal{C}$ with $b = c'\tilde{b}$, $fc = c'\tilde{f}$, $ac = \tilde{a}$.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b"', "\approx"] \arrow[r, "f"] \arrow[d, "c'"] & {} \arrow[d, "c"] & {} \arrow[l, "a"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "\tilde{b}", "\approx"'] \arrow[r, "\tilde{f}"] & {} & {} \arrow[l, "\tilde{a}", "\approx"']
\end{tikzcd}$$

As $\mathcal{C}$ is semi-saturated, the morphisms c and c' in definition (4.4) and remark (4.5) are automatically denominators in $\mathcal{C}$.

(4.6) Remark. We suppose given 3-arrows (b, f, a) and $(\tilde{b}, \tilde{f}, \tilde{a})$ in $\mathcal{C}$. If $(b, f, a) \equiv (\tilde{b}, \tilde{f}, \tilde{a})$, then f is a denominator in $\mathcal{C}$ if and only if $\tilde{f}$ is a denominator in $\mathcal{C}$.

Proof. This follows by the definition of fraction equality (4.4) and by the semi-saturatedness of $\mathcal{C}$. $\square$

Before we study a further property of the fraction equality relation, we will show that fraction equality respects the graph structure on the 3-arrow graph.

(4.7) Remark. The fraction equality relation $\equiv$ on $\operatorname{Arr} \operatorname{AG} \mathcal{C}$ defines a graph congruence on $\operatorname{AG} \mathcal{C}$. In particular, the quotient graph $(\operatorname{AG} \mathcal{C})/{\equiv}$ is defined.

Proof. For $(b, f, a) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$, $c, c' \in \operatorname{Mor} \mathcal{C}$ with $ac, c'b \in \operatorname{Den} \mathcal{C}$, we have

$$\operatorname{Source}(c'b, c'fc, ac) = \operatorname{Target}(c'b) = \operatorname{Target} b = \operatorname{Source}(b, f, a) \text{ and}$$
$$\operatorname{Target}(c'b, c'fc, ac) = \operatorname{Source}(ac) = \operatorname{Source} a = \operatorname{Target}(b, f, a).$$

Thus the assertion follows from remark (4.5)(a). $\square$

(4.8) Definition (double fraction)**.** Given a 3-arrow (b, f, a) in $\mathcal{C}$, its equivalence class in the quotient graph $(\operatorname{AG} \mathcal{C})/{\equiv}$ is denoted by $b \backslash f / a := [(b, f, a)]_{\equiv}$ and is said to be the *double fraction* of (b, f, a).

Now we will present a certain reduced form for 3-arrows. We will see that every 3-arrow is fraction equal to such a reduced form.

(4.9) Definition (normal 3-arrows)**.** A 3-arrow (p, f, i) in $\mathcal{C}$ is said to be *normal* if i is an S-denominator and p is a T-denominator in $\mathcal{C}$.

$$\begin{tikzcd}
{} & {} \arrow[l, "p"', "\Box"] \arrow[r, "f"] & {} & {} \arrow[l, "i"', "\circ"]
\end{tikzcd}$$

The following lemma and its proof is (essentially) taken from [9, sec. 36.5].

(4.10) Lemma (normalisation lemma)**.** Every 3-arrow in $\mathcal{C}$ is fraction equal to a normal 3-arrow in $\mathcal{C}$.

Proof. We suppose given an arbitrary 3-arrow (b, f, a) in $\mathcal{C}$. There exist an S-denominator i and a T-denominator p in $\mathcal{C}$ with $b = ip$, and there exist an S-denominator i' and a morphism f' in $\mathcal{C}$ with $if' = fi'$. By multiplicativity, ai' is a denominator in $\mathcal{C}$. Thus there exist an S-denominator j and a T-denominator q in $\mathcal{C}$ with $ai' = jq$, and there exist a T-denominator q' and a morphism f'' in $\mathcal{C}$ with $f''q = q'f'$. By multiplicativity, $q'p$ is a T-denominator.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b" description, "\approx"'] \arrow[r, "f"] \arrow[d, "i"] & {} \arrow[d, "i'"] & {} \arrow[l, "a" description, "\approx"'] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "p"'] \arrow[r, "f'"] & {} & {} \arrow[l, "ai'" description, "\approx"'] \arrow[d, equal] \\
{} & {} \arrow[l, "q'p"'] \arrow[r, "f''"] \arrow[u, "q'"] & {} \arrow[u, "q"] & {} \arrow[l, "j"']
\end{tikzcd}$$

Altogether, $(b, f, a) \equiv (p, f', ai') \equiv (q'p, f'', j)$, and since j is an S-denominator and $q'p$ is a T-denominator, the 3-arrow $(q'p, f'', j)$ is normal. □

(4.11) Corollary. We suppose given a uni-fractionable category $\mathcal{C}$ and 3-arrows (b_1, f_1, a_1) and (b_2, f_2, a_2) in $\mathcal{C}$.

(a) If $\operatorname{Source}(b_1, f_1, a_1) = \operatorname{Source}(b_2, f_2, a_2)$, then there exist normal 3-arrows $(p, \tilde{f}_1, i_1)$ and $(p, \tilde{f}_2, i_2)$ in $\mathcal{C}$ with $(b_1, f_1, a_1) \equiv (p, \tilde{f}_1, i_1)$ and $(b_2, f_2, a_2) \equiv (p, \tilde{f}_2, i_2)$.

(b) If $\operatorname{Target}(b_1, f_1, a_1) = \operatorname{Target}(b_2, f_2, a_2)$, then there exist normal 3-arrows $(p_1, \tilde{f}_1, i)$ and $(p_2, \tilde{f}_2, i)$ in $\mathcal{C}$ with $(b_1, f_1, a_1) \equiv (p_1, \tilde{f}_1, i)$ and $(b_2, f_2, a_2) \equiv (p_2, \tilde{f}_2, i)$.

(c) If (b_1, f_1, a_1) and (b_2, f_2, a_2) are parallel, then there exist normal 3-arrows $(p, \tilde{f}_1, i)$ and $(p, \tilde{f}_2, i)$ in $\mathcal{C}$ with $(b_1, f_1, a_1) \equiv (p, \tilde{f}_1, i)$ and $(b_2, f_2, a_2) \equiv (p, \tilde{f}_2, i)$.

Proof. By the normalisation lemma (4.10), there exist normal 3-arrows (p_k, g_k, i_k) in $\mathcal{C}$ with $(b_k, f_k, a_k) \equiv (p_k, g_k, i_k)$ for $k \in \{1, 2\}$. In particular, we have $\operatorname{Source}(p_k, g_k, i_k) = \operatorname{Source}(b_k, f_k, a_k)$ and $\operatorname{Target}(p_k, g_k, i_k) = \operatorname{Target}(b_k, f_k, a_k)$ for $k \in \{1, 2\}$. Hence $\operatorname{Source}(b_1, f_1, a_1) = \operatorname{Source}(b_2, f_2, a_2)$ implies that $\operatorname{Source}(p_1, g_1, i_1) = \operatorname{Source}(p_2, g_2, i_2)$ and $\operatorname{Target}(b_1, f_1, a_1) = \operatorname{Target}(b_2, f_2, a_2)$ implies that $\operatorname{Target}(p_1, g_1, i_1) = \operatorname{Target}(p_2, g_2, i_2)$.

(a) There exist a T-denominator p'_2 and a morphism p'_1 in $\mathcal{C}$ with $p'_2 p_1 = p'_1 p_2$. We define $p := p'_2 p_1 = p'_1 p_2$, $\tilde{f}_1 := p'_2 g_1$, $\tilde{f}_2 := p'_1 g_2$.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "p_1"'] \arrow[r, "g_1"] & {} & {} \arrow[l, "i_1"'] \\
{} \arrow[d, equal] & {} \arrow[l, "p"'] \arrow[u, "p'_2"] \arrow[d, "p'_1"'] \arrow[ur, "\tilde{f}_1"'] \arrow[dr, "\tilde{f}_2"] & {} & {} \\
{} & {} \arrow[l, "p_2"'] \arrow[r, "g_2"] & {} & {} \arrow[l, "i_2"']
\end{tikzcd}$$

By multiplicativity, $p = p'_2 p_1$ is a T-denominator in $\mathcal{C}$, and we have

$$(p, \tilde{f}_1, i_1) = (p'_2 p_1, p'_2 g_1, i_1) \equiv (p_1, g_1, i_1) \equiv (b_1, f_1, a_1) \text{ and}$$
$$(p, \tilde{f}_2, i_2) = (p'_1 p_2, p'_1 g_2, i_2) \equiv (p_2, g_2, i_2) \equiv (b_2, f_2, a_2).$$

(b) This is dual to (a).

(c) There exist a T-denominator p'_2 and a morphism p'_1 in $\mathcal{C}$ with $p'_2 p_1 = p'_1 p_2$, and there exist an S-denominator i'_1 and a morphism i'_2 in $\mathcal{C}$ with $i_1 i'_2 = i_2 i'_1$. We define $p := p'_2 p_1 = p'_1 p_2$, $i := i_1 i'_2 = i_2 i'_1$, $\tilde{f}_1 := p'_2 g_1 i'_2$, $\tilde{f}_2 := p'_1 g_2 i'_1$.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "p_1"'] \arrow[r, "g_1"] & {} \arrow[d, "i'_2"] & {} \arrow[l, "i_1"'] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "p"'] \arrow[u, "p'_2"] \arrow[d, "p'_1"'] \arrow[r, shift left, "\tilde{f}_1"] \arrow[r, shift right, "\tilde{f}_2"'] & {} & {} \arrow[l, "i"'] \arrow[d, equal] \\
{} & {} \arrow[l, "p_2"'] \arrow[r, "g_2"] & {} \arrow[u, "i'_1"'] & {} \arrow[l, "i_2"']
\end{tikzcd}$$

The assertion now follows as in (a) and (b). □

5 The fraction category

In this section, our main theorem (5.18) will be proven. We begin by constructing a localisation of a uni-fractionable category $\mathcal{C}$ (with respect to its set of denominators $\operatorname{Den}\mathcal{C}$), see proposition (5.4) and proposition (5.7). To this end, we consider the quotient graph $(\operatorname{AG}\mathcal{C})/\!\equiv$ of its 3-arrow graph $\operatorname{AG}\mathcal{C}$ with respect to fraction equality $\equiv$. The crucial point in the construction will be the following lemma and its corollaries.

(5.1) Lemma (factorisation lemma)**.** We suppose given a uni-fractionable category $\mathcal{C}$, denominators d, e and morphisms f, g in $\mathcal{C}$ with $fe = dg$. Moreover, we suppose given S-denominators i, j and T-denominators p, q in $\mathcal{C}$ with $d = ip$ and $e = jq$.

(a) There exist S-denominators $\tilde{j}$, k, a T-denominator $\tilde{q}$ and a morphism h in $\mathcal{C}$ such that $e = \tilde{j}\tilde{q}$, $f\tilde{j} = ih$, $pg = h\tilde{q}$, $\tilde{j} = jk$, $q = k\tilde{q}$.

$$\begin{tikzcd}
 & \cdot \arrow[dr, "p"] \arrow[dd, dotted, "h"] & \\
\cdot \arrow[ur, "i"] \arrow[rr, "d", "\approx"'] \arrow[dd, "f"'] & & \cdot \arrow[dd, "g"] \\
 & \cdot \arrow[dr, dotted, "\tilde{q}"] & \\
\cdot \arrow[ur, dotted, "\tilde{j}"] \arrow[rr, "e", "\approx"'] \arrow[dd, equal] & & \cdot \arrow[dd, equal] \\
 & \cdot \arrow[uu, dotted, "k"] \arrow[dr, "q"] & \\
\cdot \arrow[ur, "j"] \arrow[rr, "e", "\approx"'] & & \cdot
\end{tikzcd}$$

(b) There exist an S-denominator $\tilde{i}$, T-denominators $\tilde{p}$, r and a morphism h in $\mathcal{C}$ such that $d = \tilde{i}\tilde{p}$, $fj = \tilde{i}h$, $\tilde{p}g = hq$, $i = \tilde{i}r$, $\tilde{p} = rp$.

$$\begin{tikzcd}
 & \cdot \arrow[dr, "p"] & \\
\cdot \arrow[ur, "i"] \arrow[rr, "d", "\approx"'] \arrow[dd, equal] & & \cdot \arrow[dd, equal] \\
 & \cdot \arrow[uu, dotted, "r"] \arrow[dr, dotted, "\tilde{p}"] \arrow[dd, dotted, "h"] & \\
\cdot \arrow[ur, dotted, "\tilde{i}"] \arrow[rr, "d", "\approx"'] \arrow[dd, "f"'] & & \cdot \arrow[dd, "g"] \\
 & \cdot \arrow[dr, "q"] & \\
\cdot \arrow[ur, "j"] \arrow[rr, "e", "\approx"'] & & \cdot
\end{tikzcd}$$

Proof.

(a) We let

$$\begin{tikzcd}
\cdot \arrow[r, "i"] \arrow[d, "fj"'] & \cdot \arrow[d, "\tilde{h}"] \\
\cdot \arrow[r, "i'"] & \cdot
\end{tikzcd}$$

be a weak pushout rectangle in $\mathcal{C}$ such that i' is an S-denominator in $\mathcal{C}$. Since

$$ipg = dg = fe = fjq,$$

there exists an induced morphism a with $q = i'a$ and $pg = \tilde{h}a$. By semi-saturatedness, a is a denominator in $\mathcal{C}$, and thus there exist an S-denominator $\tilde{k}$ and a T-denominator $\tilde{q}$ with $a = \tilde{k}\tilde{q}$.

$$\begin{tikzcd}
\cdot \arrow[r, "i"] \arrow[d, "fj"'] & \cdot \arrow[d, "\tilde{h}"] \arrow[ddr, bend left, "pg"] & \\
\cdot \arrow[r, "i'"] \arrow[drr, bend right, "q"'] & \cdot \arrow[r, dotted, "\tilde{k}"] \arrow[dr, dotted, "a"'] & \cdot \arrow[d, dotted, "\tilde{q}"] \\
& & \cdot
\end{tikzcd}$$

We set $h := \tilde{h}\tilde{k}$, $k := i'\tilde{k}$, $\tilde{j} := ji'\tilde{k}$ and get $e = \tilde{j}\tilde{q}$, $f\tilde{j} = ih$, $pg = h\tilde{q}$, $\tilde{j} = jk$, $q = k\tilde{q}$. Moreover, $k = i'\tilde{k}$ and $\tilde{j} = ji'\tilde{k}$ are S-denominators in $\mathcal{C}$ by multiplicativity.

(b) This is dual to (a). □

(5.2) Corollary. We suppose given a uni-fractionable category $\mathcal{C}$, denominators d, e and morphisms f, g in $\mathcal{C}$ with $fe = dg$.

(a) Given an S-denominator i and a T-denominator p in $\mathcal{C}$ with $d = ip$, there exist an S-denominator j, a T-denominator q and a morphism h in $\mathcal{C}$ such that $e = jq$, $fj = ih$, $pg = hq$.

$$\begin{tikzcd}
& \cdot \arrow[dr, "p"] \arrow[dd, dotted, "h"] & \\
\cdot \arrow[ur, "i"] \arrow[rr, "d"] \arrow[dd, "f"'] & & \cdot \arrow[dd, "g"] \\
& \cdot \arrow[dr, dotted, "q"] & \\
\cdot \arrow[ur, dotted, "j"] \arrow[rr, "e"'] & & \cdot
\end{tikzcd}$$

(b) Given an S-denominator j and a T-denominator q in $\mathcal{C}$ with $e = jq$, there exist an S-denominator i, a T-denominator p and a morphism h in $\mathcal{C}$ such that $d = ip$, $fj = ih$, $pg = hq$.

$$\begin{tikzcd}
& \cdot \arrow[dr, dotted, "p"] \arrow[dd, dotted, "h"] & \\
\cdot \arrow[ur, dotted, "i"] \arrow[rr, "d"] \arrow[dd, "f"'] & & \cdot \arrow[dd, "g"] \\
& \cdot \arrow[dr, "q"] & \\
\cdot \arrow[ur, "j"] \arrow[rr, "e"'] & & \cdot
\end{tikzcd}$$

Proof. This follows from the factorisation axiom and the factorisation lemma (5.1). □

(5.3) Corollary. We suppose given a uni-fractionable category $\mathcal{C}$, denominators d, e and morphisms f, g in $\mathcal{C}$ with $fe = dg$. There exist S-denominators i, j, T-denominators p, q and a morphism h in $\mathcal{C}$ with $d = ip$, $e = jq$, $fj = ih$, $pg = hq$.

$$\begin{tikzcd}
& \cdot \arrow[dr, dotted, "p"] \arrow[dd, dotted, "h"] & \\
\cdot \arrow[ur, dotted, "i"] \arrow[rr, "d"] \arrow[dd, "f"'] & & \cdot \arrow[dd, "g"] \\
& \cdot \arrow[dr, dotted, "q"] & \\
\cdot \arrow[ur, dotted, "j"] \arrow[rr, "e"'] & & \cdot
\end{tikzcd}$$

Proof. This follows from the factorisation axiom and corollary (5.2). □

The following proposition will essentially prove the first part of our main theorem (5.18), cf. also proposition (5.9) below.

(5.4) Proposition. For every uni-fractionable category $\mathcal{C}$, there is a category structure on $(\mathrm{AG}\,\mathcal{C})/\!\equiv$, where the composition is constructed by the following procedure.
We suppose given $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \operatorname{Arr} \mathrm{AG}\,\mathcal{C}$ with $\operatorname{Target}(b_1, f_1, a_1) = \operatorname{Source}(b_2, f_2, a_2)$, that is, with $\operatorname{Source} a_1 = \operatorname{Target} b_2$. First, we choose an S-denominator j and a T-denominator q in $\mathcal{C}$ with $b_2 a_1 = jq$. Second, we choose a T-denominator q' and a morphism f_1' in $\mathcal{C}$ with $f_1' q = q' f_1$, and we choose an S-denominator j' and a morphism f_2' in $\mathcal{C}$ with $j f_2' = f_2 j'$.

$$\begin{tikzcd}[column sep=small]
 & & {} \arrow[dl, "q'"'] \arrow[rrr, "f_1'"] & & & {} \arrow[dl, "q"'] \arrow[rrr, "f_2'"] & & & {} & & \\
 & {} \arrow[dl, "b_1"'] \arrow[rrr, "f_1"] & & & {} & & {} \arrow[ul, "j"'] \arrow[dl, "b_2"] \arrow[rrr, "f_2"] & & & {} \arrow[ul, "j'"'] & \\
 {} & & & & & {} \arrow[ul, "a_1"] & & & & & {} \arrow[ul, "a_2"']
\end{tikzcd}$$

Then

$$(b_1 \backslash f_1 / a_1)(b_2 \backslash f_2 / a_2) = q' b_1 \backslash f_1' f_2' / a_2 j'.$$

The identity of $X \in \operatorname{Ob}(\mathrm{AG}\,\mathcal{C})/\!\equiv$ is given by

$$1_X = 1_X \backslash 1_X / 1_X.$$

Proof. We suppose given a uni-fractionable category $\mathcal{C}$. Our first aim is to show that the construction described above is independent of all choices. To this end, we first consider the particular case of choosing a weak pullback of f_1 and q and a weak pushout of f_2 and j to obtain a T-denominator q', an S-denominator j' and morphisms f_1', f_2' in $\mathcal{C}$.
We suppose given $(b_l, f_l, a_l), (\tilde{b}_l, \tilde{f}_l, \tilde{a}_l) \in \operatorname{Arr} \mathrm{AG}\,\mathcal{C}$ and $c_l, c_l' \in \operatorname{Mor} \mathcal{C}$ with $b_l = c_l \tilde{b}_l$, $f_l c_l' = c_l \tilde{f}_l$, $a_l c_l' = \tilde{a}_l$ for $l \in \{1, 2\}$, and such that $\operatorname{Target}(b_1, f_1, a_1) = \operatorname{Source}(b_2, f_2, a_2)$.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "b_1"', "\approx" marking] \arrow[r, "f_1"] \arrow[d, "c_1"] & {} \arrow[d, "c_1'"] & {} \arrow[l, "a_1"', "\approx" marking] \arrow[d, equal] & {} \arrow[l, "b_2"', "\approx" marking] \arrow[r, "f_2"] \arrow[d, "c_2"] & {} \arrow[d, "c_2'"] & {} \arrow[l, "a_2"', "\approx" marking] \arrow[d, equal] \\
{} & {} \arrow[l, "\tilde{b}_1", "\approx" marking] \arrow[r, "\tilde{f}_1"'] & {} & {} \arrow[l, "\tilde{a}_1", "\approx" marking] & {} \arrow[l, "\tilde{b}_2", "\approx" marking] \arrow[r, "\tilde{f}_2"'] & {} & {} \arrow[l, "\tilde{a}_2", "\approx" marking]
\end{tikzcd}$$

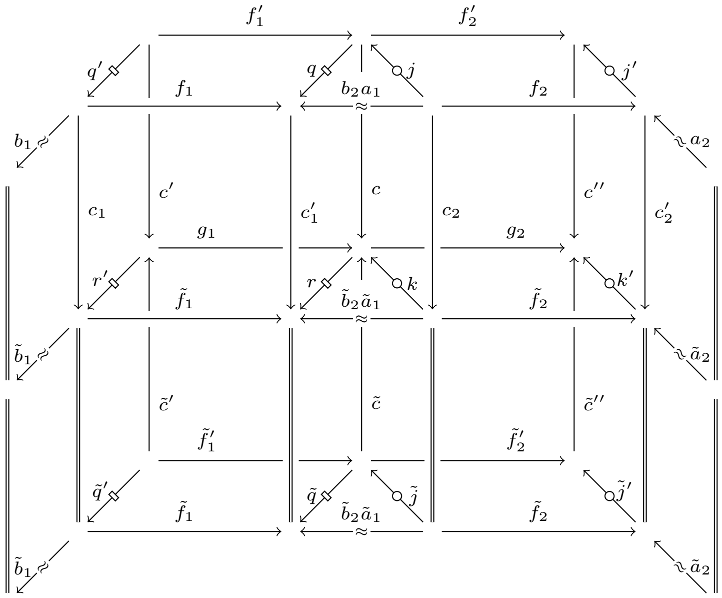
We choose S-denominators j, $\tilde{j}$ and T-denominators q, $\tilde{q}$ in $\mathcal{C}$ such that $b_2 a_1 = jq$ and $\tilde{b}_2 \tilde{a}_1 = \tilde{j} \tilde{q}$. By the factorisation lemma (5.1)(a), there exist an S-denominator k, a T-denominator r and morphisms c, $\tilde{c}$ in $\mathcal{C}$ with $\tilde{b}_2 \tilde{a}_1 = kr$, $q c_1' = cr$, $c_2 k = jc$, $\tilde{q} = \tilde{c} r$, $k = \tilde{j} \tilde{c}$. Next, we choose weak pullback rectangles

$$\begin{tikzcd}
{} \arrow[r, "f_1'"] \arrow[d, "q'"'] & {} \arrow[d, "q"] \\
{} \arrow[r, "f_1"'] & {}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[r, "g_1"] \arrow[d, "r'"'] & {} \arrow[d, "r"] \\
{} \arrow[r, "\tilde{f}_1"'] & {}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[r, "\tilde{f}_1'"] \arrow[d, "\tilde{q}'"'] & {} \arrow[d, "\tilde{q}"] \\
{} \arrow[r, "\tilde{f}_1"'] & {}
\end{tikzcd}$$

in $\mathcal{C}$ such that q', r', $\tilde{q}'$ are T-denominators, and we choose weak pushout rectangles

$$\begin{tikzcd}
{} \arrow[r, "f_2'"] & {} \\
{} \arrow[u, "j"] \arrow[r, "f_2"'] & {} \arrow[u, "j'"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[r, "g_2"] & {} \\
{} \arrow[u, "k"] \arrow[r, "\tilde{f}_2"'] & {} \arrow[u, "k'"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[r, "\tilde{f}_2'"] & {} \\
{} \arrow[u, "\tilde{j}"] \arrow[r, "\tilde{f}_2"'] & {} \arrow[u, "\tilde{j}'"']
\end{tikzcd}$$

in $\mathcal{C}$ such that j', k', $\tilde{j}'$ are S-denominators. We obtain induced morphisms c' and $\tilde{c}'$ on the weak pullbacks, that is, with $q' c_1 = c' r'$, $f_1' c = c' g_1$ and $\tilde{q}' = c' r'$, $\tilde{f}_1' \tilde{c} = \tilde{c}' g_1$, and induced morphisms c'' and $\tilde{c}''$ on the weak pushouts, that is, with $c_2'k' = j'c''$, $cg_2 = f_2'c''$ and $k' = \tilde{j}'\tilde{c}''$, $\tilde{c}g_2 = \tilde{f}_2'\tilde{c}''$.



Hence we have $q'b_1 = c'r'\tilde{b}_1$, $f_1'f_2'c'' = c'g_1g_2$, $a_2j'c'' = \tilde{a}_2k'$ and therefore $(q'b_1, f_1'f_2', a_2j') \equiv (r'\tilde{b}_1, g_1g_2, \tilde{a}_2k')$, and we get $\tilde{q}'\tilde{b}_1 = \tilde{c}'r'\tilde{b}_1$, $\tilde{f}_1'\tilde{f}_2'\tilde{c}'' = \tilde{c}'g_1g_2$, $\tilde{a}_2\tilde{j}'\tilde{c}'' = \tilde{a}_2k'$ and therefore $(r'\tilde{b}_1, g_1g_2, \tilde{a}_2k') \equiv (\tilde{q}'\tilde{b}_1, \tilde{f}_1'\tilde{f}_2', \tilde{a}_2\tilde{j}')$.

$$\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "q'b_1"', "\approx"] \arrow[r, "f_1'f_2'"] \arrow[d, "c'"] & {} \arrow[d, "c''"] & {} \arrow[l, "a_2j'"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "r'\tilde{b}_1"', "\approx"] \arrow[r, "g_1g_2"] & {} & {} \arrow[l, "\tilde{a}_2k'"', "\approx"] \\
{} \arrow[u, equal] & {} \arrow[l, "\tilde{q}'\tilde{b}_1"', "\approx"] \arrow[r, "\tilde{f}_1'\tilde{f}_2'"] \arrow[u, "\tilde{c}'"] & {} \arrow[u, "\tilde{c}''"] & {} \arrow[l, "\tilde{a}_2\tilde{j}'"', "\approx"] \arrow[u, equal]
\end{tikzcd}$$

Altogether, we have $(q'b_1, f_1'f_2', a_2j') \equiv (\tilde{q}'\tilde{b}_1, \tilde{f}_1'\tilde{f}_2', \tilde{a}_2\tilde{j}')$ in $\mathrm{AG}\,\mathcal{C}$, that is, we have

$$q'b_1\backslash f_1'f_2'/a_2j' = \tilde{q}'\tilde{b}_1\backslash \tilde{f}_1'\tilde{f}_2'/\tilde{a}_2\tilde{j}'$$

in $(\mathrm{AG}\,\mathcal{C})/\!\equiv$.

In the special case where $c_1 = 1$, $c_1' = 1$, $c_2 = 1$, $c_2' = 1$, we see that different choices of constructions via weak pullback and weak pushout rectangles lead to the same double fraction $q'b_1\backslash f_1'f_2'/a_2j' = \tilde{q}'b_1\backslash \tilde{f}_1'\tilde{f}_2'/\tilde{a}_2\tilde{j}'$. Hence we obtain a well-defined map

$$\begin{aligned} c\colon\ & \mathrm{Arr}\,\mathrm{AG}\,\mathcal{C} \;{}_{\mathrm{Target}}\!\times_{\mathrm{Source}}\, \mathrm{Arr}\,\mathrm{AG}\,\mathcal{C} \to \mathrm{Arr}\,(\mathrm{AG}\,\mathcal{C})/\!\equiv, \\ & ((b_1, f_1, a_1), (b_2, f_2, a_2)) \mapsto q'b_1\backslash f_1'f_2'/a_2j', \end{aligned}$$

where q', f_1', f_2', j' are constructed as described above. Now the general case shows that c is independent of the choice of the representatives in the equivalence classes with respect to $\equiv$, and thus we obtain an induced map

$$\bar{c}\colon\ \mathrm{Arr}\,(\mathrm{AG}\,\mathcal{C})/\!\equiv \;{}_{\mathrm{Target}}\!\times_{\mathrm{Source}}\, \mathrm{Arr}\,(\mathrm{AG}\,\mathcal{C})/\!\equiv\; \to \mathrm{Arr}\,(\mathrm{AG}\,\mathcal{C})/\!\equiv$$

given by

$$\bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2) = c((b_1, f_1, a_1), (b_2, f_2, a_2)) = q'b_1\backslash f_1'f_2'/a_2j'$$

for $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \mathrm{Arr}\,\mathrm{AG}\,\mathcal{C}$ with $\mathrm{Target}\,(b_1, f_1, a_1) = \mathrm{Source}\,(b_2, f_2, a_2)$.

We claim that arbitrary commutative quadrangles may be used instead of weak pullback and weak pushout rectangles to compute $\bar{c}$. Indeed, given a weak pullback rectangle

$$\begin{tikzcd}
{} \arrow[r, "f_1'"] \arrow[d, "q'"'] & {} \arrow[d, "q"] \\
{} \arrow[r, "f_1"] & {}
\end{tikzcd}$$

and a weak pushout rectangle

$$\begin{tikzcd}
\cdot \arrow[r, "f_2'"] & \cdot \\
\cdot \arrow[u, "j"] \arrow[r, "f_2"'] & \cdot \arrow[u, "j'"']
\end{tikzcd}$$

and arbitrary commutative quadrangles

$$\begin{tikzcd}
\cdot \arrow[r, "\tilde{f}_1'"] \arrow[d, "\tilde{q}'"'] & \cdot \arrow[d, "q"] \\
\cdot \arrow[r, "f_1"'] & \cdot
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\cdot \arrow[r, "\tilde{f}_2'"] & \cdot \\
\cdot \arrow[u, "j"] \arrow[r, "f_2"'] & \cdot \arrow[u, "\tilde{j}'"']
\end{tikzcd}$$

such that q', $\tilde{q}'$ are T-denominators and j', $\tilde{j}'$ are S-denominators in $\mathcal{C}$, we obtain induced morphisms c and c' such that $\tilde{q}' = cq'$, $\tilde{f}_1' = cf_1'$, $\tilde{f}_2' = f_2'c'$, $\tilde{j}' = j'c'$.

$$\begin{tikzcd}
 & \cdot \arrow[d, dotted, "c"] \arrow[rr, bend left, "\tilde{f}_1'"] \arrow[dd, bend right, "\tilde{q}'"'] & & & \cdot & & \\
 & \cdot \arrow[rr, "f_1'"] \arrow[d, "q'"] & & \cdot \arrow[r, "f_2'"] \arrow[ur, bend left, "\tilde{f}_2'"] \arrow[dl, "q"'] & \cdot \arrow[u, dotted, "c'"'] & & \\
 & \cdot \arrow[r, "f_1"] \arrow[dl, "b_1"'] & \cdot & \cdot \arrow[u, "j"] \arrow[rr, "f_2"] \arrow[d, "b_2"] & & \cdot \arrow[ul, "j'"'] \arrow[uul, bend right, "\tilde{j}'"'] & \\
\cdot & & \cdot \arrow[u, "a_1"'] & \cdot & & & \cdot \arrow[ul, "a_2"']
\end{tikzcd}$$

Hence $(\tilde{q}'b_1, \tilde{f}_1'\tilde{f}_2', a_2\tilde{j}') = (cq'b_1, cf_1'f_2'c', a_2j'c') \equiv (q'b_1, f_1'f_2', a_2j')$ and therefore

$$\bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2) = q'b_1\backslash f_1'f_2'/a_2j' = \tilde{q}'b_1\backslash \tilde{f}_1'\tilde{f}_2'/a_2\tilde{j}'.$$

This proves the claim.
In addition to $\bar{c}$, we define the map

$$e\colon \operatorname{Ob}(\operatorname{AG}\mathcal{C})/{\equiv} \to \operatorname{Arr}(\operatorname{AG}\mathcal{C})/{\equiv}, X \mapsto 1_X\backslash 1_X/1_X.$$

To show that $(\operatorname{AG}\mathcal{C})/{\equiv}$ is a category with composition $\bar{c}$ and identity map e, it remains to verify the category axioms. By the definitions of $\bar{c}$ and e, we have

$$\operatorname{Source}\bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2) = \operatorname{Source} q'b_1\backslash f_1'f_2'/a_2j' = \operatorname{Target}(q'b_1) = \operatorname{Target} b_1 = \operatorname{Source} b_1\backslash f_1/a_1$$

and analogously $\operatorname{Target}\bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2) = \operatorname{Target} b_2\backslash f_2/a_2$ for all $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$ with $\operatorname{Target} b_1\backslash f_1/a_1 = \operatorname{Source} b_2\backslash f_2/a_2$, as well as

$$\operatorname{Source} e(X) = \operatorname{Source} 1_X\backslash 1_X/1_X = \operatorname{Target} 1_X = X$$

and analogously $\operatorname{Target} e(X) = X$ for all $X \in \operatorname{Ob}(\operatorname{AG}\mathcal{C})/{\equiv}$.
For the associativity of $\bar{c}$, we suppose given $(b_l, f_l, a_l) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$ for $l \in \{1, 2, 3\}$ such that $\operatorname{Target} b_1\backslash f_1/a_1 = \operatorname{Source} b_2\backslash f_2/a_2$ and $\operatorname{Target} b_2\backslash f_2/a_2 = \operatorname{Source} b_3\backslash f_3/a_3$. We choose S-denominators j, $\tilde{j}$ and T-denominators q, $\tilde{q}$ with $b_2a_1 = jq$ and $b_3a_2 = \tilde{j}\tilde{q}$. Then we choose T-denominators q', $\tilde{q}'$ and morphisms f_1', $\tilde{f}_2'$ in $\mathcal{C}$ with $f_1'q = q'f_1$ and $\tilde{f}_2'\tilde{q} = \tilde{q}'f_2$, and we choose S-denominators j', $\tilde{j}'$ and morphisms f_2', $\tilde{f}_3'$ in $\mathcal{C}$ with $jf_2' = f_2j'$ and $\tilde{j}\tilde{f}_3' = f_3\tilde{j}'$. By definition of $\bar{c}$, we obtain

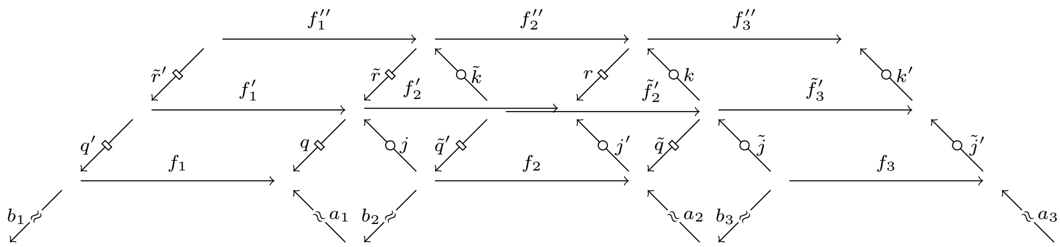
$$\bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2) = q'b_1\backslash f_1'f_2'/a_2j',$$
$$\bar{c}(b_2\backslash f_2/a_2, b_3\backslash f_3/a_3) = \tilde{q}'b_2\backslash \tilde{f}_2'\tilde{f}_3'/a_3\tilde{j}'.$$

Moreover, we have $\tilde{q}'jf_2' = \tilde{f}_2'\tilde{q}j'$, and thus by corollary (5.3) there exist S-denominators k, $\tilde{k}$, T-denominators r, $\tilde{r}$ and a morphism f_2'' in $\mathcal{C}$ with $\tilde{q}j' = kr$, $\tilde{q}'j = \tilde{k}\tilde{r}$, $\tilde{r}f_2' = f_2''r$, $\tilde{f}_2'k = \tilde{k}f_2''$. We choose a T-denominator $\tilde{r}'$ and a morphism f_1'' in $\mathcal{C}$ with $f_1''\tilde{r} = \tilde{r}'f_1'$, and we choose an S-denominator k' and a morphism f_3'' in $\mathcal{C}$ with $kf_3'' = \tilde{f}_3'k'$. Then we obtain $\tilde{r}'f_1'f_2' = f_1''f_2''r$, $\tilde{j}kf_3'' = f_3\tilde{j}'k'$, $\tilde{f}_2'\tilde{f}_3'k' = \tilde{k}f_2''f_3''$, $f_1''\tilde{r}q = \tilde{r}'q'f_1$, and therefore

$$\bar{c}(\bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2), b_3\backslash f_3/a_3) = \bar{c}(q'b_1\backslash f_1'f_2'/a_2j', b_3\backslash f_3/a_3) = \tilde{r}'q'b_1\backslash f_1''f_2''f_3''/a_3\tilde{j}'k'$$

$$= \bar{c}(b_1\backslash f_1/a_1, \tilde{q}'b_2\backslash \tilde{f}_2'\tilde{f}_3'/a_3\tilde{j}') = \bar{c}(b_1\backslash f_1/a_1, \bar{c}(b_2\backslash f_2/a_2, b_3\backslash f_3/a_3)).$$

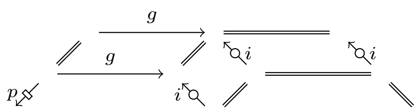
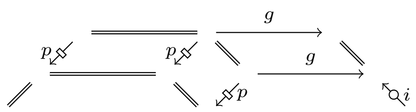
Thus $\bar{c}$ is associative.



Finally, we suppose given $(b, f, a) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$. We want to show that $\bar{c}(b\backslash f/a, e(\operatorname{Target} b\backslash f/a)) = b\backslash f/a$. By the normalisation lemma (4.10), there exists a normal arrow $(p, g, i) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ with $(b, f, a) \equiv (p, g, i)$. Since a factorisation of i into an S-denominator followed by a T-denominator is given by $i = i1$, we obtain

$$\bar{c}(b\backslash f/a, e(\operatorname{Target} b\backslash f/a)) = \bar{c}(p\backslash g/i, 1\backslash 1/1) = 1p\backslash g1/1i = p\backslash g/i = b\backslash f/a.$$

Analogously, we have $\bar{c}(e(\operatorname{Source} b\backslash f/a), b\backslash f/a) = b\backslash f/a$.

Altogether, $(\operatorname{AG} \mathcal{C})/{\equiv}$ becomes a category with $(b_1\backslash f_1/a_1)(b_2\backslash f_2/a_2) = \bar{c}(b_1\backslash f_1/a_1, b_2\backslash f_2/a_2)$ for $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ with $\operatorname{Target} b_1\backslash f_1/a_1 = \operatorname{Source} b_2\backslash f_2/a_2$ and $1_X = e(X)$ for $X \in \operatorname{Ob} (\operatorname{AG} \mathcal{C})/{\equiv}$. □

(5.5) Definition (fraction category)**.** The *fraction category* of a uni-fractionable category $\mathcal{C}$ is defined to be the category $\operatorname{Frac} \mathcal{C}$, whose underlying graph is given by the quotient graph $(\operatorname{AG} \mathcal{C})/{\equiv}$ and whose composition and identities are given as in proposition (5.4).

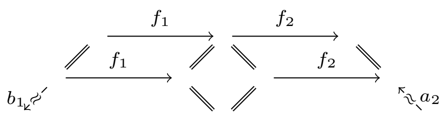
Our next aim is to show that the fraction category of a uni-fractionable category is a localisation, which is going to be the second part of our main theorem (5.18).

(5.6) Remark. Given a uni-fractionable category $\mathcal{C}$, we have

$$(b_1\backslash f_1/1)(1\backslash f_2/a_2) = b_1\backslash f_1 f_2/a_2$$

for all 3-arrows $(b_1, f_1, 1)$ and $(1, f_2, a_2)$ in $\mathcal{C}$.

Proof. This follows using the definition of the composition in proposition (5.4).



□

(5.7) Proposition (universal property of the fraction category)**.** The fraction category $\operatorname{Frac} \mathcal{C}$ of a uni-fractionable category $\mathcal{C}$ is a localisation of $\mathcal{C}$, where the localisation functor $\operatorname{loc}\colon \mathcal{C} \to \operatorname{Frac} \mathcal{C}$ is given on the objects by

$$\operatorname{loc}(X) = X$$

for $X \in \operatorname{Ob} \mathcal{C}$ and on the morphisms by

$$\operatorname{loc}(f) = 1\backslash f/1$$

for $f \in \operatorname{Mor} \mathcal{C}$. The inverse of $\operatorname{loc}(d)$ for $d \in \operatorname{Den} \mathcal{C}$ is given by

$$(\operatorname{loc}(d))^{-1} = d\backslash 1/1 = 1\backslash 1/d.$$

Given a category $\mathcal{D}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible for all $d \in \operatorname{Den}\mathcal{C}$, the unique functor $\hat{F}\colon \operatorname{Frac}\mathcal{C} \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given by

$$\hat{F}(b\backslash f/a) = (Fb)^{-1}(Ff)(Fa)^{-1}.$$

Given a category $\mathcal{D}$ and functors $F, G\colon \mathcal{C} \to \mathcal{D}$ such that Fd and Gd are invertible for all $d \in \operatorname{Den}\mathcal{C}$ and a transformation $\alpha\colon F \to G$, the unique transformation $\hat{\alpha}\colon \hat{F} \to \hat{G}$ with $\alpha_X = \hat{\alpha}_{\operatorname{loc}(X)}$ for $X \in \operatorname{Ob}\mathcal{C}$ is given by $\hat{\alpha}_X = \alpha_X$ for $X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C} = \operatorname{Ob}\mathcal{C}$.

Proof. We suppose given a uni-fractionable category $\mathcal{C}$. We define a graph morphism $L\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ on the objects by $LX := X$ for $X \in \operatorname{Ob}\mathcal{C}$ and on the arrows by $Lf := 1\backslash f/1$ for $f \in \operatorname{Mor}\mathcal{C}$. By remark (5.6), we get

$$L(fg) = 1\backslash fg/1 = (1\backslash f/1)(1\backslash g/1) = (Lf)(Lg)$$

for all $f, g \in \operatorname{Mor}\mathcal{C}$ with $\operatorname{Target} f = \operatorname{Source} g$ and

$$L1_X = 1_X\backslash 1_X/1_X = 1_{LX}$$

for all $X \in \operatorname{Ob}\mathcal{C}$, that is, L is a functor.
We want to show that $\operatorname{Frac}\mathcal{C}$ is a localisation of $\mathcal{C}$ with localisation functor L.

(Inv) We let $d \in \operatorname{Den}\mathcal{C}$ be given. By remark (5.6), we have

$$(Ld)(1\backslash 1/d) = (1\backslash d/1)(1\backslash 1/d) = 1\backslash d/d = 1\backslash 1/1 = 1 \text{ and}$$
$$(d\backslash 1/1)(Ld) = (d\backslash 1/1)(1\backslash d/1) = d\backslash d/1 = 1\backslash 1/1 = 1,$$

that is, Ld has a right inverse $1\backslash 1/d$ and a left inverse $d\backslash 1/1$. But then Ld is invertible and the left and the right inverse coincide as the unique inverse of Ld, that is,

$$(Ld)^{-1} = d\backslash 1/1 = 1\backslash 1/d.$$

(1-uni) We let $\mathcal{D}$ be a category and $F\colon \mathcal{C} \to \mathcal{D}$ be a functor such that Fd is invertible for all $d \in \operatorname{Den}\mathcal{C}$. Since

$$\operatorname{Source}((Fb)^{-1}(Ff)(Fa)^{-1}) = \operatorname{Source}(Fb)^{-1} = \operatorname{Target}(Fb) = F(\operatorname{Target} b) = F(\operatorname{Source}(b, f, a)),$$
$$\operatorname{Target}((Fb)^{-1}(Ff)(Fa)^{-1}) = \operatorname{Target}(Fa)^{-1} = \operatorname{Source}(Fa) = F(\operatorname{Source} a) = F(\operatorname{Target}(b, f, a))$$

for $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$, there is a graph morphism $F'\colon \operatorname{AG}\mathcal{C} \to \mathcal{D}$ given on the objects by $F'X = FX$ for $X \in \operatorname{Ob}\operatorname{AG}\mathcal{C}$ and on the arrows by $F'(b, f, a) = (Fb)^{-1}(Ff)(Fa)^{-1}$ for $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$. Moreover, given $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$ and $c, c' \in \operatorname{Den}\mathcal{C}$ with $\operatorname{Target} c' = \operatorname{Source} b$, $\operatorname{Source} c = \operatorname{Target} a$, we obtain

$$\begin{aligned} F'(c'b, c'fc, ac) &= (F(c'b))^{-1}(F(c'fc))(F(ac))^{-1} = ((Fc')(Fb))^{-1}((Fc')(Ff)(Fc))((Fa)(Fc))^{-1} \\ &= (Fb)^{-1}(Fc')^{-1}(Fc')(Ff)(Fc)(Fc)^{-1}(Fa)^{-1} = (Fb)^{-1}(Ff)(Fa)^{-1} = F'(b, f, a). \end{aligned}$$

Hence F' maps fraction equal 3-arrows to the same morphism and we obtain an induced graph morphism $\hat{F}\colon (\operatorname{AG}\mathcal{C})/{\equiv} \to \mathcal{D}$ with $F' = \hat{F} \circ \operatorname{quo}$.

$$\begin{tikzcd} \operatorname{AG}\mathcal{C} \arrow[r, "F'"] \arrow[d, "\operatorname{quo}"'] & \mathcal{D} \\ (\operatorname{AG}\mathcal{C})/{\equiv} \arrow[ur, dotted, "\hat{F}"'] & \end{tikzcd}$$

Given $(b_1, f_1, a_1), (b_2, f_2, a_2) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$ with $\operatorname{Target}(b_1, f_1, a_1) = \operatorname{Source}(b_2, f_2, a_2)$, we have

$$\begin{aligned} \hat{F}((b_1\backslash f_1/a_1)(b_2\backslash f_2/a_2)) &= \hat{F}(q'b_1\backslash f_1'f_2'/a_2j') = (F(q'b_1))^{-1}(F(f_1'f_2'))(F(a_2j'))^{-1} \\ &= (Fb_1)^{-1}(Fq')^{-1}(Ff_1')(Ff_2')(Fj')^{-1}(Fa_2)^{-1} \\ &= (Fb_1)^{-1}(Ff_1)(Fq)^{-1}(Fj)^{-1}(Ff_2)(Fa_2)^{-1} \\ &= (Fb_1)^{-1}(Ff_1)(Fa_1)^{-1}(Fb_2)^{-1}(Ff_2)(Fa_2)^{-1} \end{aligned}$$

$$= \hat{F}(b_1\backslash f_1/a_1)\hat{F}(b_2\backslash f_2/a_2),$$

where j, j', q, q', f_1', f_2' are supposed to be constructed as in proposition (5.4).

$$\begin{tikzcd}
 & & {} \arrow[rr, "f_1'"] \arrow[dl, "q'"'] & & {} \arrow[rr, "f_2'"] \arrow[dl, "q"'] & & {} & & \\
 & {} \arrow[rr, "f_1"] \arrow[dl, "b_1"'] & & {} & & {} \arrow[ul, "j"'] \arrow[rr, "f_2"] \arrow[dl, "b_2"] & & {} \arrow[ul, "j'"'] & \\
{} & & & & {} \arrow[ul, "a_1"] & & & & {} \arrow[ul, "a_2"']
\end{tikzcd}$$

Moreover, we have

$$\hat{F}(1_X) = \hat{F}(1_X\backslash 1_X/1_X) = (F1_X)^{-1}(F1_X)(F1_X)^{-1} = 1_{FX}^{-1}1_{FX}1_{FX}^{-1} = 1_{FX} = 1_{\hat{F}X}$$

for $X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C}$. This implies that $\hat{F}\colon \operatorname{Frac}\mathcal{C} \to \mathcal{D}$ is a functor, given by

$$\hat{F}(b\backslash f/a) = F'(b,f,a) = (Fb)^{-1}(Ff)(Fa)^{-1}$$

for every $(b,f,a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$. In particular,

$$\hat{F}Lf = \hat{F}(1\backslash f/1) = (F1)^{-1}(Ff)(F1)^{-1} = 1^{-1}(Ff)1^{-1} = Ff$$

for all $f \in \operatorname{Mor}\mathcal{C}$, that is, $\hat{F} \circ L = F$.

Conversely, given an arbitrary functor $G\colon \operatorname{Frac}\mathcal{C} \to \mathcal{D}$ with $F = G \circ L$, we conclude by remark (5.6) that

$$\begin{aligned} G(b\backslash f/a) &= G((b\backslash 1/1)(1\backslash f/1)(1\backslash 1/a)) = G((Lb)^{-1}(Lf)(La)^{-1}) = (GLb)^{-1}(GLf)(GLa)^{-1} \\ &= (Fb)^{-1}(Ff)(Fa)^{-1} \end{aligned}$$

for $(b,f,a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$.

(2-uni) We suppose given a category $\mathcal{D}$ and functors $F, G\colon \mathcal{C} \to \mathcal{D}$ such that Fd and Gd are invertible for all $d \in \operatorname{Den}\mathcal{C}$, and we let $\hat{F}, \hat{G}\colon \operatorname{Frac}\mathcal{C} \to \mathcal{D}$ be the unique functors with $F = \hat{F} \circ L$ resp. $G = \hat{G} \circ L$. Moreover, we suppose given a transformation $\alpha\colon F \to G$. We define a family $\hat{\alpha} := (\hat{\alpha}_X)_{X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C}}$ by $\hat{\alpha}_X := \alpha_X$ for $X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C} = \operatorname{Ob}\mathcal{C}$. Then $\hat{\alpha}_{LX} = \hat{\alpha}_X = \alpha_X$ for $X \in \operatorname{Ob}\mathcal{C}$. Moreover, $\hat{\alpha}$ is a transformation from $\hat{F}$ to $\hat{G}$ since for every 3-arrow $(b,f,a)\colon X \leftarrow \tilde{X} \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$, we have

$$\begin{aligned} \hat{\alpha}_X(\hat{G}(b\backslash f/a)) &= \alpha_X(Gb)^{-1}(Gf)(Ga)^{-1} = (Fb)^{-1}\alpha_{\tilde{X}}(Gf)(Ga)^{-1} = (Fb)^{-1}(Ff)\alpha_{\tilde{Y}}(Ga)^{-1} \\ &= (Fb)^{-1}(Ff)(Fa)^{-1}\alpha_Y = (\hat{F}(b\backslash f/a))\hat{\alpha}_Y. \end{aligned}$$

Conversely, given an arbitrary transformation $\beta\colon \hat{F} \to \hat{G}$ such that $\beta_{LX} = \alpha_X$ for all $X \in \operatorname{Ob}\mathcal{C}$, we necessarily have $\beta_X = \beta_{LX} = \alpha_X$ for all $X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C} = \operatorname{Ob}\mathcal{C}$.

Altogether, $\operatorname{Frac}\mathcal{C}$ is a localisation of $\mathcal{C}$ with localisation functor $\operatorname{loc}^{\operatorname{Frac}\mathcal{C}} = L$. □

(5.8) Corollary (splitting double fractions)**.** Given a uni-fractionable category $\mathcal{C}$, we have

$$b\backslash f/a = (\operatorname{loc}(b))^{-1}\operatorname{loc}(f)(\operatorname{loc}(a))^{-1}$$

for each 3-arrow (b,f,a) in $\mathcal{C}$.

Proof. By proposition (5.7), the fraction category $\operatorname{Frac}\mathcal{C}$ is a localisation of $\mathcal{C}$. In particular, $\operatorname{loc}(d)$ is invertible for all $d \in \operatorname{Den}\mathcal{C}$, and hence there exists a unique functor $\hat{L}\colon \operatorname{Frac}\mathcal{C} \to \operatorname{Frac}\mathcal{C}$ with $\operatorname{loc} = \hat{L} \circ \operatorname{loc}$, which is given by

$$\hat{L}(b\backslash f/a) = (\operatorname{loc}(b))^{-1}\operatorname{loc}(f)(\operatorname{loc}(a))^{-1}$$

for all $(b,f,a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$. But since $\operatorname{loc} = \operatorname{id}_{\operatorname{Frac}\mathcal{C}} \circ \operatorname{loc}$, we necessarily must have $\hat{L} = \operatorname{id}_{\operatorname{Frac}\mathcal{C}}$ and therefore the assertion holds. □

In the construction of the composition of the fraction category in proposition (5.4), the occurring morphisms j, j' were S-denominators, and q, q' were T-denominators. We shall now show that it suffices to have a diagram with arbitrary denominators at their places to get the correct composite.

(5.9) Proposition. We suppose given a uni-fractionable category $\mathcal{C}$.

(a) We suppose given 3-arrows (b_1, f_1, a_1) and (b_2, f_2, a_2) in $\mathcal{C}$ with $\operatorname{Target}(b_1, f_1, a_1) = \operatorname{Source}(b_2, f_2, a_2)$. Moreover, we suppose given denominators d, d', e, e' and morphisms g_1, g_2 in $\mathcal{C}$ with $b_2 a_1 = de$, $g_1 e = e' f_1$, $d g_2 = f_2 d'$.

$$\begin{tikzcd}
& & {} \arrow[rr, "g_1"] \arrow[dl, "e'"'] & & {} \arrow[rr, "g_2"] \arrow[dl, "e"'] & & {} & & \\
& {} \arrow[rr, "f_1"] \arrow[dl, "b_1"'] & & {} & & {} \arrow[rr, "f_2"] \arrow[ul, "d"'] \arrow[dl, "b_2"] & & {} \arrow[ul, "d'"'] & \\
{} & & & & {} \arrow[ul, "a_1"] & & & & {} \arrow[ul, "a_2"']
\end{tikzcd}$$

Then we have

$$(b_1 \backslash f_1 / a_1)(b_2 \backslash f_2 / a_2) = e' b_1 \backslash g_1 g_2 / a_2 d'.$$

(b) Given a 3-arrow (b, d, a) in $\mathcal{C}$ with a denominator d, the double fraction $b \backslash d / a$ is invertible in $\operatorname{Frac} \mathcal{C}$, and the inverse of $b \backslash d / a$ can be constructed as follows. We choose denominators d_1, d'_1, d_2, d'_2, a', b' in $\mathcal{C}$ with $d = d_1 d_2$, $d_1 b' = b d'_1$, $a' d_2 = d'_2 a$ ([7]).

$$\begin{tikzcd}
& {} & & {} \arrow[ll, "b'"'] \arrow[dr, "d_2"] & & {} \arrow[ll, "a'"'] \arrow[dr, "d'_2"] & \\
{} \arrow[ur, "d'_1"] & & {} \arrow[ll, "b"] \arrow[rr, "d"] \arrow[ur, "d_1"] & & {} & & {} \arrow[ll, "a"]
\end{tikzcd}$$

Then we have

$$(b \backslash d / a)^{-1} = d'_2 \backslash a' b' / d'_1.$$

Proof.

(a) We compute

$$\begin{aligned}
(b_1 \backslash f_1 / a_1)(b_2 \backslash f_2 / a_2) &= (\operatorname{loc}(b_1))^{-1} \operatorname{loc}(f_1) (\operatorname{loc}(a_1))^{-1} (\operatorname{loc}(b_2))^{-1} \operatorname{loc}(f_2) (\operatorname{loc}(a_2))^{-1} \\
&= (\operatorname{loc}(b_1))^{-1} \operatorname{loc}(f_1) (\operatorname{loc}(e))^{-1} (\operatorname{loc}(d))^{-1} \operatorname{loc}(f_2) (\operatorname{loc}(a_2))^{-1} \\
&= (\operatorname{loc}(b_1))^{-1} (\operatorname{loc}(e'))^{-1} \operatorname{loc}(g_1) \operatorname{loc}(g_2) (\operatorname{loc}(d'))^{-1} (\operatorname{loc}(a_2))^{-1} \\
&= (\operatorname{loc}(e' b_1))^{-1} \operatorname{loc}(g_1 g_2) (\operatorname{loc}(a_2 d'))^{-1} = e' b_1 \backslash g_1 g_2 / a_2 d'.
\end{aligned}$$

(b) The double fraction $b \backslash d / a = (\operatorname{loc}(b))^{-1} \operatorname{loc}(d) (\operatorname{loc}(a))^{-1}$ is invertible in $\operatorname{Frac} \mathcal{C}$ since the localisation functor $\operatorname{loc} \colon \mathcal{C} \to \operatorname{Frac} \mathcal{C}$ maps denominators in $\mathcal{C}$ to isomorphisms in $\operatorname{Frac} \mathcal{C}$.

Given denominators d_1, d'_1, d_2, d'_2, a', b' in $\mathcal{C}$ with $d = d_1 d_2$, $d_1 b' = b d'_1$, $a' d_2 = d'_2 a$, we obtain

$$\begin{aligned}
(b \backslash d / a)^{-1} &= ((\operatorname{loc}(b))^{-1} \operatorname{loc}(d) (\operatorname{loc}(a))^{-1})^{-1} = \operatorname{loc}(a) (\operatorname{loc}(d))^{-1} \operatorname{loc}(b) = \operatorname{loc}(a) (\operatorname{loc}(d_1 d_2))^{-1} \operatorname{loc}(b) \\
&= \operatorname{loc}(a) (\operatorname{loc}(d_2))^{-1} (\operatorname{loc}(d_1))^{-1} \operatorname{loc}(b) = (\operatorname{loc}(d'_2))^{-1} \operatorname{loc}(a') \operatorname{loc}(b') (\operatorname{loc}(d'_1))^{-1} \\
&= (\operatorname{loc}(d'_2))^{-1} \operatorname{loc}(a' b') (\operatorname{loc}(d'_1))^{-1} = d'_2 \backslash a' b' / d'_1.
\end{aligned}$$ □

The preceding proposition shows that the fraction category of a uni-fractionable category does not depend on the choice of S-denominators and T-denominators, as to be expected by the universal property of a localisation, cf. proposition (5.7):

(5.10) Corollary. Given uni-fractionable categories $\mathcal{C}$ and $\mathcal{C}'$ such that their underlying categories with denominators coincide, we have $\operatorname{Frac} \mathcal{C} = \operatorname{Frac} \mathcal{C}'$.

Proof. By the definition of the category structure of $\operatorname{Frac} \mathcal{C}$, see proposition (5.4), only the definition of the composition depends on the definition of $\operatorname{SDen} \mathcal{C}$ and $\operatorname{TDen} \mathcal{C}$, and proposition (5.9)(a) shows that this composition is in fact independent of $\operatorname{SDen} \mathcal{C}$ and $\operatorname{TDen} \mathcal{C}$. Analogously for $\mathcal{C}'$, and thus we have $\operatorname{Frac} \mathcal{C} = \operatorname{Frac} \mathcal{C}'$. □

Next, we want to turn the construction of the fraction category into a functor.

[7] For example, one can factorise d into a composite of an S-denominator d_1 and a T-denominator d_2 and then construct weakly universal Ore completions.

(5.11) Remark. We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\mathbf{\Theta}$ is in $\mathfrak{U}$ and a uni-fractionable category $\mathcal{C}$. If $\mathcal{C}$ is in $\mathfrak{U}$, then its fraction category $\operatorname{Frac}\mathcal{C}$ is in $\mathfrak{U}$.

Proof. Since the underlying graph of $\operatorname{Frac}\mathcal{C}$ is $\operatorname{AG}\mathcal{C}/\equiv$ and this graph is a quotient of $\operatorname{AG}\mathcal{C}$, the assertion follows from remark (4.3). □

(5.12) Proposition.

(a) Given uni-fractionable categories $\mathcal{C}$ and $\mathcal{D}$ and a morphism of categories with denominators $F\colon \mathcal{C} \to \mathcal{D}$, there exists a unique induced functor

$$\operatorname{Frac} F\colon \operatorname{Frac}\mathcal{C} \to \operatorname{Frac}\mathcal{D}$$

with $\operatorname{loc}^{\operatorname{Frac}\mathcal{D}} \circ F = (\operatorname{Frac} F) \circ \operatorname{loc}^{\operatorname{Frac}\mathcal{C}}$. It is given on the objects by

$$(\operatorname{Frac} F)X = FX$$

for $X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C}$ and on the morphisms by

$$(\operatorname{Frac} F)(b\backslash f/a) = Fb\backslash Ff/Fa$$

for $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$.

(b) We suppose given a Grothendieck universe $\mathfrak{U}$ such that $\mathbf{\Theta}$ is in $\mathfrak{U}$. The construction in (a) yields functors

$$\operatorname{Frac}\colon \mathbf{UFrCat}_{(\mathfrak{U})} \to \mathbf{Cat}_{(\mathfrak{U})} \text{ and}$$
$$\operatorname{Frac}\colon \mathbf{UFr}(\mathbf{CatD}_{(\mathfrak{U})}) \to \mathbf{Cat}_{(\mathfrak{U})}.$$

Proof.

(a) Since F preserves denominators and $\operatorname{loc}^{\operatorname{Frac}\mathcal{D}}$ maps denominators in $\mathcal{D}$ to isomorphisms in $\operatorname{Frac}\mathcal{D}$, the composite $\operatorname{loc}^{\operatorname{Frac}\mathcal{D}} \circ F$ maps denominators in $\mathcal{C}$ to isomorphisms in $\operatorname{Frac}\mathcal{D}$. Hence, by the universal property of $\operatorname{Frac}\mathcal{C}$, there exists a unique functor $\operatorname{Frac} F\colon \operatorname{Frac}\mathcal{C} \to \operatorname{Frac}\mathcal{D}$ with $\operatorname{loc}^{\operatorname{Frac}\mathcal{D}} \circ F = (\operatorname{Frac} F) \circ \operatorname{loc}^{\operatorname{Frac}\mathcal{C}}$. It follows that

$$(\operatorname{Frac} F)X = (\operatorname{Frac} F)\operatorname{loc}(X) = \operatorname{loc}(FX) = FX$$

for $X \in \operatorname{Ob}\mathcal{C}$ as well as

$$\begin{aligned}(\operatorname{Frac} F)(b\backslash f/a) &= (\operatorname{Frac} F)((\operatorname{loc}(b))^{-1}\operatorname{loc}(f)(\operatorname{loc}(a))^{-1}) \\ &= ((\operatorname{Frac} F)\operatorname{loc}(b))^{-1}((\operatorname{Frac} F)\operatorname{loc}(f))((\operatorname{Frac} F)\operatorname{loc}(a))^{-1} \\ &= (\operatorname{loc}(Fb))^{-1}\operatorname{loc}(Ff)(\operatorname{loc}(Fa))^{-1} = Fb\backslash Ff/Fa\end{aligned}$$

for $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$.

(b) We suppose given uni-fractionable categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ in $\mathfrak{U}$ and morphisms of categories with denominators $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$. Then we have

$$\operatorname{loc}^{\operatorname{Frac}\mathcal{E}} \circ G \circ F = (\operatorname{Frac} G) \circ \operatorname{loc}^{\operatorname{Frac}\mathcal{D}} \circ F = (\operatorname{Frac} G) \circ (\operatorname{Frac} F) \circ \operatorname{loc}^{\operatorname{Frac}\mathcal{C}} \text{ and}$$
$$\operatorname{loc}^{\operatorname{Frac}\mathcal{C}} \circ \operatorname{id}_{\mathcal{C}} = \operatorname{loc}^{\operatorname{Frac}\mathcal{C}} = \operatorname{id}_{\operatorname{Frac}\mathcal{C}} \circ \operatorname{loc}^{\operatorname{Frac}\mathcal{C}},$$

so by the uniqueness of the induced functor in (a), we obtain $\operatorname{Frac}(G \circ F) = (\operatorname{Frac} G) \circ (\operatorname{Frac} F)$ and $\operatorname{Frac}\operatorname{id}_{\mathcal{C}} = \operatorname{id}_{\operatorname{Frac}\mathcal{C}}$. Since every morphism of uni-fractionable categories is in particular a morphism of categories with denominators, we have a functor

$$\operatorname{Frac}\colon \mathbf{UFrCat} \to \mathbf{Cat}.$$

Moreover, since the fraction category of a uni-fractionable category does not depend on the choice of S-denominators and T-denominators by corollary (5.10), we even have a functor

$$\operatorname{Frac}\colon \mathbf{UFr}(\mathbf{CatD}) \to \mathbf{Cat}.$$

□

Here is another elementary property of the fraction category, which will be needed in section 6, when we deal with (co)products.

(5.13) Proposition. We suppose given a uni-fractionable category $\mathcal{C}$ and morphisms φ_1 and φ_2 in $\operatorname{Frac}\mathcal{C}$.

(a) If $\operatorname{Source}\varphi_1 = \operatorname{Source}\varphi_2$, then there exist normal 3-arrows (p, f_1, i_1) and (p, f_2, i_2) in $\mathcal{C}$ with $\varphi_1 = p\backslash f_1/i_1$ and $\varphi_2 = p\backslash f_2/i_2$.

(b) If $\operatorname{Target}\varphi_1 = \operatorname{Target}\varphi_2$, then there exist normal 3-arrows (p_1, f_1, i) and (p_2, f_2, i) in $\mathcal{C}$ with $\varphi_1 = p_1\backslash f_1/i$ and $\varphi_2 = p_2\backslash f_2/i$.

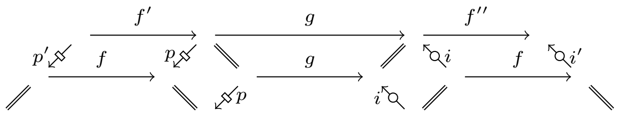
(c) If φ_1 and φ_2 are parallel, then there exist normal 3-arrows (p, f_1, i) and (p, f_2, i) in $\mathcal{C}$ with $\varphi_1 = p\backslash f_1/i$ and $\varphi_2 = p\backslash f_2/i$.

Proof. This follows from corollary (4.11). □

Our next aim is to give a sufficient (and necessary) criterion for saturatedness.

(5.14) Proposition (cf. [9, sec. 36.4])**.** A uni-fractionable category $\mathcal{C}$ is saturated if and only if it is weakly saturated.

Proof. We suppose given a uni-fractionable category $\mathcal{C}$. Since saturatedness always implies weak saturatedness, it suffices to show that if $\mathcal{C}$ is weakly saturated, then it is already saturated. So we suppose that $\mathcal{C}$ is weakly saturated and we suppose given a morphism f in $\mathcal{C}$ such that $\operatorname{loc}(f)$ is invertible in $\operatorname{Frac}\mathcal{C}$. We let (p, g, i) be a normal 3-arrow in $\mathcal{C}$ with $(\operatorname{loc}(f))^{-1} = p\backslash g/i$. Moreover, we choose a T-denominator p' and a morphism f' in $\mathcal{C}$ with $f'p = p'f$, and we choose an S-denominator i' and a morphism f'' in $\mathcal{C}$ with $if'' = fi'$.



Then we have

$$1\backslash 1/1 = (1\backslash f/1)(p\backslash g/i) = p'\backslash f'g/i \text{ and}$$
$$1\backslash 1/1 = (p\backslash g/i)(1\backslash f/1) = p\backslash gf''/i'.$$

We conclude that $f'g$ and gf'' must be denominators by remark (4.6). Hence (2 of 6) implies that f' and thus f is a denominator. Altogether, $\mathcal{C}$ is saturated. □

(5.15) Corollary. The set of isomorphisms in the fraction category of a weakly saturated uni-fractionable category $\mathcal{C}$ is given by

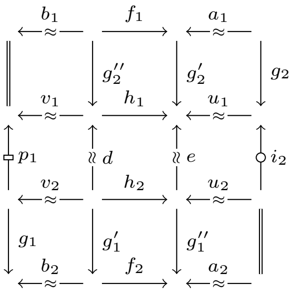
$$\operatorname{Iso}\operatorname{Frac}\mathcal{C} = \{b\backslash f/a \mid (b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C} \text{ with } f \in \operatorname{Den}\mathcal{C}\}.$$

Proof. Given a 3-arrow $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$ with $f \in \operatorname{Den}\mathcal{C}$, we have $\operatorname{loc}(b), \operatorname{loc}(f), \operatorname{loc}(a) \in \operatorname{Iso}\operatorname{Frac}\mathcal{C}$ and hence $b\backslash f/a = (\operatorname{loc}(b))^{-1}\operatorname{loc}(f)(\operatorname{loc}(a))^{-1} \in \operatorname{Iso}\operatorname{Frac}\mathcal{C}$. Conversely, we suppose given an isomorphism $\varphi \in \operatorname{Iso}\operatorname{Frac}\mathcal{C}$ and we choose a 3-arrow $(b, f, a) \in \operatorname{Arr}\operatorname{AG}\mathcal{C}$ with $\varphi = b\backslash f/a$. Since $a, b \in \operatorname{Den}\mathcal{C}$, we also have $\operatorname{loc}(b), \operatorname{loc}(a) \in \operatorname{Iso}\operatorname{Frac}\mathcal{C}$ and thus $\operatorname{loc}(f) = \operatorname{loc}(b)\,\varphi\operatorname{loc}(a) \in \operatorname{Iso}\operatorname{Frac}\mathcal{C}$. But $\mathcal{C}$ is saturated by proposition (5.14), whence $f \in \operatorname{Den}\mathcal{C}$ follows. □

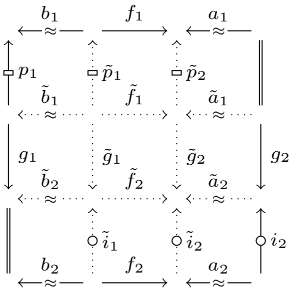
Now we come to the last part of the main theorem of this article, that is, we want to show that the uni-fractionable category $\mathcal{C}$ admits a 3-arrow calculus. It can be found in proposition (5.17). The key step of its proof is treated in the following lemma.

(5.16) Lemma (flipping lemma)**.** We suppose given a uni-fractionable category $\mathcal{C}$. Moreover, we suppose given 3-arrows (b_1, f_1, a_1), (b_2, f_2, a_2), (v_1, h_1, u_1), (v_2, h_2, u_2), morphisms g_1, g_1', g_1'', g_2, g_2', g_2'', denominators d, e, an

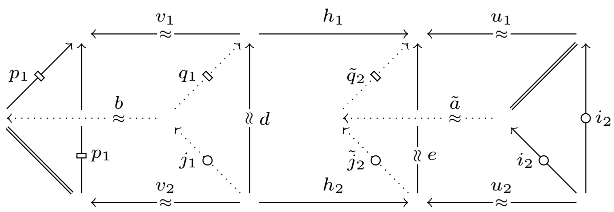
S-denominator i_2 and a T-denominator p_1 in $\mathcal{C}$, fitting into the following commutative diagram in $\mathcal{C}$.



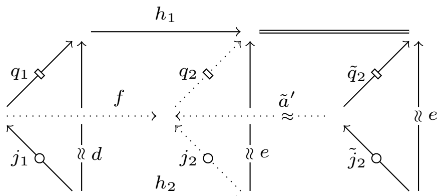
Then there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows $(\tilde{p}_1, \tilde{g}_1, \tilde{i}_1)$, $(\tilde{p}_2, \tilde{g}_2, \tilde{i}_2)$ in $\mathcal{C}$, fitting into the following commutative diagram in $\mathcal{C}$.



Proof. By corollary (5.2)(a), there exist S-denominators j_1, $\tilde{j}_2$, T-denominators q_1, $\tilde{q}_2$ and morphisms b, $\tilde{a}$ in $\mathcal{C}$ with $d = j_1 q_1$, $e = \tilde{j}_2 \tilde{q}_2$, $q_1 v_1 = b p_1$, $v_2 = j_1 b$, $u_1 = \tilde{a}\tilde{q}_2$, $u_2 \tilde{j}_2 = i_2 \tilde{a}$.



Next, using the factorisation lemma (5.1)(a), there exist an S-denominator j_2, a T-denominator q_2, a morphism f and a denominator $\tilde{a}'$ in $\mathcal{C}$ with $e = j_2 q_2$, $q_1 h_1 = f q_2$, $j_1 f = h_2 j_2$, $\tilde{q}_2 = \tilde{a}' q_2$, $j_2 = \tilde{j}_2 \tilde{a}'$.



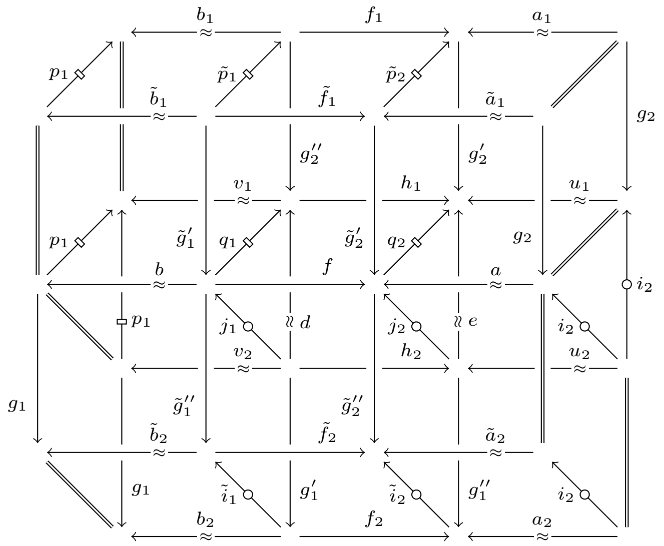
We set $a := \tilde{a}\tilde{a}'$ and obtain $u_1 = a q_2$ and $u_2 j_2 = i_2 a$.
Next, we choose weak pullback rectangles

$$\begin{tikzcd} {} \arrow[r, "\tilde{g}_1'"] \arrow[d, "\tilde{p}_1"'] & {} \arrow[d, "q_1"] \\ {} \arrow[r, "g_2''"'] & {} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {} \arrow[r, "\tilde{g}_2'"] \arrow[d, "\tilde{p}_2"'] & {} \arrow[d, "q_2"] \\ {} \arrow[r, "g_2'"'] & {} \end{tikzcd}$$

in $\mathcal{C}$ such that $\tilde{p}_1$ and $\tilde{p}_2$ are T-denominators, and we choose weak pushout rectangles

$$\begin{tikzcd} {} \arrow[r, "\tilde{g}_1''"] & {} \\ {} \arrow[u, "j_1"] \arrow[r, "g_1'"'] & {} \arrow[u, "\tilde{i}_1"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {} \arrow[r, "\tilde{g}_2''"] & {} \\ {} \arrow[u, "j_2"] \arrow[r, "g_1''"'] & {} \arrow[u, "\tilde{i}_2"'] \end{tikzcd}$$

in $\mathcal{C}$ such that $\tilde{i}_1$ and $\tilde{i}_2$ are S-denominators. We obtain induced morphisms $\tilde{b}_1$, $\tilde{f}_1$, $\tilde{a}_1$ on the weak pullbacks, that is, with $\tilde{p}_1 b_1 = \tilde{b}_1 p_1$, $\tilde{b}_1 = \tilde{g}'_1 b$, $\tilde{p}_1 f_1 = \tilde{f}_1 \tilde{p}_2$, $\tilde{f}_1 \tilde{g}'_2 = \tilde{g}'_1 f$, $a_1 = \tilde{a}_1 \tilde{p}_2$, $\tilde{a}_1 \tilde{g}'_2 = g_2 a$, and induced morphisms $\tilde{b}_2$, $\tilde{f}_2$, $\tilde{a}_2$ on the weak pushouts, that is, with $b g_1 = \tilde{g}''_1 \tilde{b}_2$, $\tilde{i}_1 \tilde{b}_2 = b_2$, $f \tilde{g}''_2 = \tilde{g}''_1 \tilde{f}_2$, $\tilde{i}_1 \tilde{f}_2 = f_2 \tilde{i}_2$, $a \tilde{g}''_2 = \tilde{a}_2$, $i_2 \tilde{a}_2 = a_2 \tilde{i}_2$.



Setting $\tilde{g}_1 := \tilde{g}'_1 \tilde{g}''_1$ and $\tilde{g}_2 := \tilde{g}'_2 \tilde{g}''_2$ yields $\tilde{b}_1 g_1 = \tilde{g}_1 \tilde{b}_2$, $\tilde{f}_1 \tilde{g}_2 = \tilde{g}_1 \tilde{f}_2$, $\tilde{a}_1 \tilde{g}_2 = g_2 \tilde{a}_2$. Moreover, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{b}_1$, $\tilde{b}_2$ are denominators in $\mathcal{C}$ by semi-saturatedness. $\square$

(5.17) Proposition (3-arrow calculus, cf. [9, sec. 36.3])**.** We suppose given a uni-fractionable category $\mathcal{C}$.

(a) Given parallel 3-arrows (b_1, f_1, a_1) and (b_2, f_2, a_2) in $\mathcal{C}$, we have

$$b_1 \backslash f_1 / a_1 = b_2 \backslash f_2 / a_2$$

in $\operatorname{Frac} \mathcal{C}$ if and only if there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows (p_1, d_1, i_1), (p_2, d_2, i_2) with denominators d_1, d_2, fitting into the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
\cdot \arrow[d, equal] & \cdot \arrow[l, "b_1"', "\approx"] \arrow[r, "f_1"] & \cdot & \cdot \arrow[l, "a_1"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[d, equal] & \cdot \arrow[l, dotted, "\tilde{b}_1"', "\approx"] \arrow[r, dotted, "\tilde{f}_1"] \arrow[u, dotted, "p_1"'] \arrow[d, dotted, "d_1", "\approx"'] & \cdot \arrow[u, dotted, "p_2"'] \arrow[d, dotted, "d_2", "\approx"'] & \cdot \arrow[l, dotted, "\tilde{a}_1"', "\approx"] \arrow[d, equal] \\
\cdot \arrow[d, equal] & \cdot \arrow[l, dotted, "\tilde{b}_2"', "\approx"] \arrow[r, dotted, "\tilde{f}_2"] & \cdot & \cdot \arrow[l, dotted, "\tilde{a}_2"', "\approx"] \arrow[d, equal] \\
\cdot & \cdot \arrow[l, "b_2"', "\approx"] \arrow[r, "f_2"] \arrow[u, dotted, "i_1"'] & \cdot \arrow[u, dotted, "i_2"'] & \cdot \arrow[l, "a_2"', "\approx"]
\end{tikzcd}$$

If (b_1, f_1, a_1) and (b_2, f_2, a_2) are normal 3-arrows, then $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$ and $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ can be chosen to be normal, too.

(b) Given 3-arrows (b_1, f_1, a_1), (b_2, f_2, a_2) and normal 3-arrows (p_1, g_1, i_1), (p_2, g_2, i_2) in $\mathcal{C}$, we have

$$(b_1 \backslash f_1 / a_1)(p_2 \backslash g_2 / i_2) = (p_1 \backslash g_1 / i_1)(b_2 \backslash f_2 / a_2)$$

in $\operatorname{Frac} \mathcal{C}$ if and only if there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows $(\tilde{p}_1, \tilde{g}_1, \tilde{i}_1)$,

$(\tilde{p}_2, \tilde{g}_2, \tilde{i}_2)$, fitting into the following commutative diagram in $\mathcal{C}$.

$$
\begin{tikzcd}
{} & {} \arrow[l, "b_1"', "\approx"] \arrow[r, "f_1"] & {} & {} \arrow[l, "a_1"', "\approx"] \\
{} \arrow[u, "p_1"'] \arrow[d, "g_1"'] & {} \arrow[l, dotted, "\tilde{b}_1"', "\approx"] \arrow[r, dotted, "\tilde{f}_1"] \arrow[u, dotted, "\tilde{p}_1"'] \arrow[d, dotted, "\tilde{g}_1"'] & {} \arrow[u, dotted, "\tilde{p}_2"'] \arrow[d, dotted, "\tilde{g}_2"'] & {} \arrow[l, dotted, "\tilde{a}_1"', "\approx"] \arrow[u, "p_2"'] \arrow[d, "g_2"'] \\
{} & {} \arrow[l, dotted, "\tilde{b}_2"', "\approx"] \arrow[r, dotted, "\tilde{f}_2"] & {} & {} \arrow[l, dotted, "\tilde{a}_2"', "\approx"] \\
{} \arrow[u, "i_1"'] & {} \arrow[l, "b_2"', "\approx"] \arrow[r, "f_2"] \arrow[u, dotted, "\tilde{i}_1"'] & {} \arrow[u, dotted, "\tilde{i}_2"'] & {} \arrow[l, "a_2"', "\approx"] \arrow[u, "i_2"']
\end{tikzcd}
$$

Proof.

(a) If we have a commutative diagram as stated, then we have

$$(b_1, f_1, a_1) \equiv (\tilde{b}_1, \tilde{f}_1, \tilde{a}_1) \equiv (\tilde{b}_2, \tilde{f}_2, \tilde{a}_2) \equiv (b_2, f_2, a_2)$$

and thus $b_1 \backslash f_1 / a_1 = b_2 \backslash f_2 / a_2$ in $\operatorname{Frac} \mathcal{C}$.

So we suppose conversely that $b_1 \backslash f_1 / a_1 = b_2 \backslash f_2 / a_2$ in $\operatorname{Frac} \mathcal{C}$, that is, we suppose that $(b_1, f_1, a_1) \equiv (b_2, f_2, a_2)$ in $\operatorname{AG} \mathcal{C}$. By remark (4.5)(b), there exist $n \in \mathbb{N}_0$, $(v_l, h_l, u_l) \in \operatorname{Arr} \operatorname{AG} \mathcal{C}$ for $l \in [0, 2n+1]$, $c_l, c'_l \in \operatorname{Mor} \mathcal{C}$ for $l \in [0, n]$, $w_l, w'_l \in \operatorname{Mor} \mathcal{C}$ for $l \in [0, n-1]$, with $(v_0, h_0, u_0) = (b_1, f_1, a_1)$ and $(v_{2n+1}, h_{2n+1}, u_{2n+1}) = (b_2, f_2, a_2)$ as well as $v_{2l} = c_l v_{2l+1}$, $h_{2l} c'_l = c_l h_{2l+1}$, $u_{2l} c'_l = u_{2l+1}$ for $l \in [0, n]$ and $v_{2l+2} = w_l v_{2l+1}$, $w_l h_{2l+1} = h_{2l+2} w'_l$, $u_{2l+2} w'_l = u_{2l+1}$ for $l \in [0, n-1]$.

$$
\begin{tikzcd}
{} \arrow[d, equal] & {} \arrow[l, "v_{2l}"', "\approx"] \arrow[r, "h_{2l}"] \arrow[d, "c_l"] & {} \arrow[d, "c'_l"] & {} \arrow[l, "u_{2l}"', "\approx"] \arrow[d, equal] \\
{} & {} \arrow[l, "v_{2l+1}"', "\approx"] \arrow[r, "h_{2l+1}"] & {} & {} \arrow[l, "u_{2l+1}"', "\approx"] \\
{} \arrow[u, equal] & {} \arrow[l, "v_{2l+2}"', "\approx"] \arrow[r, "h_{2l+2}"] \arrow[u, "w_l"] & {} \arrow[u, "w'_l"] & {} \arrow[l, "u_{2l+2}"', "\approx"] \arrow[u, equal]
\end{tikzcd}
$$

By semi-saturatedness, c_l and c'_l are denominators for all $l \in [0, n]$ and w_l, w'_l are denominators for all $l \in [0, n-1]$. Using the flipping lemma (5.16) and induction on $n \in \mathbb{N}_0$ yields the first assertion.

Now let us suppose that (b_1, f_1, a_1) and (b_2, f_2, a_2) are normal 3-arrows. By multiplicativity, $\tilde{b}_1 = p_1 b_1$ is a T-denominator and $\tilde{a}_2 = a_2 i_2$ is an S-denominator in $\mathcal{C}$. We choose S-denominators j_1, j_2 and T-denominators q_1, q_2 with $\tilde{a}_1 = j_1 q_1$ and $\tilde{b}_2 = j_2 q_2$. Moreover, we choose a T-denominator q'_1 and a morphism $\tilde{f}'_1$ in $\mathcal{C}$ with $\tilde{f}'_1 q_1 = q'_1 \tilde{f}_1$, and we choose an S-denominator j'_2 and a morphism $\tilde{f}'_2$ in $\mathcal{C}$ with $j_2 \tilde{f}'_2 = \tilde{f}_2 j'_2$.

$$
\begin{tikzcd}
& {} \arrow[rr, "\tilde{f}'_1"] \arrow[dl, "q'_1"'] & & {} \arrow[dl, "q_1"'] & \\
{} \arrow[rr, "\tilde{f}_1"'] & & {} & & {} \arrow[ll, "\tilde{a}_1", "\approx"'] \arrow[ul, "j_1"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& {} \arrow[rr, "\tilde{f}'_2"] \arrow[dl, "q_2"'] & & {} & \\
{} & & {} \arrow[ll, "\tilde{b}_2", "\approx"'] \arrow[rr, "\tilde{f}_2"'] \arrow[ul, "j_2"'] & & {} \arrow[ul, "j'_2"']
\end{tikzcd}
$$

We obtain the following commutative diagram.

$$
\begin{tikzcd}
\bullet \arrow[d, equal] & \bullet \arrow[l, "b_1"'] \arrow[r, "f_1"] & \bullet & \bullet \arrow[l, "a_1"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "\tilde{b}_1"'] \arrow[r, "\tilde{f}_1"] \arrow[u, "p_1"'] & \bullet \arrow[u, "p_2"'] & \bullet \arrow[l, "\tilde{a}_1"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "q_1'\tilde{b}_1"'] \arrow[r, "\tilde{f}_1'"] \arrow[u, "q_1'"'] \arrow[d, "q_1'"] & \bullet \arrow[u, "q_1"'] \arrow[d, "q_1"] & \bullet \arrow[l, "j_1"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "\tilde{b}_1"'] \arrow[r, "\tilde{f}_1"] \arrow[d, "d_1"] & \bullet \arrow[d, "d_2"] & \bullet \arrow[l, "\tilde{a}_1"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "\tilde{b}_2"'] \arrow[r, "\tilde{f}_2"] \arrow[d, "j_2"] & \bullet \arrow[d, "j_2'"] & \bullet \arrow[l, "\tilde{a}_2"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "q_2"'] \arrow[r, "\tilde{f}_2'"] & \bullet & \bullet \arrow[l, "\tilde{a}_2 j_2'"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "\tilde{b}_2"'] \arrow[r, "\tilde{f}_2"] \arrow[u, "j_2"'] & \bullet \arrow[u, "j_2'"'] & \bullet \arrow[l, "\tilde{a}_2"'] \arrow[d, equal] \\
\bullet & \bullet \arrow[l, "b_2"'] \arrow[r, "f_2"] \arrow[u, "i_1"'] & \bullet \arrow[u, "i_2"'] & \bullet \arrow[l, "a_2"']
\end{tikzcd}
$$

By multiplicativity, $q_1'\tilde{b}_1 = q_1'p_1b_1$, $q_1'p_1$, q_1p_2 are T-denominators and $\tilde{a}_2j_2' = a_2i_2j_2'$, i_1j_2, i_2j_2' are S-denominators in $\mathcal{C}$. Altogether, the diagram

$$
\begin{tikzcd}
\bullet \arrow[d, equal] & \bullet \arrow[l, "b_1"'] \arrow[r, "f_1"] & \bullet & \bullet \arrow[l, "a_1"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "q_1'\tilde{b}_1"'] \arrow[r, "\tilde{f}_1'"] \arrow[u, "q_1'p_1"'] \arrow[d, "q_1'd_1j_2"] & \bullet \arrow[u, "q_1p_2"'] \arrow[d, "q_1d_2j_2'"] & \bullet \arrow[l, "j_1"'] \arrow[d, equal] \\
\bullet \arrow[d, equal] & \bullet \arrow[l, "q_2"'] \arrow[r, "\tilde{f}_2'"] & \bullet & \bullet \arrow[l, "\tilde{a}_2j_2'"'] \arrow[d, equal] \\
\bullet & \bullet \arrow[l, "b_2"'] \arrow[r, "f_2"] \arrow[u, "i_1j_2"'] & \bullet \arrow[u, "i_2j_2'"'] & \bullet \arrow[l, "a_2"']
\end{tikzcd}
$$

commutes, and $(q_1'\tilde{b}_1, \tilde{f}_1', j_1)$, $(q_2, \tilde{f}_2', \tilde{a}_2j_2')$, $(q_1'p_1, q_1'd_1j_2, i_1j_2)$, $(q_1p_2, q_1d_2j_2', i_2j_2')$ are normal 3-arrows.

(b) If we have a commutative diagram as stated, then proposition (5.9)(a) implies that

$$(b_1\backslash f_1/a_1)(p_2\backslash g_2/i_2) = \tilde{p}_1b_1\backslash \tilde{f}_1\tilde{g}_2/i_2\tilde{a}_2 = \tilde{b}_1p_1\backslash \tilde{g}_1\tilde{f}_2/a_2\tilde{i}_2 = (p_1\backslash g_1/i_1)(b_2\backslash f_2/a_2).$$

So we suppose conversely that $(b_1\backslash f_1/a_1)(p_2\backslash g_2/i_2) = (p_1\backslash g_1/i_1)(b_2\backslash f_2/a_2)$. We construct the composites $(b_1\backslash f_1/a_1)(p_2\backslash g_2/i_2) = q'b_1\backslash f_1'g_2'/i_2j'$ and $(p_1\backslash g_1/i_1)(b_2\backslash f_2/a_2) = \tilde{q}'p_1\backslash g_1'f_2'/a_2\tilde{j}'$ as in proposition (5.4).

$$
\begin{tikzcd}
\bullet & \bullet \arrow[l, "b_1"'] \arrow[r, "f_1"] & \bullet & \bullet \arrow[l, "a_1"'] \\
 & \bullet \arrow[u, "q'"'] \arrow[r, "f_1'"] & \bullet \arrow[u, "q"'] \arrow[d, "g_2'"] & \bullet \arrow[l, "j"'] \arrow[u, "p_2"'] \arrow[d, "g_2"] \\
 & & \bullet & \bullet \arrow[l, "j'"'] \\
 & & & \bullet \arrow[u, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & & & \\
\bullet \arrow[u, "p_1"'] \arrow[d, "g_1"] & \bullet \arrow[l, "\tilde{q}'"'] \arrow[d, "g_1'"] & & \\
\bullet & \bullet \arrow[l, "\tilde{q}"'] \arrow[r, "f_2'"] & \bullet & \\
\bullet \arrow[u, "i_1"'] & \bullet \arrow[l, "b_2"'] \arrow[r, "f_2"] \arrow[u, "\tilde{j}"'] & \bullet \arrow[u, "\tilde{j}'"'] & \bullet \arrow[l, "a_2"']
\end{tikzcd}
$$

Hence the following diagrams commute.

$$
\begin{tikzcd}
\bullet & \bullet \arrow[l, "b_1"', "\approx" description] \arrow[r, "f_1"] & \bullet & \bullet \arrow[l, "a_1"', "\approx" description] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "q'b_1"', "\approx" description] \arrow[r, "f_1'"] \arrow[u, "q'"'] & \bullet \arrow[u, "q"'] \arrow[d, "g_2'"'] & \bullet \arrow[l, "j"'] \arrow[u, "p_2"'] \arrow[d, "g_2"] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "q'b_1"', "\approx" description] \arrow[r, "f_1'g_2'"] \arrow[u, equal] & \bullet & \bullet \arrow[l, "j'"'] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "q'b_1"', "\approx" description] \arrow[r, "f_1'g_2'"] \arrow[u, equal] & \bullet \arrow[u, equal] & \bullet \arrow[l, "i_2j'"'] \arrow[u, "i_2"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet & \bullet \arrow[l, "\tilde{q}'p_1"'] \arrow[r, "g_1'f_2'"] & \bullet & \bullet \arrow[l, "a_2\tilde{\tilde{j}}'"', "\approx" description] \\
\bullet \arrow[u, "p_1"'] \arrow[d, "g_1"] & \bullet \arrow[l, "\tilde{q}'"'] \arrow[r, "g_1'f_2'"] \arrow[u, equal] \arrow[d, "g_1'"] & \bullet \arrow[u, equal] & \bullet \arrow[l, "a_2\tilde{\tilde{j}}'"', "\approx" description] \arrow[u, equal] \\
\bullet & \bullet \arrow[l, "\tilde{q}"'] \arrow[r, "f_2'"] & \bullet \arrow[u, equal] & \bullet \arrow[l, "a_2\tilde{\tilde{j}}'"', "\approx" description] \arrow[u, equal] \\
\bullet \arrow[u, "i_1"'] & \bullet \arrow[l, "b_2"', "\approx" description] \arrow[r, "f_2"] \arrow[u, "\tilde{j}"'] & \bullet \arrow[u, "\tilde{j}'"'] & \bullet \arrow[l, "a_2"', "\approx" description] \arrow[u, equal]
\end{tikzcd}
$$

By (a), since

$$q'b_1\backslash f_1'g_2'/i_2j' = (b_1\backslash f_1/a_1)(p_2\backslash g_2/i_2) = (p_1\backslash g_1/i_1)(b_2\backslash f_2/a_2) = \tilde{q}'p_1\backslash g_1'f_2'/a_2\tilde{\tilde{j}}',$$

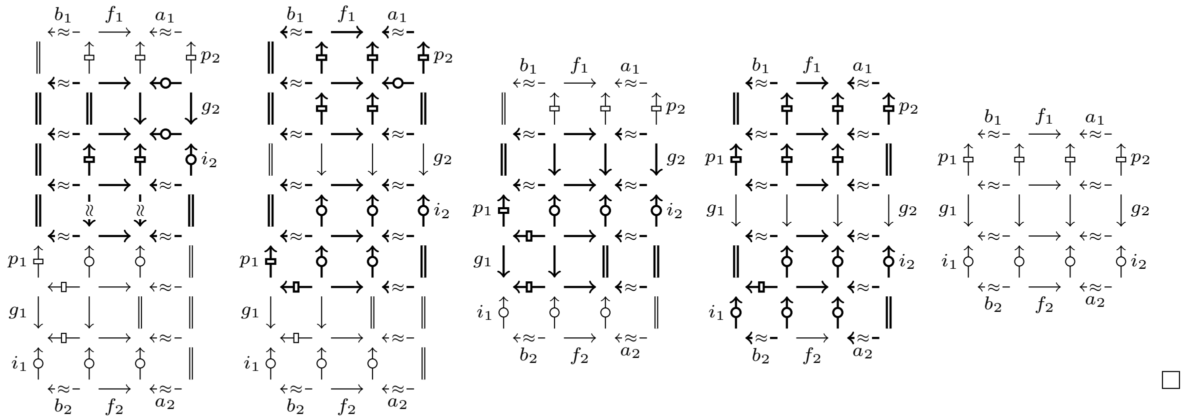
there exist 3-arrows (v_1, h_1, u_1), (v_2, h_2, u_2) and normal 3-arrows (r_1, d_1, k_1), (r_2, d_2, k_2) in $\mathcal{C}$ with denominators d_1, d_2, fitting into the following commutative diagram.

$$
\begin{tikzcd}
\bullet & \bullet \arrow[l, "q'b_1"', "\approx" description] \arrow[r, "f_1'g_2'"] & \bullet & \bullet \arrow[l, "i_2j'"'] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, dotted, "v_1"', "\approx" description] \arrow[r, dotted, "h_1"] \arrow[u, dotted, "r_1"'] \arrow[d, dotted, "d_1", "\approx"'] & \bullet \arrow[u, dotted, "r_2"'] \arrow[d, dotted, "d_2", "\approx"'] & \bullet \arrow[l, dotted, "u_1"', "\approx" description] \arrow[u, equal] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, dotted, "v_2"', "\approx" description] \arrow[r, dotted, "h_2"] & \bullet & \bullet \arrow[l, dotted, "u_2"', "\approx" description] \arrow[u, equal] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "\tilde{q}'p_1"'] \arrow[r, "g_1'f_2'"] \arrow[u, dotted, "k_1"'] & \bullet \arrow[u, dotted, "k_2"'] & \bullet \arrow[l, "a_2\tilde{\tilde{j}}'"', "\approx" description] \arrow[u, equal]
\end{tikzcd}
$$

Altogether, the following diagram commutes.

$$
\begin{tikzcd}
\bullet & \bullet \arrow[l, "b_1"', "\approx" description] \arrow[r, "f_1"] & \bullet & \bullet \arrow[l, "a_1"', "\approx" description] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "q'b_1"', "\approx" description] \arrow[r, "f_1'"] \arrow[u, "q'"'] & \bullet \arrow[u, "q"'] \arrow[d, "g_2'"'] & \bullet \arrow[l, "j"'] \arrow[u, "p_2"'] \arrow[d, "g_2"] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "q'b_1"', "\approx" description] \arrow[r, "f_1'g_2'"] \arrow[u, equal] & \bullet & \bullet \arrow[l, "j'"'] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "v_1"', "\approx" description] \arrow[r, "h_1"] \arrow[u, "r_1"'] \arrow[d, "d_1", "\approx"'] & \bullet \arrow[u, "r_2"'] \arrow[d, "d_2", "\approx"'] & \bullet \arrow[l, "u_1"', "\approx" description] \arrow[u, "i_2"'] \\
\bullet \arrow[u, equal] & \bullet \arrow[l, "v_2"', "\approx" description] \arrow[r, "h_2"] & \bullet & \bullet \arrow[l, "u_2"', "\approx" description] \arrow[u, equal] \\
\bullet \arrow[u, "p_1"'] \arrow[d, "g_1"] & \bullet \arrow[l, "\tilde{q}'"'] \arrow[r, "g_1'f_2'"] \arrow[u, "k_1"'] \arrow[d, "g_1'"] & \bullet \arrow[u, "k_2"'] & \bullet \arrow[l, "a_2\tilde{\tilde{j}}'"', "\approx" description] \arrow[u, equal] \\
\bullet & \bullet \arrow[l, "\tilde{q}"'] \arrow[r, "f_2'"] & \bullet \arrow[u, equal] & \bullet \arrow[l, "a_2\tilde{\tilde{j}}'"', "\approx" description] \arrow[u, equal] \\
\bullet \arrow[u, "i_1"'] & \bullet \arrow[l, "b_2"', "\approx" description] \arrow[r, "f_2"] \arrow[u, "\tilde{j}"'] & \bullet \arrow[u, "\tilde{j}'"'] & \bullet \arrow[l, "a_2"', "\approx" description] \arrow[u, equal]
\end{tikzcd}
$$

Applying the flipping lemma (5.16) twice and composing yields the assertion:



□

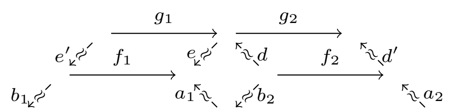
Altogether, we have proven the following main theorem of this article.

(5.18) Theorem. The fraction category $\operatorname{Frac}\mathcal{C}$ of a uni-fractionable category $\mathcal{C}$ (see definition (3.1)(a)) fulfills the following properties.

(a) The object set of $\operatorname{Frac}\mathcal{C}$ is the object set of $\mathcal{C}$. The morphism set of $\operatorname{Frac}\mathcal{C}$ consists of double fractions, that is, equivalence classes of 3-arrows with respect to fraction equality, where a 3-arrow (b, f, a) is a diagram

$$\xleftarrow[\approx]{b} \quad \xrightarrow{f} \quad \xleftarrow[\approx]{a}$$

in $\mathcal{C}$ with denominators a and b. For every 3-arrow (b, f, a) in $\mathcal{C}$, source and target of the double fraction $b\backslash f/a$ are given by $\operatorname{Source} b\backslash f/a = \operatorname{Target} b$ and $\operatorname{Target} b\backslash f/a = \operatorname{Source} a$. Given 3-arrows (b_1, f_1, a_1) and (b_2, f_2, a_2) in $\mathcal{C}$ with $\operatorname{Target} b_1\backslash f_1/a_1 = \operatorname{Source} b_2\backslash f_2/a_2$, the composite of the double fractions can be constructed as follows: One chooses denominators d, d', e, e' and morphisms g_1, g_2 in $\mathcal{C}$ with $b_2a_1 = de$, $g_1e = e'f_1$, $dg_2 = f_2d'$. Then $(b_1\backslash f_1/a_1)(b_2\backslash f_2/a_2) = e'b_1\backslash g_1g_2/a_2d'$.



The identity of an object X in $\operatorname{Frac}\mathcal{C}$ is given by $1_X = 1_X\backslash 1_X/1_X$.

(b) The fraction category $\operatorname{Frac}\mathcal{C}$ is a localisation of $\mathcal{C}$, where the localisation functor $\operatorname{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ is given on the objects by $\operatorname{loc}(X) = X$ for $X \in \operatorname{Ob}\mathcal{C}$ and on the morphisms by $\operatorname{loc}(f) = 1\backslash f/1$ for $f \in \operatorname{Mor}\mathcal{C}$. The inverse of $\operatorname{loc}(d)$ for $d \in \operatorname{Den}\mathcal{C}$ is given by $(\operatorname{loc}(d))^{-1} = d\backslash 1/1 = 1\backslash 1/d$.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ such that Fd is invertible for all $d \in \operatorname{Den}\mathcal{C}$, the unique functor $\hat{F}\colon \operatorname{Frac}\mathcal{C} \to \mathcal{D}$ with $F = \hat{F} \circ \operatorname{loc}$ is given by $\hat{F}(b\backslash f/a) = (Fb)^{-1}(Ff)(Fa)^{-1}$.

Given functors $F, G\colon \mathcal{C} \to \mathcal{D}$ such that Fd and Gd are invertible for all $d \in \operatorname{Den}\mathcal{C}$, and given a transformation $\alpha\colon F \to G$, the unique transformation $\hat{\alpha}\colon \hat{F} \to \hat{G}$ with $\alpha_X = \hat{\alpha}_{\operatorname{loc}(X)}$ for $X \in \operatorname{Ob}\mathcal{C}$ is given by $\hat{\alpha}_X = \alpha_X$ for $X \in \operatorname{Ob}\operatorname{Frac}\mathcal{C} = \operatorname{Ob}\mathcal{C}$.

(c) Given 3-arrows (b_1, f_1, a_1), (b_2, f_2, a_2) and normal 3-arrows (p_1, g_1, i_1), (p_2, g_2, i_2) in $\mathcal{C}$, we have

$$(b_1\backslash f_1/a_1)(p_2\backslash g_2/i_2) = (p_1\backslash g_1/i_1)(b_2\backslash f_2/a_2)$$

if and only if there exist 3-arrows $(\tilde{b}_1, \tilde{f}_1, \tilde{a}_1)$, $(\tilde{b}_2, \tilde{f}_2, \tilde{a}_2)$ and normal 3-arrows $(\tilde{p}_1, \tilde{g}_1, \tilde{i}_1)$, $(\tilde{p}_2, \tilde{g}_2, \tilde{i}_2)$ in $\mathcal{C}$, fitting into the following commutative diagram in $\mathcal{C}$.

$$\begin{tikzcd}
{} & {} \arrow[l, "b_1"', "\approx" description] \arrow[r, "f_1"] & {} & {} \arrow[l, "a_1"', "\approx" description] \\
{} \arrow[u, "p_1"'] \arrow[d, "g_1"] & {} \arrow[l, dotted, "\tilde{b}_1"', "\approx" description] \arrow[r, dotted, "\tilde{f}_1"] \arrow[u, dotted, "\tilde{p}_1"'] \arrow[d, dotted, "\tilde{g}_1"] & {} \arrow[u, dotted, "\tilde{p}_2"'] \arrow[d, dotted, "\tilde{g}_2"] & {} \arrow[l, dotted, "\tilde{a}_1"', "\approx" description] \arrow[u, "p_2"'] \arrow[d, "g_2"] \\
{} & {} \arrow[l, dotted, "\tilde{b}_2"', "\approx" description] \arrow[r, dotted, "\tilde{f}_2"] & {} & {} \arrow[l, dotted, "\tilde{a}_2"', "\approx" description] \\
{} \arrow[u, "i_1"'] & {} \arrow[l, "b_2"', "\approx" description] \arrow[r, "f_2"] \arrow[u, dotted, "\tilde{i}_1"'] & {} \arrow[u, dotted, "\tilde{i}_2"'] & {} \arrow[l, "a_2"', "\approx" description] \arrow[u, "i_2"']
\end{tikzcd}$$

Proof. This follows from propositions (5.4) and (5.9)(a), proposition (5.7) and proposition (5.17)(b). □

As a consequence of 3-arrow calculus, we get the following criterion. For a related 2-arrow version of this result, cf. [34, ch. 1, §2, th. 4-2] and [13, III.2.10].

(5.19) Proposition. We suppose given a uni-fractionable category $\mathcal{C}$ and a category with denominators $\mathcal{U}$ such that $\mathcal{U}$ is a full subcategory of $\mathcal{C}$ and $\operatorname{Den}\mathcal{U} = (\operatorname{Den}\mathcal{C}) \cap (\operatorname{Mor}\mathcal{U})$. Moreover, we suppose that $\mathcal{U}$ fulfills one of the following two dual conditions.

(a) For every object X in $\mathcal{C}$, there exist an object $\tilde{X}$ in $\mathcal{U}$ and a denominator $d\colon \tilde{X} \to X$ in $\mathcal{C}$. Moreover, for every S-denominator $i\colon U \to \tilde{U}$ with U in $\mathcal{U}$, it follows that $\tilde{U}$ is in $\mathcal{U}$.

(b) For every object X in $\mathcal{C}$, there exist an object $\tilde{X}$ in $\mathcal{U}$ and a denominator $d\colon X \to \tilde{X}$ in $\mathcal{C}$. Moreover, for every T-denominator $p\colon \tilde{U} \to U$ with U in $\mathcal{U}$, it follows that $\tilde{U}$ is in $\mathcal{U}$.

Then the inclusion functor $\operatorname{inc}\colon \mathcal{U} \to \mathcal{C}$ induces an equivalence $\operatorname{Frac}\operatorname{inc}\colon \operatorname{Frac}\mathcal{U} \to \operatorname{Frac}\mathcal{C}$.

Proof. We suppose that $\mathcal{U}$ fulfills (a), the other case follows by duality. To show that Frac inc is an equivalence of categories, we will verify that Frac inc is full, faithful and dense. Since for every $X \in \operatorname{Ob}\mathcal{C}$ there exist $\tilde{X} \in \operatorname{Ob}\mathcal{U}$ and a denominator $d\colon \tilde{X} \to X$ in $\mathcal{C}$, we have $X \cong \tilde{X} = (\operatorname{Frac}\operatorname{inc})\tilde{X}$ in $\operatorname{Frac}\mathcal{C}$. Hence Frac inc is dense. To prove that Frac inc is full and faithful, we have to show that the map

$$_{\operatorname{Frac}\mathcal{U}}(U, V) \to {}_{\operatorname{Frac}\mathcal{C}}(U, V), \varphi \mapsto (\operatorname{Frac}\operatorname{inc})\varphi$$

is bijective for $U, V \in \operatorname{Ob}\mathcal{U}$.
To show surjectivity, we suppose given a morphism $\psi \in {}_{\operatorname{Frac}\mathcal{C}}(U, V)$ and a normal 3-arrow $(p, f, i)\colon U \leftarrow X \to Y \leftarrow V$ in $\mathcal{C}$ with $\psi = p\backslash f/i$. Since i is an S-denominator and V is an object in $\mathcal{U}$, it follows that Y is an object in $\mathcal{U}$. Moreover, there exists an object $\tilde{X}$ in $\mathcal{U}$ and a denominator $d\colon \tilde{X} \to X$.

$$\begin{tikzcd}
U \arrow[d, equal] & \tilde{X} \arrow[l, "dp"', "\approx" description] \arrow[r, "df"] \arrow[d, "d", "\wr"'] & Y \arrow[d, equal] & V \arrow[l, "i"', "\approx" description] \arrow[d, equal] \\
U & X \arrow[l, "p"'] \arrow[r, "f"] & Y & V \arrow[l, "i"']
\end{tikzcd}$$

It follows that $(p, f, i) \equiv (dp, df, i)$, and as (dp, df, i) is a 3-arrow in $\mathcal{U}$, we have

$$\psi = p\backslash f/i = dp\backslash df/i = (\operatorname{Frac}\operatorname{inc})(dp\backslash df/i).$$

Thus $_{\operatorname{Frac}\mathcal{U}}(U, V) \to {}_{\operatorname{Frac}\mathcal{C}}(U, V), \varphi \mapsto (\operatorname{Frac}\operatorname{inc})\varphi$ is surjective.
To show injectivity, we suppose given $\varphi_1, \varphi_2 \in {}_{\operatorname{Frac}\mathcal{U}}(U, V)$ with $(\operatorname{Frac}\operatorname{inc})\varphi_1 = (\operatorname{Frac}\operatorname{inc})\varphi_2$. We choose normal 3-arrows $(p_1, f_1, i_1)\colon U \leftarrow U_1 \to V_1 \leftarrow V$ and $(p_2, f_2, i_2)\colon U \leftarrow U_2 \to V_2 \leftarrow V$ in $\mathcal{U}$ with $\varphi_1 = p_1\backslash f_1/i_1$ and $\varphi_2 = p_2\backslash f_2/i_2$. By proposition (5.17)(a), there exist normal 3-arrows $(\tilde{p}_1, \tilde{f}_1, \tilde{i}_1)\colon U \leftarrow X_1 \to Y_1 \leftarrow V$, $(\tilde{p}_2, \tilde{f}_2, \tilde{i}_2)\colon U \leftarrow X_2 \to Y_2 \leftarrow V$, $(q_1, d_1, j_1)\colon U_1 \leftarrow X_1 \to X_2 \leftarrow U_2$, $(q_2, d_2, j_2)\colon V_1 \leftarrow Y_1 \to Y_2 \leftarrow V_2$ in $\mathcal{C}$ with

denominators d_1, d_2, fitting into a commutative diagram as follows.

$$\begin{tikzcd}
U \arrow[d, equal] & U_1 \arrow[l, "p_1"'] \arrow[r, "f_1"] & V_1 & V \arrow[l, "i_1"'] \arrow[d, equal] \\
U \arrow[d, equal] & X_1 \arrow[l, dotted, "\tilde{p}_1"'] \arrow[r, dotted, "\tilde{f}_1"] \arrow[u, dotted, "q_1"'] \arrow[d, dotted, "\wr\, d_1"] & Y_1 \arrow[u, dotted, "q_2"'] \arrow[d, dotted, "\wr\, d_2"] & V \arrow[l, dotted, "\tilde{i}_1"'] \arrow[d, equal] \\
U \arrow[d, equal] & X_2 \arrow[l, dotted, "\tilde{p}_2"'] \arrow[r, dotted, "\tilde{f}_2"] & Y_2 & V \arrow[l, dotted, "\tilde{i}_2"'] \arrow[d, equal] \\
U & U_2 \arrow[l, "p_2"'] \arrow[r, "f_2"] \arrow[u, dotted, "j_1"'] & V_2 \arrow[u, dotted, "j_2"'] & V \arrow[l, "i_2"']
\end{tikzcd}$$

Since $\tilde{i}_1$ resp. j_1 resp. j_2 is an S-denominator and V resp. U_2 resp. V_2 is an object in $\mathcal{U}$, it follows that Y_1 resp. X_2 resp. Y_2 is an object in $\mathcal{U}$. Moreover, there exists an object $\tilde{X}_1$ in $\mathcal{U}$ and a denominator $d\colon \tilde{X}_1 \to X_1$ in $\mathcal{C}$. Thus we obtain the following commutative diagram in which all objects – and hence all morphisms – are in $\mathcal{U}$, and where $d\tilde{p}_1$ is a denominator by multiplicativity.

$$\begin{tikzcd}
U \arrow[d, equal] & U_1 \arrow[l, "p_1"'] \arrow[r, "f_1"] & V_1 & V \arrow[l, "i_1"'] \arrow[d, equal] \\
U \arrow[d, equal] & \tilde{X}_1 \arrow[l, "d\tilde{p}_1"', "\approx"] \arrow[r, "d\tilde{f}_1"] \arrow[u, "\wr\, dq_1"'] \arrow[d, "\wr\, dd_1"] & Y_1 \arrow[u, "q_2"'] \arrow[d, "\wr\, d_2"] & V \arrow[l, "\tilde{i}_1"'] \arrow[d, equal] \\
U \arrow[d, equal] & X_2 \arrow[l, "\tilde{p}_2"'] \arrow[r, "\tilde{f}_2"] & Y_2 & V \arrow[l, "\tilde{i}_2"'] \arrow[d, equal] \\
U & U_2 \arrow[l, "p_2"'] \arrow[r, "f_2"] \arrow[u, "j_1"'] & V_2 \arrow[u, "j_2"'] & V \arrow[l, "i_2"']
\end{tikzcd}$$

But this implies

$$\varphi_1 = p_1 \backslash f_1 / i_1 = p_2 \backslash f_2 / i_2 = \varphi_2$$

in $\operatorname{Frac}\mathcal{U}$. Therefore the map ${}_{\operatorname{Frac}\mathcal{U}}(U, V) \to {}_{\operatorname{Frac}\mathcal{C}}(U, V), \varphi \mapsto (\operatorname{Frac}\operatorname{inc})\varphi$ is injective. □

6 (Co)products and additive uni-fractionable categories

Some of our examples of uni-fractionable categories in section 7 have finite coproducts or products or are even additive categories, so it is a natural question to ask whether these features are preserved when passing to the fraction category.

(6.1) Proposition. We suppose given a uni-fractionable category $\mathcal{C}$.

(a) We suppose that $\mathcal{C}$ admits finite coproducts.

  (i) If $\operatorname{Den}\mathcal{C}$ is closed under finite coproducts, then the fraction category $\operatorname{Frac}\mathcal{C}$ admits finite coproducts and the localisation functor $\operatorname{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite coproducts. In this case, we have $\operatorname{ini}^{\operatorname{loc}(¡)}_{\operatorname{loc}(X)} = \operatorname{loc}(\operatorname{ini}^{¡}_{X})\colon \operatorname{loc}(¡) \to \operatorname{loc}(X)$ for $X \in \operatorname{Ob}\mathcal{C}$, and we have

$$\begin{pmatrix} b_1 \backslash f_1 / a \\ b_2 \backslash f_2 / a \end{pmatrix}^{\operatorname{loc}(X_1 \amalg X_2)} = (b_1 \amalg b_2) \backslash \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2} / a \colon \operatorname{loc}(X_1 \amalg X_2) \to \operatorname{loc}(Y)$$

  for 3-arrows $(b_1, f_1, a)\colon X_1 \leftarrow \tilde{X}_1 \to \tilde{Y} \leftarrow Y$ and $(b_2, f_2, a)\colon X_2 \leftarrow \tilde{X}_2 \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$.

  (ii) If $\mathcal{C}$ is saturated and the localisation functor $\operatorname{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite coproducts, then $\operatorname{Den}\mathcal{C}$ is closed under finite coproducts.

(b) We suppose that $\mathcal{C}$ admits finite products.

  (i) If $\operatorname{Den}\mathcal{C}$ is closed under finite products, then the fraction category $\operatorname{Frac}\mathcal{C}$ admits finite products and the localisation functor $\mathrm{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite products. In this case, we have $\mathrm{ter}^{\mathrm{loc}(!)}_{\mathrm{loc}(X)} = \mathrm{loc}(\mathrm{ter}^{!}_{X})\colon \mathrm{loc}(X) \to \mathrm{loc}(!)$ for $X \in \operatorname{Ob}\mathcal{C}$, and we have

$$\left(\, b\backslash f_1/a_1 \;\; b\backslash f_2/a_2 \,\right)^{\mathrm{loc}(Y_1 \amalg Y_2)} = b\backslash\left(\, f_1 \;\; f_2 \,\right)^{\tilde{Y}_1 \amalg \tilde{Y}_2}/(a_1 \amalg a_2)\colon \mathrm{loc}(X) \to \mathrm{loc}(Y_1 \amalg Y_2)$$

  for 3-arrows $(b, f_1, a_1)\colon X \leftarrow \tilde{X} \to \tilde{Y}_1 \leftarrow Y_1$ and $(b, f_2, a_2)\colon X \leftarrow \tilde{X} \to \tilde{Y}_2 \leftarrow Y_2$ in $\mathcal{C}$.

  (ii) If $\mathcal{C}$ is saturated and the localisation functor $\mathrm{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite products, then $\operatorname{Den}\mathcal{C}$ is closed under finite products.

(c) We suppose that $\mathcal{C}$ admits finite sums.

  (i) If $\operatorname{Den}\mathcal{C}$ is closed under finite sums, then the fraction category $\operatorname{Frac}\mathcal{C}$ admits finite sums and the localisation functor $\mathrm{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite sums. In this case, we have $0 = \mathrm{loc}(0)\colon \mathrm{loc}(X) \to \mathrm{loc}(Y)$ for $X, Y \in \operatorname{Ob}\mathcal{C}$. Moreover, we have

$$\begin{pmatrix} b_1\backslash f_1/a \\ b_2\backslash f_2/a \end{pmatrix}^{\mathrm{loc}(X_1 \oplus X_2)} = (b_1 \oplus b_2)\backslash\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \oplus \tilde{X}_2}/a\colon \mathrm{loc}(X_1 \oplus X_2) \to \mathrm{loc}(Y)$$

  for 3-arrows $(b_1, f_1, a)\colon X_1 \leftarrow \tilde{X}_1 \to \tilde{Y} \leftarrow Y$ and $(b_2, f_2, a)\colon X_2 \leftarrow \tilde{X}_2 \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$, and we have

$$\left(\, b\backslash f_1/a_1 \;\; b\backslash f_2/a_2 \,\right)^{\mathrm{loc}(Y_1 \oplus Y_2)} = b\backslash\left(\, f_1 \;\; f_2 \,\right)^{\tilde{Y}_1 \oplus \tilde{Y}_2}/(a_1 \oplus a_2)\colon \mathrm{loc}(X) \to \mathrm{loc}(Y_1 \oplus Y_2)$$

  for 3-arrows $(b, f_1, a_1)\colon X \leftarrow \tilde{X} \to \tilde{Y}_1 \leftarrow Y_1$ and $(b, f_2, a_2)\colon X \leftarrow \tilde{X} \to \tilde{Y}_2 \leftarrow Y_2$ in $\mathcal{C}$.

  (ii) If $\mathcal{C}$ is saturated and the localisation functor $\mathrm{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite sums, then $\operatorname{Den}\mathcal{C}$ is closed under finite sums.

Proof.

(a) (i) We suppose that $\operatorname{Den}\mathcal{C}$ is closed under finite coproducts. Moreover, we suppose given $X \in \operatorname{Ob}\mathcal{C}$. Then $\mathrm{loc}(\mathrm{ini}^{¡}_{X})$ is a morphism from $\mathrm{loc}(¡)$ to $\mathrm{loc}(X)$. So let us suppose given an arbitrary morphism $\varphi\colon \mathrm{loc}(¡) \to X$ in $\operatorname{Frac}\mathcal{C}$, and we let $(b, f, a)\colon ¡ \leftarrow I \to \tilde{X} \leftarrow X$ be a 3-arrow in $\mathcal{C}$ with $\varphi = b\backslash f/a$. By the universal property of ¡, we have $\mathrm{ini}^{¡}_{I} b = 1_{¡}$ and $\mathrm{ini}^{¡}_{I} f = \mathrm{ini}^{¡}_{\tilde{X}} = \mathrm{ini}^{¡}_{X} a$, and therefore

$$\varphi = b\backslash f/a = \mathrm{ini}^{¡}_{I} b\backslash \mathrm{ini}^{¡}_{I} f/a = 1_{¡}\backslash \mathrm{ini}^{¡}_{X}/1 = \mathrm{loc}(\mathrm{ini}^{¡}_{X}).$$

Hence $\mathrm{loc}(¡)$ is an initial object in $\operatorname{Frac}\mathcal{C}$ with $\mathrm{ini}^{\mathrm{loc}(¡)}_{\mathrm{loc}(X)} = \mathrm{loc}(\mathrm{ini}^{¡}_{X})$ for all $X \in \operatorname{Ob}\mathcal{C}$.

$$\begin{tikzcd}
¡ \arrow[r, equal] \arrow[d, equal] & ¡ \arrow[r, "\mathrm{ini}^{¡}_{X}"] \arrow[d, dashed, "\wr\, \mathrm{ini}^{¡}_{I}"] & X \arrow[r, equal] \arrow[d, dashed, "\wr\, a"] & X \arrow[d, equal] \\
¡ & I \arrow[l, "b"', "\approx"] \arrow[r, "f"] & \tilde{X} & X \arrow[l, "a"', "\approx"]
\end{tikzcd}$$

Next, we suppose given morphisms $\varphi_1\colon X_1 \to Y$ and $\varphi_2\colon X_2 \to Y$ in $\operatorname{Frac}\mathcal{C}$. By proposition (5.13), there exist 3-arrows $(b_1, f_1, a)\colon X_1 \leftarrow \tilde{X}_1 \to \tilde{Y} \leftarrow Y$ and $(b_2, f_2, a)\colon X_2 \leftarrow \tilde{X}_2 \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $\varphi_1 = b_1\backslash f_1/a$ and $\varphi_2 = b_2\backslash f_2/a$. As $b_1 \amalg b_2$ is a denominator in $\mathcal{C}$ by assumption, we have the 3-arrow $(b_1 \amalg b_2, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, a)$ in $\mathcal{C}$. Moreover, since $\mathrm{emb}^{\tilde{X}_1 \amalg \tilde{X}_2}_{k}(b_1 \amalg b_2) = b_k \mathrm{emb}^{X_1 \amalg X_2}_{k}$, we have

$$\mathrm{loc}(\mathrm{emb}^{X_1 \amalg X_2}_{k})((b_1 \amalg b_2)\backslash\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}/a) = b_k\backslash \mathrm{emb}^{\tilde{X}_1 \amalg \tilde{X}_2}_{k}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}/a = b_k\backslash f_k/a = \varphi_k$$

for $k \in \{1, 2\}$.

$$\begin{tikzcd}
 & & \tilde{X}_k \arrow[rr, "\mathrm{emb}^{\tilde{X}_1 \amalg \tilde{X}_2}_{k}"] \arrow[dl, "b_k"', "\approx"] & & \tilde{X}_1 \amalg \tilde{X}_2 \arrow[rr, "\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}"] \arrow[dl, "b_1 \amalg b_2"', "\approx"] \arrow[dr, equal] & & \tilde{Y} \arrow[dr, equal] & & \\
 & X_k \arrow[rr, "\mathrm{emb}^{X_1 \amalg X_2}_{k}"] \arrow[dl, equal] & & X_1 \amalg X_2 \arrow[dr, equal] & & \tilde{X}_1 \amalg \tilde{X}_2 \arrow[rr, "\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^{\tilde{X}_1 \amalg \tilde{X}_2}"] \arrow[dl, "b_1 \amalg b_2", "\approx"'] & & \tilde{Y} & \\
X_k & & & & X_1 \amalg X_2 & & & & Y \arrow[ul, "a"', "\approx"]
\end{tikzcd}$$

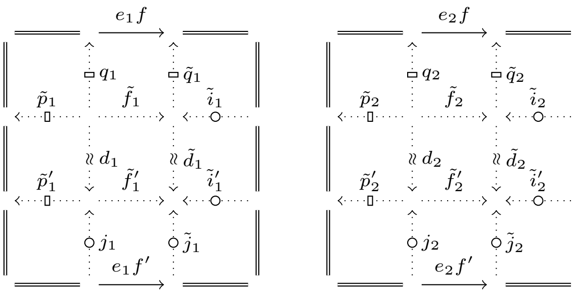
Conversely, we suppose given morphisms $\varphi, \varphi' \colon \mathrm{loc}(X_1 \amalg X_2) \to \mathrm{loc}(Y)$ in $\mathrm{Frac}\,\mathcal{C}$ such that $\mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2})\varphi = \mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2})\varphi' = \varphi_k$ for $k \in \{1,2\}$. By proposition (5.13), there exist normal 3-arrows $(p, f, i), (p, f', i) \colon X_1 \amalg X_2 \leftarrow \tilde{X} \to \tilde{Y} \leftarrow Y$ in $\mathcal{C}$ with $\varphi = p \backslash f / i$ and $\varphi' = p \backslash f' / i$. For $k \in \{1,2\}$, we choose a T-denominator $p_k \colon \tilde{X}_k \to X_k$ and a morphism $e_k \colon \tilde{X}_k \to \tilde{X}$ in $\mathcal{C}$ with $p_k \mathrm{emb}_k^{X_1 \amalg X_2} = e_k p$. Then we have

$$\varphi_k = \mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2})\varphi = \mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2})(p \backslash f / i) = p_k \backslash e_k f / i$$

for $k \in \{1,2\}$.

$$\begin{tikzcd}
 & & \tilde{X}_k \arrow[dl, "p_k"'] \arrow[rr, "e_k"] & & \tilde{X} \arrow[dl, "p"'] \arrow[dr, equal] \arrow[rr, "f"] & & \tilde{Y} \arrow[dr, equal] & & \\
 & X_k \arrow[dl, equal] \arrow[rr, "\mathrm{emb}_k^{X_1 \amalg X_2}"] & & X_1 \amalg X_2 \arrow[dr, equal] & & \tilde{X} \arrow[dl, "p"] \arrow[rr, "f"] & & \tilde{Y} & \\
X_k & & & & X_1 \amalg X_2 & & & & Y \arrow[ul, "i"']
\end{tikzcd}$$

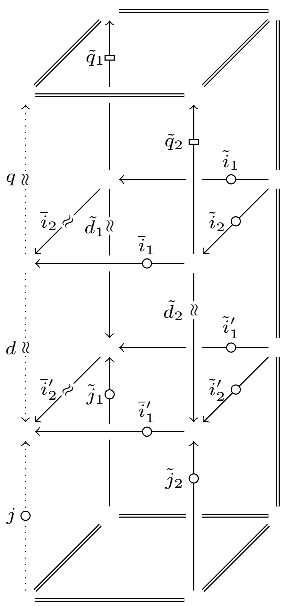
Analogously, we also have $\varphi_k = p_k \backslash e_k f' / i$ and therefore $\mathrm{loc}(e_k f) = \mathrm{loc}(e_k f')$ for $k \in \{1,2\}$. By proposition (5.17)(a), there exist normal 3-arrows $(\tilde{p}_k, \tilde{f}_k, \tilde{i}_k)$, $(\tilde{p}'_k, \tilde{f}'_k, \tilde{i}'_k)$, (q_k, d_k, j_k), $(\tilde{q}_k, \tilde{d}_k, \tilde{j}_k)$ in $\mathcal{C}$ with denominators $d_k, \tilde{d}_k$ for $k \in \{1,2\}$, fitting into the following commutative diagrams in $\mathcal{C}$.



We let

$$\begin{tikzcd}
{} \arrow[d, "\bar{i}_2"', "\wr"] & {} \arrow[l, "\tilde{i}_1"'] \arrow[d, "\tilde{i}_2"] \\
{} & {} \arrow[l, "\bar{i}_1"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[d, "\bar{i}'_2"', "\wr"] & {} \arrow[l, "\tilde{i}'_1"'] \arrow[d, "\tilde{i}'_2"] \\
{} & {} \arrow[l, "\bar{i}'_1"']
\end{tikzcd}$$

be weak pushout rectangles in $\mathcal{C}$ such that $\bar{i}_1$ and $\bar{i}'_1$ are S-denominators, so that we obtain morphisms q, d, j such that the following diagram commutes.



Thus we obtain the following commutative diagrams.

$$
\begin{tikzcd}
{} \arrow[r, equal] & {} \arrow[r, "e_1 f"] & {} \arrow[r, equal] & {} \\
{} \arrow[u, equal] \arrow[d, equal] & {} \arrow[l, "\tilde{p}_1"'] \arrow[u, "q_1"'] \arrow[r, "\tilde{f}_1 \bar{i}_2"] \arrow[d, "\wr\, d_1"'] & {} \arrow[u, "\wr\, q"'] \arrow[d, "\wr\, d"'] & {} \arrow[l, "\tilde{i}_1 \bar{i}_2"'] \arrow[u, equal] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "\tilde{p}'_1"'] \arrow[r, "\tilde{f}'_1 \bar{i}'_2"] & {} & {} \arrow[l, "\tilde{i}'_1 \bar{i}'_2"'] \arrow[d, equal] \\
{} \arrow[r, equal] & {} \arrow[u, "j_1"'] \arrow[r, "e_1 f'"'] & {} \arrow[u, "j"'] \arrow[r, equal] & {}
\end{tikzcd}
\qquad
\begin{tikzcd}
{} \arrow[r, equal] & {} \arrow[r, "e_2 f"] & {} \arrow[r, equal] & {} \\
{} \arrow[u, equal] \arrow[d, equal] & {} \arrow[l, "\tilde{p}_2"'] \arrow[u, "q_2"'] \arrow[r, "\tilde{f}_2 \bar{i}_1"] \arrow[d, "\wr\, d_2"'] & {} \arrow[u, "\wr\, q"'] \arrow[d, "\wr\, d"'] & {} \arrow[l, "\tilde{i}_2 \bar{i}_1"'] \arrow[u, equal] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "\tilde{p}'_2"'] \arrow[r, "\tilde{f}'_2 \bar{i}'_1"] & {} & {} \arrow[l, "\tilde{i}'_2 \bar{i}'_1"'] \arrow[d, equal] \\
{} \arrow[r, equal] & {} \arrow[u, "j_2"'] \arrow[r, "e_2 f'"'] & {} \arrow[u, "j"'] \arrow[r, equal] & {}
\end{tikzcd}
$$

Using coproducts, these diagrams provide in turn the following commutative diagram.

$$
\begin{tikzcd}
{} \arrow[r, equal] & {} \arrow[r, "\left(\begin{smallmatrix} e_1 f \\ e_2 f \end{smallmatrix}\right)"] & {} \arrow[r, equal] & {} \\
{} \arrow[u, equal] \arrow[d, equal] & {} \arrow[l, "\tilde{p}_1 \amalg \tilde{p}_2"', "\approx"] \arrow[u, "q_1 \amalg q_2", "\wr"'] \arrow[r, "\left(\begin{smallmatrix} \tilde{f}_1 \bar{i}_2 \\ \tilde{f}_2 \bar{i}_1 \end{smallmatrix}\right)"] \arrow[d, "d_1 \amalg d_2", "\wr"'] & {} \arrow[u, "\wr\, q"'] \arrow[d, "\wr\, d"'] & {} \arrow[l, "\tilde{i}_1 \bar{i}_2"'] \arrow[u, equal] \arrow[d, equal] \\
{} \arrow[d, equal] & {} \arrow[l, "\tilde{p}'_1 \amalg \tilde{p}'_2"', "\approx"] \arrow[r, "\left(\begin{smallmatrix} \tilde{f}'_1 \bar{i}'_2 \\ \tilde{f}'_2 \bar{i}'_1 \end{smallmatrix}\right)"] & {} & {} \arrow[l, "\tilde{i}'_1 \bar{i}'_2"'] \arrow[d, equal] \\
{} \arrow[r, equal] & {} \arrow[u, "j_1 \amalg j_2", "\wr"'] \arrow[r, "\left(\begin{smallmatrix} e_1 f' \\ e_2 f' \end{smallmatrix}\right)"] & {} \arrow[u, "j"'] \arrow[r, equal] & {}
\end{tikzcd}
$$

We finally have

$$\mathrm{loc}(\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right))\mathrm{loc}(f) = \mathrm{loc}(\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right) f) = \mathrm{loc}(\begin{pmatrix} e_1 f \\ e_2 f \end{pmatrix}) = \mathrm{loc}(\begin{pmatrix} e_1 f' \\ e_2 f' \end{pmatrix}) = \mathrm{loc}(\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right) f') = \mathrm{loc}(\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right))\mathrm{loc}(f').$$

On the other hand,

$$\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right) p = \left(\begin{smallmatrix} e_1 p \\ e_2 p \end{smallmatrix}\right) = \begin{pmatrix} p_1 \mathrm{emb}_1^{X_1 \amalg X_2} \\ p_2 \mathrm{emb}_2^{X_1 \amalg X_2} \end{pmatrix} = p_1 \amalg p_2$$

implies that $\left(\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right)$ is a denominator in $\mathcal{C}$ by semi-saturatedness, so we have $\mathrm{loc}(f) = \mathrm{loc}(f')$ and therefore

$$\varphi = p\backslash f/i = p\backslash f'/i = \varphi'.$$

Altogether, $\mathrm{loc}(X_1 \amalg X_2)$ is a coproduct of $\mathrm{loc}(X_1)$ and $\mathrm{loc}(X_2)$ with embeddings $\mathrm{emb}_k^{\mathrm{loc}(X_1 \amalg X_2)} = \mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2})$ for $k \in \{1, 2\}$.

(ii) We suppose that $\mathcal{C}$ is saturated and that loc preserves finite coproducts. Moreover, we suppose given denominators $d_1 \colon X_1 \to Y_1$ and $d_2 \colon X_2 \to Y_2$ in $\mathcal{C}$. Then we have

$$\begin{aligned} \mathrm{loc}(d_k)\mathrm{emb}_k^{\mathrm{loc}(Y_1 \amalg Y_2)} &= \mathrm{loc}(d_k)\mathrm{loc}(\mathrm{emb}_k^{Y_1 \amalg Y_2}) = \mathrm{loc}(d_k \mathrm{emb}_k^{Y_1 \amalg Y_2}) = \mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2}(d_1 \amalg d_2)) \\ &= \mathrm{loc}(\mathrm{emb}_k^{X_1 \amalg X_2})\mathrm{loc}(d_1 \amalg d_2) = \mathrm{emb}_k^{\mathrm{loc}(X_1 \amalg X_2)}\mathrm{loc}(d_1 \amalg d_2). \end{aligned}$$

Since d_1 and d_2 are denominators, $\mathrm{loc}(d_1)$ and $\mathrm{loc}(d_2)$ are isomorphisms. But then $\mathrm{loc}(d_1 \amalg d_2)$ is also an isomorphism and hence $d_1 \amalg d_2$ is a denominator since $\mathcal{C}$ is saturated.

(b) This is dual to (a).

(c) This follows from (a) and (b). □

The preceding criterion motivates the next definition.

(6.2) Definition. An *additive uni-fractionable category* is a uni-fractionable category $\mathcal{A}$ such that the underlying category of $\mathcal{A}$ is equipped with the structure of an additive category and such that $\operatorname{Den}\mathcal{A}$ is closed under finite sums.

(6.3) Remark. We suppose given a uni-fractionable category $\mathcal{C}$.

(a) We suppose that $\mathcal{C}$ admits finite coproducts. Then $\operatorname{Den}\mathcal{C}$ is closed under finite coproducts if and only if $i \amalg j$ is a denominator for all S-denominators i, j in $\mathcal{C}$ and $p \amalg q$ is a denominator for all T-denominators p, q in $\mathcal{C}$.

(b) We suppose that $\mathcal{C}$ admits finite products. Then $\operatorname{Den}\mathcal{C}$ is closed under finite products if and only if $i \Pi j$ is a denominator for all S-denominators i, j in $\mathcal{C}$ and $p \Pi q$ is a denominator for all T-denominators p, q in $\mathcal{C}$.

Proof.

(a) If $\operatorname{Den}\mathcal{C}$ is closed under finite coproducts, then in particular $i \amalg j$ is a denominator for all S-denominators i, j, and $p \amalg q$ is a denominator for all T-denominators p, q in $\mathcal{C}$. So let us conversely suppose that $i \amalg j$ is a denominator for all S-denominators i, j and that $p \amalg q$ is a denominator for all T-denominators p, q, and let us suppose given denominators d, e in $\mathcal{C}$. Then there exist S-denominators i, j and T-denominators p, q with $d = ip$ and $e = jq$, and hence

$$d \amalg e = (ip) \amalg (jq) = (i \amalg j)(p \amalg q)$$

is a denominator in $\mathcal{C}$ by multiplicativity. Thus $\operatorname{Den}\mathcal{C}$ is closed under finite coproducts. □

Recall that every hom-set in a category that admits finite sums carries a unique structure of a commutative monoid such that addition of morphisms becomes compatible with composition [25, ch. VIII, sec. 2, ex. 4(a)]. In modern terms: Such a category is enriched over the category of commutative monoids in a unique way. Moreover, every hom-set becomes an abelian group, that is, the category under consideration is additive, if and only if every identity has a negative element with respect to the addition on its hom-set [25, ch. VIII, sec. 2, ex. 4(b)]. The latter condition is equivalent to the condition that the morphism $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ is always an isomorphism. A functor between additive categories is additive if and only if it preserves finite sums, that is, if and only if the image of every (chosen) finite sum is a finite sum of the images, such that the embeddings resp. projections are the images of the embeddings resp. projections. Cf. also [24, sec. 18–19], [23, sec. 3.1–3.2].

(6.4) Proposition. Given an additive uni-fractionable category $\mathcal{A}$, the additive structure of $\mathcal{A}$ induces an additive structure on the fraction category $\operatorname{Frac}\mathcal{A}$ such that the localisation functor $\operatorname{loc}\colon \mathcal{A} \to \operatorname{Frac}\mathcal{A}$ becomes an additive functor. For parallel 3-arrows (b, f, a) and (b, g, a) in $\mathcal{A}$ (cf. proposition (5.13)), we have

$$b\backslash f/a + b\backslash g/a = b\backslash(f+g)/a.$$

Proof. By proposition (6.1)(c)(i), $\operatorname{loc}(0)$ is a zero object in $\operatorname{Frac}\mathcal{A}$, and for objects X_1, X_2 in $\mathcal{A}$, the object $\operatorname{loc}(X_1 \oplus X_2)$ is a sum of $\operatorname{loc}(X_1)$ and $\operatorname{loc}(X_2)$ in $\operatorname{Frac}\mathcal{A}$ with $\operatorname{emb}_k^{\operatorname{loc}(X_1 \oplus X_2)} = \operatorname{loc}(\operatorname{emb}_k^{X_1 \oplus X_2})$ and $\operatorname{pr}_k^{\operatorname{loc}(X_1 \oplus X_2)} = \operatorname{loc}(\operatorname{pr}_k^{X_1 \oplus X_2})$ for $k \in \{1, 2\}$. Thus $\operatorname{Frac}\mathcal{A}$ admits finite sums. For the purpose of this proof, let us choose $\operatorname{loc}(X_1) \oplus \operatorname{loc}(X_2) := \operatorname{loc}(X_1 \oplus X_2)$ for $X_1, X_2 \in \operatorname{Ob}\mathcal{A}$, so that we can use matrix notation for induced morphisms between those objects. Then we have $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \operatorname{loc}(\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right))\colon \operatorname{loc}(X) \oplus \operatorname{loc}(X) \to \operatorname{loc}(X) \oplus \operatorname{loc}(X)$ for every object X in $\mathcal{A}$, and so $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ is an isomorphism. Altogether, $\operatorname{Frac}\mathcal{A}$ is an additive category and $\operatorname{loc}\colon \mathcal{A} \to \operatorname{Frac}\mathcal{A}$ is an additive functor. In particular, we obtain

$$\begin{aligned} b\backslash f/a + b\backslash g/a &= (\operatorname{loc}(b))^{-1}\operatorname{loc}(f)(\operatorname{loc}(a))^{-1} + (\operatorname{loc}(b))^{-1}\operatorname{loc}(g)(\operatorname{loc}(a))^{-1} \\ &= (\operatorname{loc}(b))^{-1}(\operatorname{loc}(f) + \operatorname{loc}(g))(\operatorname{loc}(a))^{-1} = (\operatorname{loc}(b))^{-1}\operatorname{loc}(f+g)(\operatorname{loc}(a))^{-1} \\ &= b\backslash(f+g)/a \end{aligned}$$

for parallel 3-arrows (b, f, a) and (b, g, a) in $\mathcal{A}$. □

7 Applications

In this final section, we consider some examples and applications.

Quillen model categories

Given a Quillen model category $\mathcal{M}$ [29, ch. I, §1, def. 1], we denote by $\mathbf{Cof}(\mathcal{M})$ the full subcategory of cofibrant objects, by $\mathbf{Fib}(\mathcal{M})$ the full subcategory of fibrant objects and by $\mathbf{Bif}(\mathcal{M})$ the full subcategory of bifibrant (that is, cofibrant and fibrant) objects.

(7.1) Example. Given a Quillen model category $\mathcal{M}$, the categories $\mathcal{M}$, $\mathbf{Cof}(\mathcal{M})$, $\mathbf{Fib}(\mathcal{M})$, $\mathbf{Bif}(\mathcal{M})$ carry the structure of uni-fractionable categories, where

$$\begin{aligned} \operatorname{Den}\mathcal{C} &= \{w \in \operatorname{Mor}\mathcal{C} \mid w \text{ is a weak equivalence}\}, \\ \operatorname{SDen}\mathcal{C} &= \{i \in \operatorname{Mor}\mathcal{C} \mid i \text{ is an acyclic cofibration}\}, \\ \operatorname{TDen}\mathcal{C} &= \{p \in \operatorname{Mor}\mathcal{C} \mid p \text{ is an acyclic fibration}\} \end{aligned}$$

for $\mathcal{C} \in \{\mathcal{M}, \mathbf{Cof}(\mathcal{M}), \mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$. In particular, the homotopy category $\operatorname{Ho}\mathcal{M}$ is isomorphic to $\operatorname{Frac}\mathcal{M}$. If $\mathcal{M}$ is a closed Quillen model category, then $\operatorname{Den}\mathcal{C}$ is saturated for $\mathcal{C} \in \{\mathcal{M}, \mathbf{Cof}(\mathcal{M}), \mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$. The localisation functor $\operatorname{loc}\colon \mathcal{C} \to \operatorname{Frac}\mathcal{C}$ preserves finite coproducts for $\mathcal{C} \in \{\mathbf{Cof}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$ and finite products for $\mathcal{C} \in \{\mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$. ([8])

Proof.

(a) We consider $\mathcal{M}$ and verify the axioms of a uni-fractionable category.

- (Cat) By definition of a Quillen model category, weak equivalences, cofibrations and fibrations are closed under composition and contain all isomorphisms. Hence in particular weak equivalences, acyclic cofibrations and acyclic fibrations are closed under composition and contain all identities.
- (2 of 3) This holds by definition of a Quillen model category.
- (WU) We suppose given an acyclic cofibration $i\colon X \to X'$ and a morphism $f\colon X \to Y$ in $\mathcal{M}$, and we let

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

be a pushout rectangle in $\mathcal{C}$. Then i' is an acyclic cofibration.
The other assertion follows by duality.
- (Fac) Since every morphism decomposes into a composite of a cofibration followed by an acyclic fibration, the assertion follows by semi-saturatedness.

Altogether, $\mathcal{M}$ becomes a uni-fractionable category with

$$\begin{aligned} \operatorname{Den}\mathcal{M} &= \{w \in \operatorname{Mor}\mathcal{M} \mid w \text{ is a weak equivalence}\}, \\ \operatorname{SDen}\mathcal{M} &= \{i \in \operatorname{Mor}\mathcal{M} \mid i \text{ is an acyclic cofibration}\}, \\ \operatorname{TDen}\mathcal{M} &= \{p \in \operatorname{Mor}\mathcal{M} \mid p \text{ is an acyclic fibration}\}. \end{aligned}$$

The assertion on the saturatedness of $\mathcal{M}$ is proven in [29, ch. I, §5, prop. 1].

(b) We consider $\mathbf{Cof}(\mathcal{M})$ and have to verify the axioms of a uni-fractionable category. Since (Cat) and (2 of 3) hold for $\mathcal{M}$ by (a), they hold in particular for $\mathbf{Cof}(\mathcal{M})$.

[8] In general, the localisation functor $\operatorname{loc}\colon \mathcal{M} \to \operatorname{Frac}\mathcal{M}$ does not preserve finite coproducts or finite products since the set of denominators in a closed Quillen model category need not be closed under finite (co)products. A counterexample is provided by $(\mathbb{Z}/4 \downarrow \mathbf{mod}(\mathbb{Z}/4))$, cf. [11, rem. 3.11], as considered in [10, ex.]: The coproduct of $2\colon (\mathbb{Z}/4, 1) \to (\mathbb{Z}/4, 2)$ with itself is given by $\begin{pmatrix} 2 & 0 \end{pmatrix} \colon (\mathbb{Z}/4, 1) \to (\mathbb{Z}/4 \oplus \mathbb{Z}/2, \begin{pmatrix} 2 & 0 \end{pmatrix})$; the former is a weak equivalence, but the latter is not since $\mathbb{Z}/4$ is a bijective object and $\mathbb{Z}/4 \oplus \mathbb{Z}/2$ is not a bijective object in $\mathbf{mod}(\mathbb{Z}/4)$.

(WU) We suppose given an acyclic cofibration $i\colon X \to X'$ and a morphism $f\colon X \to Y$ in $\mathbf{Cof}(\mathcal{M})$, and we let

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

be a pushout rectangle in $\mathcal{C}$. Then i' is an acyclic cofibration, and since Y is cofibrant and i' is in particular a cofibration, it follows that Y' is also cofibrant.

Now we suppose given an acyclic fibration $p\colon Y' \to Y$ and a morphism $f\colon X \to Y$ in $\mathbf{Cof}(\mathcal{M})$, and we let

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a pullback rectangle in $\mathcal{C}$. Then p' is an acyclic fibration. We consider a strong cofibrant approximation of X', that is, we let $\tilde{X}'$ be a cofibrant object together with an acyclic fibration $q\colon \tilde{X}' \to X'$. The composite qp' is an acyclic fibration by multiplicativity. We will show that

$$\begin{tikzcd} \tilde{X}' \arrow[r, "qf'"] \arrow[d, "qp'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a weak pullback of f along p. To this end, we suppose given an object $T \in \operatorname{Ob}\mathbf{Cof}(\mathcal{M})$ and morphisms $s\colon T \to X$, $t\colon T \to Y'$ with $sf = tp$. By the universal property of X', there exists a (unique) morphism $u\colon T \to X'$ such that $up' = s$ and $uf' = t$.

$$\begin{tikzcd} T \arrow[dr, dotted, "u"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & X' \arrow[r, "f'"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

Moreover, since T is cofibrant and q is an acyclic fibration, there exists a lift $\hat{u}\colon T \to \tilde{X}'$ such that $u = \hat{u}q$.

$$\begin{tikzcd} & \tilde{X}' \arrow[d, "q"] \\ T \arrow[ur, dotted, "\hat{u}"] \arrow[r, "u"] & X' \end{tikzcd}$$

Now we have $\hat{u}qp' = up' = s$ and $\hat{u}qf' = uf' = t$.

$$\begin{tikzcd} T \arrow[dr, "\hat{u}"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & \tilde{X}' \arrow[r, "qf'"] \arrow[d, "qp'"'] & Y' \arrow[d, "p"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

(Fac) We let $w\colon X \to Y$ be a weak equivalence in $\mathbf{Cof}(\mathcal{M})$. Then there exists an acyclic cofibration $i\colon X \to Z$ and an acyclic fibration $p\colon Z \to Y$ in $\mathcal{M}$ with $w = ip$.

$$\begin{tikzcd} & Z \arrow[dr, dotted, "p"] & \\ X \arrow[ur, dotted, "i"] \arrow[rr, "w"] & & Y \end{tikzcd}$$

But since X is cofibrant and i is a cofibration, Z is cofibrant, too.

Altogether, $\mathbf{Cof}(\mathcal{M})$ becomes a uni-fractionable category with

$$\begin{aligned} \operatorname{Den}\mathbf{Cof}(\mathcal{M}) &= \{w \in \operatorname{Mor}\mathbf{Cof}(\mathcal{M}) \mid w \text{ is a weak equivalence}\}, \\ \operatorname{SDen}\mathbf{Cof}(\mathcal{M}) &= \{i \in \operatorname{Mor}\mathbf{Cof}(\mathcal{M}) \mid i \text{ is an acyclic cofibration}\}, \\ \operatorname{TDen}\mathbf{Cof}(\mathcal{M}) &= \{p \in \operatorname{Mor}\mathbf{Cof}(\mathcal{M}) \mid p \text{ is an acyclic fibration}\}. \end{aligned}$$

The assertion on the saturatedness of $\mathbf{Cof}(\mathcal{M})$ follows from (a) since if $\operatorname{loc}^{\operatorname{Frac}\mathbf{Cof}(\mathcal{M})}(f)$ is an isomorphism, then also $\operatorname{loc}^{\operatorname{Frac}\mathcal{M}}(f)$ is an isomorphism. The fact that the localisation functor $\operatorname{loc}\colon \mathbf{Cof}(\mathcal{M}) \to \operatorname{Frac}\mathbf{Cof}(\mathcal{M})$ preserves finite coproducts follows from the gluing lemma [16, lem. 7.4], cf. also [14, ch. II, lem. 8.8], and proposition (6.1)(a)(i).

(c) This is dual to (b).

(d) This is a combination of (b) and (c). □

As an application of our abstract machinery, we obtain the following part of Quillen's homotopy category theorem [29, ch. I, §1, th. 1]. Given a Quillen model category $\mathcal{M}$, we (re-)define the *homotopy category* of $\mathcal{C} \in \{\mathcal{M}, \mathbf{Cof}(\mathcal{M}), \mathbf{Fib}(\mathcal{M}), \mathbf{Bif}(\mathcal{M})\}$ by $\operatorname{Ho}\mathcal{C} := \operatorname{Frac}\mathcal{C}$, using the uni-fractionable category structures from the preceding example.

(7.2) Example. We suppose given a Quillen model category $\mathcal{M}$. The commutative diagram of inclusion functors

$$\begin{tikzcd} & \mathbf{Cof}(\mathcal{M}) \arrow[dr, "\mathrm{inc}"] & \\ \mathbf{Bif}(\mathcal{M}) \arrow[ur, "\mathrm{inc}"] \arrow[dr, "\mathrm{inc}"'] & & \mathcal{M} \\ & \mathbf{Fib}(\mathcal{M}) \arrow[ur, "\mathrm{inc}"'] & \end{tikzcd}$$

induces a commutative diagram of equivalences

$$\begin{tikzcd} & \operatorname{Ho}\mathbf{Cof}(\mathcal{M}) \arrow[dr, "\simeq"] & \\ \operatorname{Ho}\mathbf{Bif}(\mathcal{M}) \arrow[ur, "\simeq"] \arrow[dr, "\simeq"'] & & \operatorname{Ho}\mathcal{M} \\ & \operatorname{Ho}\mathbf{Fib}(\mathcal{M}) \arrow[ur, "\simeq"'] & \end{tikzcd} .$$

In particular, $\operatorname{Ho}\mathbf{Bif}(\mathcal{M}) \simeq \operatorname{Ho}\mathcal{M}$.

Proof. This follows from proposition (5.19), using criterion (a) for $\mathrm{inc}\colon \mathbf{Cof}(\mathcal{M}) \to \mathcal{M}$ and $\mathrm{inc}\colon \mathbf{Bif}(\mathcal{M}) \to \mathbf{Fib}(\mathcal{M})$, and using criterion (b) for $\mathrm{inc}\colon \mathbf{Fib}(\mathcal{M}) \to \mathcal{M}$ and $\mathrm{inc}\colon \mathbf{Bif}(\mathcal{M}) \to \mathbf{Cof}(\mathcal{M})$. □

The proof of Quillen's homotopy category theorem, which states in particular that the homotopy category $\operatorname{Ho}\mathcal{M}$ is equivalent to the quotient category $\mathbf{Bif}(\mathcal{M})/\sim$, where $\sim$ denotes the homotopy congruence, can now be completed as in [20, cor. 1.2.9] by showing that $\mathbf{Bif}(\mathcal{M})/\sim$ fulfills the universal property of a localisation, which is essentially a corollary of Whitehead's theorem [20, prop. 1.2.8].

Derivable categories

Recall that a *derivable category* in the sense of CISINSKI [7, sec. 2.25] consists of the same data as a Quillen model category, that is, a category $\mathcal{C}$ together with three distinguished subsets of morphisms, called *cofibrations*, *fibrations* and *weak equivalences*, subject to the following axioms, where (co)fibrant objects and acyclic (co)fibrations are defined as in the Quillen model category case: The set of weak equivalences is supposed to be semi-saturated. The set of cofibrations is supposed to be closed under (binary) composition. There exists an initial object in $\mathcal{C}$, which is supposed to be cofibrant. The set of cofibrant objects is supposed to be closed under isomorphisms. The set of cofibrations between cofibrant objects and the subset of acyclic cofibrations therein are supposed to be stable under pushouts along morphisms between cofibrant objects. Every morphism with cofibrant source object factors into a cofibration followed by a weak equivalence. And dually for the fibrations and fibrant objects.
For homotopical algebra in derivable categories, cf. also the manuscript of RĂDULESCU-BANU [31], who uses the terminology *Anderson-Brown-Cisinski premodel category*.
Derivable categories are a natural generalisation of *categories of fibrant objects* in the sense of K. BROWN [4, sec. 1]. More precisely: Given a derivable category, then its full subcategory of fibrant objects is a category of fibrant objects in this sense, and its full subcategory of cofibrant objects fulfills the dual properties. Conversely, given a category $\mathcal{C}$ together with distinguished subsets of cofibrations, fibrations and weak equivalences such that there exists an terminal object in $\mathcal{C}$ and such that the full subcategory of fibrant objects is a category of fibrant objects in the sense of K. BROWN, and dually, then $\mathcal{C}$ fulfills all axioms of a derivable category except possibly for the stronger factorisation axioms of CISINSKI. These stronger factorisation axioms are sufficient to obtain the desirable equivalences between the homotopy categories of the full subcategories of (co)fibrant objects in $\mathcal{C}$ and the homotopy category of $\mathcal{C}$, see [7, prop. 1.8].
In the proof of example (7.1), we have not used the existence of general finite limits and colimits [29, ch. I, §1, def. 1, ax. M0]. Moreover, to show that a Quillen model category carries the structure of a uni-fractionable category, we also did not use the lifting axioms [29, ch. I, §1, def. 1, ax. M1]. Thus we obtain the following more general example.

(7.3) Example. We let $\mathcal{C}$ be a derivable category such that the following properties hold.

- Every identity in $\mathcal{C}$ is a cofibration and a fibration. ([9])
- Given an acyclic cofibration $i\colon X \to X'$ and a morphism $f\colon X \to Y$ in $\mathcal{C}$, there exists a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

  in $\mathcal{C}$ such that i' is an acyclic cofibration. Dually, given an acyclic fibration $p\colon Y' \to Y$ and a morphism $f\colon X \to Y$ in $\mathcal{C}$, there exists a pullback rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

  in $\mathcal{C}$ such that p' is an acyclic fibration.
- For every weak equivalence $w\colon X \to Y$ in $\mathcal{C}$ there exists an acyclic cofibration $i\colon X \to Z$ and an acyclic fibration $p\colon Z \to Y$ with $w = ip$.

$$\begin{tikzcd} & Z \arrow[dr, dashed, "p"] & \\ X \arrow[ur, dashed, "i"] \arrow[rr, "w"] & & Y \end{tikzcd}$$

[9]This is no restriction: Given a derivable category $\mathcal{C}$ with set of cofibrations C, set of fibrations F and set of weak equivalences W, the underlying category of $\mathcal{C}$ also becomes a derivable category with set of cofibrations $C \cup \{1_X \mid X \in \operatorname{Ob}\mathcal{C}\}$, set of fibrations $F \cup \{1_X \mid X \in \operatorname{Ob}\mathcal{C}\}$ and set of weak equivalences W.

Then $\mathcal{C}$ carries the structure of a uni-fractionable category, where

$$\begin{aligned}
\operatorname{Den}\mathcal{C} &= \{w \in \operatorname{Mor}\mathcal{C} \mid w \text{ is a weak equivalence}\},\\
\operatorname{SDen}\mathcal{C} &= \{i \in \operatorname{Mor}\mathcal{C} \mid i \text{ is an acyclic cofibration}\},\\
\operatorname{TDen}\mathcal{C} &= \{p \in \operatorname{Mor}\mathcal{C} \mid p \text{ is an acyclic fibration}\}.
\end{aligned}$$

Proof. This is the same proof as for a Quillen model category, see part (a) of the proof of example (7.1). □

Complexes and exact categories

Our next example yields a construction for the derived category of an arbitrary abelian category $\mathcal{A}$. We denote by $\mathrm{H}\colon \mathrm{C}(\mathcal{A}) \to \mathcal{A}^{\operatorname{Disc}\mathbb{Z}}$ the cohomology functor, where $\operatorname{Disc}\mathbb{Z}$ denotes the discrete category associated to the set $\mathbb{Z}$ of integers.

(7.4) Example. The category $\mathrm{C}(\mathcal{A})$ of complexes in an abelian category $\mathcal{A}$ carries the structure of a saturated additive uni-fractionable category, where

$$\begin{aligned}
\operatorname{Den}\mathrm{C}(\mathcal{A}) &= \{f \in \operatorname{Mor}\mathrm{C}(\mathcal{A}) \mid \mathrm{H}(f) \text{ is an isomorphism}\},\\
\operatorname{SDen}\mathrm{C}(\mathcal{A}) &= \{i \in \operatorname{Den}\mathrm{C}(\mathcal{A}) \mid i \text{ is a monomorphism}\},\\
\operatorname{TDen}\mathrm{C}(\mathcal{A}) &= \{p \in \operatorname{Den}\mathrm{C}(\mathcal{A}) \mid p \text{ is an epimorphism}\}.
\end{aligned}$$

In particular, the derived category $\mathrm{D}(\mathcal{A})$ is isomorphic to $\operatorname{Frac}\mathrm{C}(\mathcal{A})$.

Proof. First of all, the set $\{f \in \operatorname{Mor}\mathrm{C}(\mathcal{A}) \mid \mathrm{H}(f) \text{ is an isomorphism}\}$ is closed under finite sums since the cohomology functor $\mathrm{H}\colon \mathrm{C}(\mathcal{A}) \to \mathcal{A}^{\operatorname{Disc}\mathbb{Z}}$ is additive. We verify the axioms of a uni-fractionable category.

(2 of 3) Given morphisms $f, g \in \operatorname{Mor}\mathrm{C}(\mathcal{A})$ with $\operatorname{Target} f = \operatorname{Source} g$, we have $\mathrm{H}(f)\mathrm{H}(g) = \mathrm{H}(fg)$. Hence if two out of the morphisms $\mathrm{H}(f)$, $\mathrm{H}(g)$, $\mathrm{H}(fg)$ are isomorphisms, then so is the third.

(Cat) For every $X \in \operatorname{Ob}\mathrm{C}(\mathcal{A})$, we have $\mathrm{H}(1_X) = 1_{\mathrm{H}(X)}$, so in particular $\mathrm{H}(1_X)$ is an isomorphism. Thus the set $\{f \in \operatorname{Mor}\mathrm{C}(\mathcal{A}) \mid \mathrm{H}(f) \text{ is an isomorphism}\}$ is multiplicative. But then also its subsets of monomorphisms resp. epimorphisms are multiplicative since monomorphisms compose to monomorphisms resp. epimorphisms compose to epimorphisms.

(WU) We suppose given a monomorphism $i\colon X \to X'$ and a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{A})$, and we let

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \\
X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"']
\end{tikzcd}$$

be a pushout rectangle in $\mathrm{C}(\mathcal{A})$. Then we obtain an induced isomorphism $\operatorname{Coker} i \to \operatorname{Coker} i'$. A consideration of the long exact cohomology sequence induced by

$$X \xrightarrow{i} X' \xrightarrow{\text{quo}} \operatorname{Coker} i$$

shows that $\mathrm{H}(i)$ is an isomorphism if and only if $\mathrm{H}(\operatorname{Coker} i) \cong 0$, and analogously for i'. So if $\mathrm{H}(i)$ is an isomorphism, then also $\mathrm{H}(i')$ is an isomorphism.

The other assertion follows by duality.

(Fac) This follows from the fact that every morphism $f\colon X \to Y$ in $\mathrm{C}(\mathcal{A})$ can be factorised into the morphism $\begin{pmatrix} f & \text{ins} \end{pmatrix} : X \to Y \oplus \operatorname{Cone} X$, where $\text{ins}\colon X \to \operatorname{Cone} X$ is the insertion of X into the cone of X, followed by the (split) epimorphism $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : Y \oplus \operatorname{Cone} X \to Y$, cf. [13, III.3.2–3].

$$\begin{tikzcd}
 & Y \oplus \operatorname{Cone} X \arrow[dr, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \\
X \arrow[ur, "\begin{pmatrix} f & \text{ins} \end{pmatrix}"] \arrow[rr, "f"] & & Y
\end{tikzcd}$$

The morphism $\begin{pmatrix} f & \mathrm{ins} \end{pmatrix}$ is a monomorphism as ins is a monomorphism. Moreover, $\mathrm{H}(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))\colon \mathrm{H}(Y \oplus \mathrm{Cone}\, X) \to \mathrm{H}(Y)$ is an isomorphism since H is an additive functor and $\mathrm{H}(\mathrm{Cone}\, X) \cong 0$. Hence if $\mathrm{H}(f)$ is an isomorphism, then also $\mathrm{H}(\begin{pmatrix} f & \mathrm{ins} \end{pmatrix})$ is an isomorphism. ([10]) □

Altogether, $\mathrm{C}(\mathcal{A})$ becomes an additive uni-fractionable category with

$$\begin{aligned} \mathrm{Den}\,\mathrm{C}(\mathcal{A}) &= \{f \in \mathrm{Mor}\,\mathrm{C}(\mathcal{A}) \mid \mathrm{H}(f) \text{ is an isomorphism}\}, \\ \mathrm{SDen}\,\mathrm{C}(\mathcal{A}) &= \{i \in \mathrm{Den}\,\mathrm{C}(\mathcal{A}) \mid i \text{ is a monomorphism}\}, \\ \mathrm{TDen}\,\mathrm{C}(\mathcal{A}) &= \{p \in \mathrm{Den}\,\mathrm{C}(\mathcal{A}) \mid p \text{ is an epimorphism}\}. \end{aligned}$$

To show that $\mathrm{C}(\mathcal{A})$ is saturated, we suppose given morphisms $f, g, h \in \mathrm{Mor}\,\mathrm{C}(\mathcal{A})$ with $\mathrm{Target}\, f = \mathrm{Source}\, g$ and $\mathrm{Target}\, g = \mathrm{Source}\, h$ and such that fg and gh are denominators in $\mathrm{C}(\mathcal{A})$, that is, $\mathrm{H}(fg)$ and $\mathrm{H}(gh)$ are isomorphisms in $\mathcal{A}^{\mathrm{Disc}\,\mathbb{Z}}$. Then also $\mathrm{H}(g)$ is an isomorphism with $\mathrm{H}(g)^{-1} = \mathrm{H}(fg)^{-1}\mathrm{H}(f) = \mathrm{H}(h)\mathrm{H}(gh)^{-1}$ and hence g is a denominator in $\mathrm{C}(\mathcal{A})$. But since $\mathrm{C}(\mathcal{A})$ is semi-saturated, this already implies that $\mathrm{C}(\mathcal{A})$ is weakly saturated and therefore saturated by proposition (5.14).

The preceding example of complexes can be generalised to exact categories in the sense of Quillen [30, §2, pp. 99–100] as follows in example (7.5). A denominator in example (7.4) is a morphism of complexes such that the induced morphisms on the cohomology level are isomorphisms. This is what one usually calls a *quasi-isomorphism*, and can be characterised as follows: A morphism of complexes with entries in an abelian category is a quasi-isomorphism if and only if its cone is acyclic [35, cor. 1.5.4]. Since we have no cohomology functor for exact categories, we have to clarify first what we want to understand by a (formal) cone and a quasi-isomorphism in an exact category. We will develop these notions and some facts in appendix A, cf. in particular definitions (A.1) resp. (A.3) resp. (A.4) resp. (A.5) for the definitions of formal cones resp. having enough formal cones resp. quasi-isomorphisms resp. closedness under pure short sequences.

(7.5) Example. We suppose given an exact category $\mathcal{E}$ and a non-empty full subcategory $\mathcal{U}$ of $\mathcal{E}$ that is closed under pure short exact sequences and such that $\mathcal{E}$ has enough formal $\mathcal{U}$-cones. Then $\mathcal{E}$ carries the structure of a uni-fractionable category with

$$\begin{aligned} \mathrm{Den}\,\mathcal{E} &= \{f \in \mathrm{Mor}\,\mathcal{E} \mid f \text{ is a } \mathcal{U}\text{-quasi-isomorphism}\}, \\ \mathrm{SDen}\,\mathcal{E} &= \{i \in \mathrm{Den}\,\mathcal{E} \mid i \text{ is a pure monomorphism}\} \\ &= \{i \in \mathrm{Mor}\,\mathcal{E} \mid i \text{ is a pure monomorphism with } \mathrm{Coker}\, i \in \mathrm{Ob}\,\mathcal{U}\}, \\ \mathrm{TDen}\,\mathcal{E} &= \{p \in \mathrm{Den}\,\mathcal{E} \mid p \text{ is a pure epimorphism}\} \\ &= \{p \in \mathrm{Mor}\,\mathcal{E} \mid p \text{ is a pure epimorphism with } \mathrm{Ker}\, p \in \mathrm{Ob}\,\mathcal{U}\}. \end{aligned}$$

If moreover $\mathcal{U}$ is closed under summands, then $\mathcal{E}$ is saturated.

Proof. The set of quasi-isomorphisms in $\mathcal{E}$ is closed under finite sums by remark (A.7) and semi-saturated by proposition (A.11), hence in particular multiplicative. The set of pure monomorphisms that are quasi-isomorphisms is also multiplicative since the set of pure monomorphisms in an exact category is multiplicative [5, def. 2.1]. Dually, the set of pure epimorphisms that are quasi-isomorphisms is multiplicative. Axiom (WU) is fulfilled by corollary (A.9) since pure monomorphisms in an exact category are stable under pushouts and pure epimorphisms are stable under pullbacks [5, def. 2.1]. Finally, every quasi-isomorphism factors into a pure monomorphism that is a quasi-isomorphism and a pure epimorphism that is a quasi-isomorphism by corollary (A.12). Altogether, $\mathcal{E}$ becomes an additive uni-fractionable category with

$$\begin{aligned} \mathrm{Den}\,\mathcal{E} &= \{f \in \mathrm{Mor}\,\mathcal{E} \mid f \text{ is a } \mathcal{U}\text{-quasi-isomorphism}\}, \\ \mathrm{SDen}\,\mathcal{E} &= \{i \in \mathrm{Den}\,\mathcal{E} \mid i \text{ is a pure monomorphism}\}, \\ \mathrm{TDen}\,\mathcal{E} &= \{p \in \mathrm{Den}\,\mathcal{E} \mid p \text{ is a pure epimorphism}\}. \end{aligned}$$

Moreover, $\mathrm{SDen}\,\mathcal{E} = \{i \in \mathrm{Mor}\,\mathcal{E} \mid i \text{ is a pure monomorphism with } \mathrm{Coker}\, i \in \mathrm{Ob}\,\mathcal{U}\}$ and $\mathrm{TDen}\,\mathcal{E} = \{p \in \mathrm{Mor}\,\mathcal{E} \mid p \text{ is a pure epimorphism with } \mathrm{Ker}\, p \in \mathrm{Ob}\,\mathcal{U}\}$ by proposition (A.8). The assertion on the saturatedness of $\mathcal{E}$ follows from proposition (A.13) and proposition (5.14). □

[10] Alternatively, one can give a factorisation using the cylinder of f instead of $Y \oplus \mathrm{Cone}\, X$, cf. for example [13, III.3.2–3].

Recall that an additive category is said to be *idempotent splitting* if every morphism e with $e^2 = e$ is split. (BÜHLER uses the notion "idempotent complete", see [5, def. 6.1].) Moreover, recall that the category of complexes with entries in an exact category becomes an exact category with degreewise pure short exact sequences, see [5, lem. 9.1].
As an application of example (7.5), we obtain in particular a 3-arrow calculus for the derived category of an idempotent splitting exact category without passing to the homotopy category in advance, cf. [26], [5, sec. 10].

(7.6) Example. The category $\mathrm{C}(\mathcal{E})$ of complexes in an idempotent splitting exact category $\mathcal{E}$ carries the structure of a saturated additive uni-fractionable category, where

$$\begin{aligned}
\operatorname{Den}\mathrm{C}(\mathcal{E}) &= \{f \in \operatorname{Mor}\mathrm{C}(\mathcal{E}) \mid f \text{ is a quasi-isomorphism}\},\\
\operatorname{SDen}\mathrm{C}(\mathcal{E}) &= \{i \in \operatorname{Den}\mathrm{C}(\mathcal{E}) \mid i \text{ is a pure monomorphism}\}\\
&= \{i \in \operatorname{Mor}\mathrm{C}(\mathcal{E}) \mid i \text{ is a pure monomorphism with } \operatorname{Coker} i \text{ purely acyclic}\},\\
\operatorname{TDen}\mathrm{C}(\mathcal{E}) &= \{p \in \operatorname{Den}\mathrm{C}(\mathcal{E}) \mid p \text{ is a pure epimorphism}\}\\
&= \{p \in \operatorname{Mor}\mathrm{C}(\mathcal{E}) \mid p \text{ is a pure epimorphism with } \operatorname{Ker} p \text{ purely acyclic}\}.
\end{aligned}$$

In particular, the derived category $\mathrm{D}(\mathcal{E})$ is isomorphic to $\operatorname{Frac}\mathrm{C}(\mathcal{E})$.

Proof. By [5, rem. 10.17] and corollary (A.12), a quasi-isomorphism, in the sense of [5, def. 10.16], is precisely a quasi-isomorphism with respect to the full subcategory of pure acyclic complexes [6, def. 4(2)] in the sense of definition (A.4). Moreover, the full subcategory of pure acyclic complexes is closed under pure short exact sequences [6, cor. 29] and under summands [5, lem. 10.7], and the exact category $\mathrm{C}(\mathcal{E})$ has enough formal cones with respect to this full subcategory [5, def. 9.2, rem. 9.9, prop. 10.9]. Thus the assertion follows from example (7.5). □

Classical examples

We finish this article by considering some classical examples, which yield in fact a 2-arrow calculus, but nonetheless fit in our framework. Example (7.7)(b) implicitly occurs in GROTHENDIECK's Tôhoku article [15, sec. 1.11]. This example differs from the others since here the S-denominators are epimorphisms and the T-denominators are monomorphisms, while in all our other examples the S-denominators are "mono-like" (certain cofibrations resp. monomorphisms resp. pure monomorphisms) and the T-denominators are "epi-like" (certain fibrations resp. epimorphisms resp. pure epimorphisms).
Recall that a *thick subcategory* of an abelian category $\mathcal{A}$ is a non-empty full (abelian) subcategory $\mathcal{U}$ that is closed under extensions, subobjects and quotient objects. Recall that a *thick subcategory* of a Verdier triangulated category $\mathcal{V}$ is a (non-empty) full triangulated subcategory $\mathcal{U}$ that is closed under taking summands.

(7.7) Example.

(a) We suppose given an abelian category $\mathcal{A}$ and a thick subcategory $\mathcal{U}$ in $\mathcal{A}$. Then $\mathcal{A}$ carries the structure of an additive uni-fractionable category, where

$$\operatorname{Den}\mathcal{A} = \operatorname{SDen}\mathcal{A} = \operatorname{TDen}\mathcal{A} = \{f \in \operatorname{Mor}\mathcal{A} \mid \operatorname{Ker} f \text{ and } \operatorname{Coker} f \text{ are in } \mathcal{U}\}.$$

In particular, the Serre quotient of $\mathcal{A}$ by $\mathcal{U}$ is isomorphic to $\operatorname{Frac}\mathcal{A}$.

(b) We suppose given an abelian category $\mathcal{A}$ and a thick subcategory $\mathcal{U}$ in $\mathcal{A}$. Then $\mathcal{A}$ carries the structure of an additive uni-fractionable category, where

$$\begin{aligned}
\operatorname{Den}\mathcal{A} &= \{f \in \operatorname{Mor}\mathcal{A} \mid \operatorname{Ker} f \text{ and } \operatorname{Coker} f \text{ are in } \mathcal{U}\},\\
\operatorname{SDen}\mathcal{A} &= \{s \in \operatorname{Den}\mathcal{A} \mid s \text{ is an epimorphism}\},\\
\operatorname{TDen}\mathcal{A} &= \{t \in \operatorname{Den}\mathcal{A} \mid t \text{ is a monomorphism}\}.
\end{aligned}$$

In particular, the Serre quotient of $\mathcal{A}$ by $\mathcal{U}$ is isomorphic to $\operatorname{Frac}\mathcal{A}$.

(c) We suppose given a Verdier triangulated category $\mathcal{V}$ and a thick subcategory $\mathcal{U}$ in $\mathcal{V}$. Then $\mathcal{V}$ carries the structure of an additive uni-fractionable category, where

$$\operatorname{Den}\mathcal{V} = \operatorname{SDen}\mathcal{V} = \operatorname{TDen}\mathcal{V} = \{f \in \operatorname{Mor}\mathcal{V} \mid \text{a cone of } f \text{ is in } \mathcal{U}\}.$$

In particular, the Verdier quotient of $\mathcal{V}$ by $\mathcal{U}$ is isomorphic to $\operatorname{Frac}\mathcal{V}$.

Proof.

(a) We set $D := \{f \in \operatorname{Mor}\mathcal{A} \mid \operatorname{Ker} f \text{ and } \operatorname{Coker} f \text{ are in } \mathcal{U}\}$ and have to verify the axioms (Cat), (2 of 3) and (WU).

(Cat) We suppose given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{A}$ with $f, g \in D$, so that $\operatorname{Ker} f$, $\operatorname{Coker} f$, $\operatorname{Ker} g$, $\operatorname{Coker} g$ are in $\mathcal{U}$. By the circumference lemma [23, Lem. 132], we have an exact sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker}(fg) \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Coker} f \longrightarrow \operatorname{Coker}(fg) \longrightarrow \operatorname{Coker} g \longrightarrow 0.$$

Since $\operatorname{Ker} f$ and $\operatorname{Ker} g$ are in $\mathcal{U}$, it follows that $\operatorname{Ker}(fg)$ is in $\mathcal{U}$, and since $\operatorname{Coker} f$ and $\operatorname{Coker} g$ are in $\mathcal{U}$, it follows that $\operatorname{Coker}(fg)$ is in $\mathcal{U}$. Thus we have $fg \in D$.

Moreover, $0 \in \operatorname{Ob}\mathcal{U}$ and therefore $1_X \in D$ for all $X \in \operatorname{Ob}\mathcal{A}$.

(2 of 3) We suppose given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{A}$ with $f, fg \in D$, so that the objects $\operatorname{Ker} f$, $\operatorname{Coker} f$, $\operatorname{Ker}(fg)$, $\operatorname{Coker}(fg)$ are in $\mathcal{U}$. The circumference lemma [23, Lem. 132] implies that $\operatorname{Ker} g$ is in $\mathcal{U}$ since $\operatorname{Ker}(fg)$ and $\operatorname{Coker} f$ are in $\mathcal{U}$, and that $\operatorname{Coker} g$ is in $\mathcal{U}$ since $\operatorname{Coker}(fg)$ is in $\mathcal{U}$. Thus we have $g \in D$.

The other case follows by duality.

(WU) We suppose given morphisms $d\colon X \to X'$ and $f\colon X \to Y$ in $\mathcal{A}$ with $d \in D$, and we let

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "d"] \arrow[r, "f"] & X' \arrow[u, "d'"'] \end{tikzcd}$$

be a pushout rectangle in $\mathcal{A}$. Then the induced morphism $\operatorname{Coker} d \to \operatorname{Coker} d'$ is an isomorphism and the induced morphism $\operatorname{Ker} d \to \operatorname{Ker} d'$ is an epimorphism. Thus since $\operatorname{Ker} d$ and $\operatorname{Coker} d$ are in $\mathcal{U}$, it follows that $\operatorname{Ker} d'$ and $\operatorname{Coker} d'$ are in $\mathcal{U}$, that is, $d' \in D$.

The other property follows by duality.

Altogether, there is a structure of a uni-fractionable category on $\mathcal{A}$ with $\operatorname{Den}\mathcal{A} = \operatorname{SDen}\mathcal{A} = \operatorname{TDen}\mathcal{A} = D$. Moreover, $\operatorname{Den}\mathcal{A}$ is closed under finite sums since $\mathcal{U}$ is an additive subcategory, and hence $\mathcal{A}$ is an additive uni-fractionable category.

(b) The axioms (Cat), (2 of 3) and (WU) as well as additivity follow from (a), taking into account that epimorphisms are stable under composition and pushouts, and dually. Moreover, (Fac) holds since every morphism in an abelian category factorises into an epimorphism with the same kernel followed by a monomorphism with the same cokernel.

(c) We set $D := \{f \in \operatorname{Mor}\mathcal{V} \mid \text{a cone of } f \text{ is in } \mathcal{U}\}$ and have to verify the axioms of a uni-fractionable category. In the following, given a morphism $f \in \operatorname{Mor}\mathcal{V}$, we denote by C_f a chosen cone of f. Since every cone of f is isomorphic to C_f [13, IV.1.4 b)], we have $f \in D$ if and only if C_f is in $\mathcal{U}$. The shift functor in $\mathcal{V}$ is denoted by T.

(Cat) We suppose given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ with $f, g \in D$, so that C_f and C_g are in $\mathcal{U}$. By the octahedral axiom, we get a distinguished triangle

$$\dots \xrightarrow{\mathrm{T}^{-1}v} \mathrm{T}^{-1}C_g \xrightarrow{\mathrm{T}^{-1}w} C_f \xrightarrow{u} C_{fg} \xrightarrow{v} C_g \xrightarrow{w} \mathrm{T}C_f \xrightarrow{\mathrm{T}u} \dots,$$

and in particular, C_{fg} is a cone of $\mathrm{T}^{-1}w$. Since C_g is in $\mathcal{U}$, it follows that $\mathrm{T}^{-1}C_g$ is in $\mathcal{U}$. But then $\mathrm{T}^{-1}w$ is a morphism in $\mathcal{U}$ and thus C_{fg} is an object in $\mathcal{U}$.

Moreover, $0 \in \operatorname{Ob}\mathcal{U}$ implies $1_X \in D$ for all $X \in \operatorname{Ob}\mathcal{V}$.

(2 of 3) We suppose given morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{V}$, and we use the distinguished triangle obtained by the octahedral axiom from above. If $f, fg \in D$, then C_f and C_{fg} are in $\mathcal{U}$, hence u is a morphism in $\mathcal{U}$ and therefore C_g is in $\mathcal{U}$, that is, $g \in D$. Analogously, $fg, g \in D$ implies that C_{fg} and C_g are in $\mathcal{U}$, hence v is a morphism in $\mathcal{U}$ and $\mathrm{T}C_f$ is an object in $\mathcal{U}$, and therefore C_f is in $\mathcal{U}$, that is, $f \in D$.

(WU) We suppose given morphisms $d\colon X \to X'$ and $f\colon X \to Y$ in $\mathcal{V}$ with $d \in D$, and we let

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "d"] \arrow[r, "f"] & Y \arrow[u, "d'"'] \end{tikzcd}$$

be a dweak square in $\mathcal{V}$, that is, a quadrangle whose diagonal sequence fits in a distinguished triangle, and hence in particular a weak pushout. Then C_d is a cone of d', cf. [27, lem. 1.4.4]. Hence $d' \in D$ as C_d is in $\mathcal{U}$.

The other property follows by duality.

Altogether, there is a structure of a uni-fractionable category on $\mathcal{V}$ with $\operatorname{Den}\mathcal{V} = \operatorname{SDen}\mathcal{V} = \operatorname{TDen}\mathcal{V} = D$. Moreover, additivity of $\mathcal{V}$ follows from the additivity of $\mathcal{U}$. □

A Formal cones in exact categories

In this appendix, we develop a theory about "formal cones" and "quasi-isomorphisms" in an exact category relative to a suitable subcategory. This will be used to generalise example (7.4), where we have shown that the category of complexes in an abelian category carries a uni-fractionable category structure in such a way that the fraction category becomes the derived category, to the case of idempotent splitting exact categories, see example (7.6).

We consider an exact category $\mathcal{E}$ in the sense of Quillen [30, §2, pp. 99–100], cf. also [22, app. A], [5, def. 2.1]. The distinguished short exact sequences in $\mathcal{E}$ will be called *pure short exact sequences*. Likewise, the monomorphisms occurring in a pure short exact sequence are called *pure monomorphisms*, and the epimorphisms occurring in a pure short exact sequence are called *pure epimorphisms*.

During this appendix, we suppose given an exact category $\mathcal{E}$ and a non-empty full subcategory $\mathcal{U}$ of $\mathcal{E}$. From remark (A.7) on, we suppose that $\mathcal{U}$ is closed under pure short exact sequences, see definition (A.5), and that $\mathcal{E}$ has enough formal $\mathcal{U}$-cones, see definition (A.3).

(A.1) Definition (formal cone)**.**

(a) We suppose given an object X in $\mathcal{E}$. A *formal cone* with respect to $\mathcal{U}$ (or *formal $\mathcal{U}$-cone* or just *formal cone*) of X consists of an object C in $\mathcal{U}$ together with a pure monomorphism $i\colon X \to C$. By abuse of notation, we denote the formal cone as well as its underlying object by C. The pure monomorphism i is called the *insertion* in C. Given a formal cone C of X with insertion i, we write $\operatorname{ins} = \operatorname{ins}^C := i$.

(b) We suppose given a morphism $f\colon X \to Y$ in $\mathcal{E}$. Given a formal $\mathcal{U}$-cone C_X of X, a *formal cone* with respect to $\mathcal{U}$ (or *formal $\mathcal{U}$-cone* or just *formal cone*) of f *corresponding to C_X* consists of an object C in $\mathcal{U}$ together with a pure monomorphism $i\colon Y \to C$ such that there exists a pushout rectangle

$$\begin{tikzcd} C_X \arrow[r, "f'"] & C \\ X \arrow[u, "\operatorname{ins}^{C_X}"] \arrow[r, "f"] & Y \arrow[u, "i"'] \end{tikzcd}$$

in $\mathcal{E}$. By abuse of notation, we denote the formal cone as well as its underlying object by C. The pure monomorphism $i\colon Y \to C$ is called the *insertion* in C. Given a formal cone C of f corresponding to C_X with insertion i, we write $\operatorname{ins} = \operatorname{ins}^C := i$.

A *formal cone* with respect to $\mathcal{U}$ (or *formal $\mathcal{U}$-cone* or just *formal cone*) of f is a formal $\mathcal{U}$-cone of f corresponding to some formal $\mathcal{U}$-cone of X.

The cone of a complex resp. of a morphism of complexes with entries in an additive category as defined for example in [5, def. 9.2] is a formal cone with respect to the full subcategory of acyclic complexes (or even of split acyclic complexes).

(A.2) Remark. We suppose that $\mathcal{U}$ is an additive subcategory of $\mathcal{E}$.

(a) We suppose given objects X_1 and X_2 in $\mathcal{E}$, a formal $\mathcal{U}$-cone C_1 of X_1 and a formal $\mathcal{U}$-cone C_2 of X_2. Then $C_1 \oplus C_2$ is a formal $\mathcal{U}$-cone of $X_1 \oplus X_2$ with $\mathrm{ins}^{C_1 \oplus C_2} = \mathrm{ins}^{C_1} \oplus \mathrm{ins}^{C_2}$.

(b) We suppose given morphisms $f_1 \colon X_1 \to Y_1$ and $f_2 \colon X_2 \to Y_2$ in $\mathcal{E}$, a formal $\mathcal{U}$-cone C_1 of f_1 and a formal $\mathcal{U}$-cone C_2 of f_2. Then $C_1 \oplus C_2$ is a formal $\mathcal{U}$-cone of $f_1 \oplus f_2$ with $\mathrm{ins}^{C_1 \oplus C_2} = \mathrm{ins}^{C_1} \oplus \mathrm{ins}^{C_2}$.

(A.3) Definition (having enough formal cones)**.** The exact category $\mathcal{E}$ is said to *have enough formal cones* with respect to $\mathcal{U}$ (or to *have enough formal $\mathcal{U}$-cones* or just to *have enough formal cones*) if there exists a formal $\mathcal{U}$-cone of every object in $\mathcal{E}$.

(A.4) Definition (quasi-isomorphism)**.** We suppose that $\mathcal{E}$ has enough formal $\mathcal{U}$-cones. A *quasi-isomorphism* with respect to $\mathcal{U}$ (or *$\mathcal{U}$-quasi-isomorphism* or just *quasi-isomorphism*) is a morphism f in $\mathcal{E}$ such that there exists a formal cone of f that is in $\mathcal{U}$.

(A.5) Definition (closed under pure short exact sequences)**.** The full subcategory $\mathcal{U}$ of $\mathcal{E}$ is said to be *closed under pure short exact sequences* in $\mathcal{E}$ if it fulfills the following axiom.

(Seq) Given a pure short exact sequence

$$X' \longrightarrow X \longrightarrow X''$$

in $\mathcal{E}$ such that two out of the objects X', X, X'' are in $\mathcal{U}$, then so is the third.

(A.6) Remark. If $\mathcal{U}$ is closed under pure short exact sequences, then $\mathcal{U}$ is closed under isomorphisms and an additive subcategory of $\mathcal{E}$.

Proof. Since $\mathcal{U}$ is supposed to be non-empty, there exists an object X in $\mathcal{U}$. But then

$$X \xrightarrow{1_X} X \xrightarrow{0} 0$$

is a pure short exact sequence [5, def. 2.1] and hence 0 is in $\mathcal{U}$. Given objects X_1 and X_2 in $\mathcal{U}$, we have the pure short exact sequence

$$X_1 \xrightarrow{(1\ 0)} X_1 \oplus X_2 \xrightarrow{\binom{0}{1}} X_2$$

by [5, lem. 2.7] and therefore $X_1 \oplus X_2$ is in $\mathcal{U}$. Thus $\mathcal{U}$ is an additive subcategory of $\mathcal{E}$.
Finally, given an object X in $\mathcal{U}$ and an isomorphism $f \colon X \to Y$ in $\mathcal{E}$, we have the pure short exact sequence

$$X \xrightarrow{f} Y \xrightarrow{0} 0$$

and hence Y is in $\mathcal{U}$. □

From now on, we suppose that $\mathcal{U}$ is closed under pure short exact sequences and that $\mathcal{E}$ has enough formal $\mathcal{U}$-cones.

(A.7) Remark. The set of $\mathcal{U}$-quasi-isomorphisms in $\mathcal{E}$ is closed under finite sums.

Proof. This follows from remark (A.6) and remark (A.2)(b). □

(A.8) Proposition.

(a) A pure monomorphism i in $\mathcal{E}$ is a $\mathcal{U}$-quasi-isomorphism if and only if $\operatorname{Coker} i$ is in $\mathcal{U}$.

(b) A pure epimorphism p in $\mathcal{E}$ is a $\mathcal{U}$-quasi-isomorphism if and only if $\operatorname{Ker} p$ is in $\mathcal{U}$.

Proof. We let $f \colon X \to Y$ be a morphism in $\mathcal{E}$ and we let C_X be a formal cone of X. Moreover, we let C_f be a formal cone of f corresponding to C_X, that is, we suppose that there exists a pushout rectangle

$$\begin{tikzcd} C_X \arrow[r, "f'"] & C_f \\ X \arrow[u, "\mathrm{ins}^{C_X}"] \arrow[r, "f"] & Y \arrow[u, "\mathrm{ins}^{C_f}"'] \end{tikzcd}$$

in $\mathcal{E}$. By [5, prop. 2.12], this rectangle is also a pullback.
Let us first suppose that f is a pure monomorphism. Then f' is also a pure monomorphism by [5, def. 2.1], and since $\mathcal{U}$ is closed under pure short exact sequences, the formal cone C_f is in $\mathcal{U}$ if and only if $\operatorname{Coker} f'$ is in $\mathcal{U}$. But by [5, prop. 2.12], we have $\operatorname{Coker} f \cong \operatorname{Coker} f'$, so C_f is in $\mathcal{U}$ if and only if $\operatorname{Coker} f$ is in $\mathcal{U}$. Since C_X and C_f were chosen arbitrarily, this means that f is a quasi-isomorphism if and only if $\operatorname{Coker} f$ is in $\mathcal{U}$.
Next, let us suppose that f is a pure epimorphism. Then f' is also a pure epimorphism by [5, dual of prop. 2.15], and since $\mathcal{U}$ is closed under pure short exact sequences, the formal cone C_f is in $\mathcal{U}$ if and only if $\operatorname{Ker} f'$ is in $\mathcal{U}$. By [5, dual of prop. 2.12], we have $\operatorname{Ker} f \cong \operatorname{Ker} f'$, so C_f is in $\mathcal{U}$ if and only if $\operatorname{Ker} f$ is in $\mathcal{U}$. Since C_X and C_f were chosen arbitrarily, this means that f is a quasi-isomorphism if and only if $\operatorname{Ker} f$ is in $\mathcal{U}$. □

(A.9) Corollary.

(a) Given a pushout rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] & Y' \\ X \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "i'"'] \end{tikzcd}$$

in $\mathcal{E}$ with pure monomorphism i, then i is a $\mathcal{U}$-quasi-isomorphism if and only if i' is a $\mathcal{U}$-quasi-isomorphism.

(b) Given a pullback rectangle

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "p'"'] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{E}$ with pure epimorphism p, then p is a $\mathcal{U}$-quasi-isomorphism if and only if p' is a $\mathcal{U}$-quasi-isomorphism.

Proof. This follows by proposition (A.8) since these pushouts induce isomorphisms on the cokernels of the pure monomorphisms and these pullbacks induce isomorphisms on the kernels of the pure epimorphisms [5, prop. 2.12 and its dual]. □

(A.10) Proposition. We let $f\colon X \to Y$ be a morphism in $\mathcal{E}$. For every formal $\mathcal{U}$-cone C_X of X, we have the factorisation $f = \begin{pmatrix} f & \mathrm{ins} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $\begin{pmatrix} f & \mathrm{ins} \end{pmatrix} : X \to Y \oplus C_X$ is a pure monomorphism such that $\operatorname{Coker} \begin{pmatrix} f & \mathrm{ins} \end{pmatrix}$ carries the structure of a formal $\mathcal{U}$-cone of f corresponding to C_X and where $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : Y \oplus C_X \to Y$ is a split pure epimorphism with $\operatorname{Ker} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cong C_X$. In particular, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a $\mathcal{U}$-quasi-isomorphism.

$$\begin{tikzcd} & Y \oplus C_X \arrow[dr, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \\ X \arrow[ur, "\begin{pmatrix} f & \mathrm{ins} \end{pmatrix}"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

Proof. We let C_X be a formal cone of X and we let C_f be a formal cone of f corresponding to C_X, so that there exists a pushout rectangle

$$\begin{tikzcd} C_X \arrow[r, "f'"] & C_f \\ X \arrow[u, "\mathrm{ins}^{C_X}"] \arrow[r, "f"] & Y \arrow[u, "\mathrm{ins}^{C_f}"'] \end{tikzcd}$$

in $\mathcal{E}$. We get the factorisation $f = \begin{pmatrix} f & \mathrm{ins}^{C_X} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\begin{tikzcd} & Y \oplus C_X \arrow[dr, dotted, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"] & \\ X \arrow[ur, dotted, "\begin{pmatrix} f & \mathrm{ins}^{C_X} \end{pmatrix}"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

By [5, prop. 2.12], we have the pure short exact sequence

$$X \xrightarrow{\left(\, f \;\; \mathrm{ins}^{C_X} \,\right)} Y \oplus C_X \xrightarrow{\left(\begin{smallmatrix} \mathrm{ins}^{C_f} \\ -f' \end{smallmatrix}\right)} C_f.$$

Hence $(\, f \;\; \mathrm{ins}^{C_X} \,) : X \to Y \oplus C_X$ is a pure monomorphism and $\operatorname{Coker}(\, f \;\; \mathrm{ins}^{C_X} \,) \cong C_f$. Moreover, since C_X is in $\mathcal{U}$ and the split short exact sequence

$$C_X \xrightarrow{(\,0\;1\,)} Y \oplus C_X \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} Y$$

is a pure short exact sequence by [5, lem. 2.7], the morphism $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) : Y \oplus C_X \to Y$ is a quasi-isomorphism by proposition (A.8)(b). $\square$

(A.11) Proposition. The set of $\mathcal{U}$-quasi-isomorphisms in $\mathcal{E}$ is a semi-saturated denominator set in $\mathcal{E}$.

Proof. For every object X in $\mathcal{E}$, the identity 1_X is a pure monomorphism [5, def. 2.1] with $\operatorname{Coker} 1_X \cong 0$. As 0 is in $\mathcal{U}$ by remark (A.6), it follows that 1_X is a quasi-isomorphism for all $X \in \operatorname{Ob}\mathcal{E}$ by proposition (A.8)(a).
We suppose given morphisms $f\colon X_0 \to X_1$ and $g\colon X_1 \to X_2$ in $\mathcal{E}$. By proposition (A.10), there exists for every formal cone C_f of f a pure monomorphism $i\colon X_0 \to Y_0$ and a pure epimorphism $p\colon Y_0 \to X_1$ with $f = ip$ and such that $\operatorname{Coker} i \cong C_f$ and p is a quasi-isomorphism. Analogously, there exists for every formal cone C_g of g a pure monomorphism $j\colon X_1 \to Y_1$ and a pure epimorphism $q\colon Y_1 \to X_2$ with $g = jq$ and such that $\operatorname{Coker} j \cong C_g$ and q is a quasi-isomorphism. Every formal cone C_0 for X_0 leads to a diagram as follows, where all quadrangles are pushout rectangles and hence also pullback rectangles [5, prop. 2.12], and where D_0 is a formal cone of i corresponding to C_0, where C_1 is a formal cone of $ip = f$ corresponding to C_0, where D_1 is a formal cone of ipj corresponding to C_0, and where C_2 is a formal cone of $ipjq = fg$ corresponding to C_0.

$$\begin{tikzcd}
C_0 \arrow[r, "i'"] & D_0 \arrow[r, "p'"] & C_1 \arrow[r, "j'"] & D_1 \arrow[r, "q'"] & C_2 \\
X_0 \arrow[r, "i"] \arrow[u, "\mathrm{ins}^{C_0}"'] & Y_0 \arrow[r, "p"] \arrow[u, "\mathrm{ins}^{D_0}"'] & X_1 \arrow[r, "j"] \arrow[u, "\mathrm{ins}^{C_1}"'] & Y_1 \arrow[r, "q"] \arrow[u, "\mathrm{ins}^{D_1}"'] & X_2 \arrow[u, "\mathrm{ins}^{C_2}"']
\end{tikzcd}$$

In particular, i' and j' are pure monomorphisms [5, def. 2.1] and p' and q' are pure epimorphisms [5, dual of prop. 2.15].
So let us first suppose that f and g are quasi-isomorphisms. We choose formal cones C_f of f and C_g of g such that C_f and C_g are in $\mathcal{U}$, and we choose an arbitrary formal cone C_0 of X_0. Then C_0 is in $\mathcal{U}$ and hence D_0 is in $\mathcal{U}$ since $\operatorname{Coker} i' \cong \operatorname{Coker} i \cong C_f$ is in $\mathcal{U}$ by [5, prop. 2.12] and $\mathcal{U}$ is closed under pure short exact sequences. By proposition (A.8)(b), $\operatorname{Ker} p$ is in $\mathcal{U}$ as p is a quasi-isomorphism. But then C_1 is in $\mathcal{U}$ since D_0 and $\operatorname{Ker} p' \cong \operatorname{Ker} p$ are in $\mathcal{U}$. Analogously, C_1 in $\mathcal{U}$ implies that D_1 is in $\mathcal{U}$, and this in turn implies that C_2 is in $\mathcal{U}$. But C_2 is a formal cone of fg corresponding to C_0, whence fg is a quasi-isomorphism.
Next, we suppose that f and fg are quasi-isomorphisms. We choose formal cones C_f of f and C_{fg} of fg such that C_f and C_{fg} are in $\mathcal{U}$. Moreover, we choose the formal cone C_0 of X_0 such that C_{fg} is corresponding to C_0. As shown above, C_0 in $\mathcal{U}$ implies that D_0 is in $\mathcal{U}$, and D_0 in $\mathcal{U}$ implies that C_1 is in $\mathcal{U}$. But then C_1 is a formal cone of X_2 and hence C_2 is a formal cone of $jq = g$ of corresponding to C_1. Since $C_2 \cong C_{fg}$ is in $\mathcal{U}$ by our choice of C_0, this implies that g is a quasi-isomorphism.
Finally, let us suppose that g and fg are quasi-isomorphisms. We choose formal cones C_g of g and C_{fg} of fg such that C_g and C_{fg} are in $\mathcal{U}$. Moreover, we choose the formal cone C_0 of X_0 such that C_{fg} is corresponding to C_0. Then $C_2 \cong C_{fg}$ is in $\mathcal{U}$ and $\operatorname{Ker} q' \cong \operatorname{Ker} q$ is in $\mathcal{U}$, and therefore D_1 is in $\mathcal{U}$. This in turn implies that C_1 is in $\mathcal{U}$ since $\operatorname{Coker} j' \cong \operatorname{Coker} j \cong C_g$ is in $\mathcal{U}$. But C_1 is a formal cone of f corresponding to C_0, whence f is a quasi-isomorphism.
Altogether, the set of quasi-isomorphisms is a semi-saturated denominator set in $\mathcal{E}$. $\square$

(A.12) Corollary. We suppose given a morphism f in $\mathcal{E}$. The following conditions are equivalent.

(a) The morphism f is a quasi-isomorphism with respect to $\mathcal{U}$.

(b) Every formal $\mathcal{U}$-cone of f is in $\mathcal{U}$.

(c) There exist a pure monomorphism i and a pure epimorphism p with $f = ip$ and such that i and p are $\mathcal{U}$-quasi-isomorphisms.

Proof. First, we let f be a quasi-isomorphism and we suppose given an arbitrary formal cone C_f of f. By proposition (A.10), there exist a pure monomorphism i and a pure epimorphism p with $f = ip$ and such that p is a quasi-isomorphism and $\operatorname{Coker} i \cong C_f$. But i is a quasi-isomorphism since f is a quasi-isomorphism and since the set of quasi-isomorphisms is semi-saturated by proposition (A.11), so $C_f \cong \operatorname{Coker} i$ is in $\mathcal{U}$ by proposition (A.8)(a).
If every formal cone of f is an object of $\mathcal{U}$, then f is a $\mathcal{U}$-quasi-isomorphism since $\mathcal{E}$ has enough formal $\mathcal{U}$-cones. Finally, if there exist a pure monomorphism i and a pure epimorphism p with $f = ip$ and such that i and p are quasi-isomorphisms, then f is a quasi-isomorphism since the set of quasi-isomorphisms is multiplicative by proposition (A.11). □

Corollary (A.12)(c) and proposition (A.8) show that the notion of a quasi-isomorphism is self-dual, provided $\mathcal{E}$ fulfills also the dual of definition (A.3).

(A.13) Proposition. If $\mathcal{U}$ is closed under taking summands, then the set of $\mathcal{U}$-quasi-isomorphisms in $\mathcal{E}$ is a weakly saturated denominator set in $\mathcal{E}$.

Proof. We suppose that $\mathcal{U}$ is closed under taking summands, and we suppose given morphisms $f\colon X_0 \to X_1$, $g\colon X_1 \to X_2$, $h\colon X_2 \to X_3$ in $\mathcal{E}$ such that fg and gh are quasi-isomorphisms. We let C_0 be a formal cone of X_0 and construct iteratively pushouts as in the following diagram, so that C_1 is a formal cone of f corresponding to C_0, so that C_2 is a formal cone of fg corresponding to C_0, and so that C_3 is a formal cone of fgh corresponding to C_0.

$$\begin{tikzcd}
C_0 \arrow[r, "f'"] & C_1 \arrow[r, "g'"] & C_2 \arrow[r, "h'"] & C_3 \\
X_0 \arrow[r, "f"] \arrow[u, "\mathrm{ins}^{C_0}"'] & X_1 \arrow[r, "g"] \arrow[u, "\mathrm{ins}^{C_1}"'] & X_2 \arrow[r, "h"] \arrow[u, "\mathrm{ins}^{C_2}"'] & X_3 \arrow[u, "\mathrm{ins}^{C_3}"']
\end{tikzcd}$$

Next, we let D_1 be a formal cone of C_1 and construct again iteratively pushouts as in the following diagram, so that D_2 is a formal cone of g' corresponding to D_1, and so that D_3 is a formal cone of $g'h'$ corresponding to D_1.

$$\begin{tikzcd}
& D_1 \arrow[r, "g''"] & D_2 \arrow[r, "h''"] & D_3 \\
C_0 \arrow[r, "f'"] & C_1 \arrow[r, "g'"] \arrow[u, "\mathrm{ins}^{D_1}"'] & C_2 \arrow[r, "h'"] \arrow[u, "\mathrm{ins}^{D_2}"'] & C_3 \arrow[u, "\mathrm{ins}^{D_3}"'] \\
X_0 \arrow[r, "f"] \arrow[u, "\mathrm{ins}^{C_0}"'] & X_1 \arrow[r, "g"] \arrow[u, "\mathrm{ins}^{C_1}"'] & X_2 \arrow[r, "h"] \arrow[u, "\mathrm{ins}^{C_2}"'] & X_3 \arrow[u, "\mathrm{ins}^{C_3}"']
\end{tikzcd}$$

Then D_1 is also a formal cone of X_1 and therefore D_3 is a formal cone of gh corresponding to D_1. Since fg and gh are quasi-isomorphisms, C_2 and D_3 are in $\mathcal{U}$ by corollary (A.12). But since we have the pure short exact sequence

$$C_2 \xrightarrow{\left(\, h' \;\, \mathrm{ins}^{D_2} \,\right)} C_3 \oplus D_2 \xrightarrow{\left(\begin{smallmatrix} \mathrm{ins}^{D_3} \\ -h'' \end{smallmatrix}\right)} D_3$$

by [5, prop. 2.12], we conclude that $C_3 \oplus D_2$ is in $\mathcal{U}$ and therefore that D_2 is in $\mathcal{U}$ since $\mathcal{U}$ is closed under pure short exact sequences and taking summands. Thus g is a quasi-isomorphism as D_2 is a formal cone of g corresponding to D_1, and hence also f, h, fgh are quasi-isomorphisms by proposition (A.11). □

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