

The third cohomology group classifies crossed module extensions

Sebastian Thomas

November 15, 2009 *

Abstract

We give an elementary proof of the well-known fact that the third cohomology group $H^3(G, M)$ of a group G with coefficients in an abelian G -module M is in bijection to the set $\text{Ext}^2(G, M)$ of equivalence classes of crossed module extensions of G with M .

1 Introduction

This manuscript does not claim originality.

Perhaps the best-known result from group cohomology is the Schreier theorem, which gives an interpretation of the second cohomology group $H^2(G, M)$ of a group G with coefficients in an abelian G -module M . More precisely, it states that $H^2(G, M)$ classifies group extensions of G with M in the sense that there is a bijection from $H^2(G, M)$ to the set of extension classes $\text{Ext}^1(G, M)$ of group extensions of G with M . By such a group extension we mean a short exact sequence of groups

$$M \xrightarrow{\iota} E \xrightarrow{\pi} G$$

for which the induced G -module structure on M coincides with the given one.

To give an interpretation of $H^3(G, M)$, one has to consider crossed module extensions of G with M instead of group extensions. Roughly said, a crossed module extension of G with M is a four term exact sequence equipped with extra data such that the middle two terms form a crossed module.

The aim of this manuscript is to prove the following well-known theorem.

Theorem (cf. [7, th. 4.5], [8, p. 310], [14, th. 9.4]). Given a group G and an abelian G -module M , we have

$$\text{Ext}^2(G, M) \cong H^3(G, M).$$

A priori, the set of crossed module extension classes $\text{Ext}^2(G, M)$ is actually only a set, while the third cohomology group $H^3(G, M)$ is an abelian group. So “isomorphic” in this theorem means that there is a bijection between $\text{Ext}^2(G, M)$ and $H^3(G, M)$, which can of course be used to transport a group structure to $\text{Ext}^2(G, M)$. However, we will not pursue that possibility in this manuscript.

The proof presented here follows a sketch of BROWN [2, ch. IV, sec. 5], while the techniques involved originally go back to EILENBERG and MAC LANE [4], [5], [12]. The cohomology class associated to a given crossed module extension class is constructed using certain lifts or sections in the underlying exact sequence of a representing crossed module extension. Conversely, to a given cohomology class we attach the extension class of a standard extension. Our proof allows to conclude that extensions in the same extension class are connected by at most two elementary steps (see corollary (6.8), cf. also [7, lem. 3.3]).

This manuscript sets the stage for [16], where we study the second cohomology group of a crossed module and, more generally, of a simplicial group. In that article, we will make explicit use of the constructions presented in this manuscript, in particular of the chosen sections and the 3-cocycle constructed from them.

The result also appears in [14, th. 9.4]. There is a more general result giving an interpretation of $H^{n+1}(G, M)$ in terms of extensions for all $n \geq 1$. It has been independently proven by HOLT [7, th. 4.5] and HUEBSCHMANN [8,

Mathematics Subject Classification 2010: 20J06, 18D35.

*This is a slightly revised version from May 9, 2011.

p. 310] and states that there is a bijection between $H^{n+1}(G, M)$ and the set of equivalence classes of so-called crossed n -fold extensions. To prove this theorem, HOLT uses universal delta functors, while HUEBSCHMANN works with projective crossed resolutions. The author does not know whether there exists a proof of this more general result using lifts and sections in the spirit of SCHREIER and of EILENBERG and MAC LANE. A summary of the development leading to this result can be found in the historical note of MAC LANE [13]. A version in terms of LODAYS n -categorical groups can be found in [11, th. 4.2]. Finally, an interpretation of $H^4(G, M)$ using CONDUCHÉS 2-crossed modules can be found in [3, th. 4.7].

Outline We start in section 2 with some preliminaries on groups and crossed modules. We show in section 3 that group cohomology can be expressed using componentwise pointed cochains. In section 4, we give the definitions of crossed module extensions and consider some examples. Thereafter, we show in section 5 how a 3-cohomology class can be associated to a given crossed module extension. Conversely, in section 6 we construct a standard extension with respect to a given cocycle and show that both constructions are mutually inverse. This finally proves the classification theorem.

Conventions and notations

- The composite of morphisms $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ is usually denoted by $fg: X \rightarrow Z$.
- Given a complex of abelian groups A such that $A^n = 0$ for $n < 0$, we usually do not denote these zero objects.
- We use the notations $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- Given a map $f: X \rightarrow Y$ and subsets $X' \subseteq X, Y' \subseteq Y$ with $X'f \subseteq Y'$, we write $f|_{X'}^{Y'}: X' \rightarrow Y', x' \mapsto x'f$. Moreover, we abbreviate $f|_{X'} := f|_{X'}^Y$ and $f|^{Y'} := f|_X^{Y'}$.
- Given integers $a, b \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ for the set of integers lying between a and b . If we need to specify orientation, then we write $\lceil a, b \rceil := \{z \in \mathbb{Z} \mid a \leq z \leq b\}$ for the *ascending interval* and $\lfloor a, b \rfloor := \{z \in \mathbb{Z} \mid a \geq z \geq b\}$ for the *descending interval*. Whereas we formally deal with tuples, we use the element notation, for example we write $\prod_{i \in [1, 3]} g_i = g_1 g_2 g_3$ and $\prod_{i \in [3, 1]} g_i = g_3 g_2 g_1$ or $(g_i)_{i \in [3, 1]} = (g_3, g_2, g_1)$ for group elements g_1, g_2, g_3 .
- Given tuples $(x_i)_{i \in I}$ and $(x_j)_{j \in J}$ with disjoint index sets I and J , we write $(x_i)_{i \in I} \cup (x_j)_{j \in J}$ for their concatenation.
- Given groups G and H , we denote by $\text{triv}: G \rightarrow H$ the trivial group homomorphism $g \mapsto 1$.
- Given a group G , a subgroup U of G and a quotient group Q of G , we denote by $\text{inc} = \text{inc}^U: U \rightarrow G$ the inclusion $u \mapsto u$ and by $\text{quo} = \text{quo}^Q: G \rightarrow Q$ the quotient morphism.
- Given a group homomorphism $\varphi: G \rightarrow H$, we denote its kernel by $\text{Ker } \varphi$, its cokernel by $\text{Coker } \varphi$ and its image by $\text{Im } \varphi$.
- The distinguished point in a pointed set X will be denoted by $*$ or $*^X$.
- The Kronecker delta is defined by

$$\delta_{x,y} = \begin{cases} 1 & \text{for } x = y, \\ 0 & \text{for } x \neq y, \end{cases}$$

where x and y are elements of some set.

A remark on Grothendieck universes To avoid set-theoretical difficulties, we work with Grothendieck universes [1, exp. I, sec. 0] in this manuscript. In particular, every category has an object *set* and a morphism *set*.

We suppose given a Grothendieck universe \mathfrak{U} . A \mathfrak{U} -*set* is a set that is an element of \mathfrak{U} , a \mathfrak{U} -*map* is a map between \mathfrak{U} -sets. The *category of \mathfrak{U} -sets* consisting of the set of \mathfrak{U} -sets, that is, of \mathfrak{U} , as object set and the set of \mathfrak{U} -maps as morphism set will be denoted by $\mathbf{Set}_{(\mathfrak{U})}$. A \mathfrak{U} -*group* is a group whose underlying set is a \mathfrak{U} -set, a \mathfrak{U} -*group*

homomorphism is a group homomorphism between \mathfrak{U} -groups. The *category of \mathfrak{U} -groups* consisting of the set of \mathfrak{U} -groups as object set and the set of \mathfrak{U} -group homomorphisms as morphism set will be denoted by $\mathbf{Grp}_{(\mathfrak{U})}$. Similarly for (abelian) G -modules, etc.

Because we do not want to overload our text with the usage of Grothendieck universes, we may suppress them in notation, provided we work with a single fixed Grothendieck universe. For example, instead of

Remark. We suppose given a Grothendieck universe \mathfrak{U} . The forgetful functor $\mathbf{Grp}_{(\mathfrak{U})} \rightarrow \mathbf{Set}_{(\mathfrak{U})}$ is faithful.

we may just write

Remark. The forgetful functor $\mathbf{Grp} \rightarrow \mathbf{Set}$ is faithful.

Grothendieck universes will play a role when we consider extension classes of crossed module extensions, cf. section 4.

2 Preliminaries

Sections and lifts

We suppose given a category \mathcal{C} , objects $X, Y, Z \in \text{Ob } \mathcal{C}$ and morphisms $f \in \mathcal{C}(X, Y)$, $g \in \mathcal{C}(Z, Y)$. A *section* of f is a morphism $s \in \mathcal{C}(Y, X)$ such that $sf = 1_Y$. A *lift* of g along f is a morphism $l \in \mathcal{C}(Z, X)$ such that $g = lf$. If f is a retraction, then the sections of f are exactly the lifts of 1_Y along f . Moreover, every section s of f defines a lift l of g along f by $l := gs$.

Free groups

We suppose given a set X . Recall that a *free group* on X consists of a group F together with a map $e: X \rightarrow F$ such that for every group G and every map $f: X \rightarrow G$ there exists a unique group homomorphism $\varphi: F \rightarrow G$ with $f = e\varphi$.

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ e \uparrow & \nearrow f & \\ X & & \end{array}$$

By abuse of notation, we often refer to the free group (consisting of the group F and the map e) as well as to its underlying group by F . The map e is said to be the (*ordered*) *basis* of the free group F . Given a free group F on X with basis e , we write $e = e^F := e$. The elements of $\text{Im } e^F$ are called *free generators* of F .

There exists a free group on every set X , see for example [10, ch. I, prop. 12.1]. Moreover, the Nielsen-Schreier Theorem states that every subgroup U of a free group F is again a free group, and it describes explicitly a set of free generators of U , see for example [9, §36, p. 36]. We will apply this theorem in proposition (6.3).

Since a group has a natural underlying pointed set with the neutral element as distinguished point, we can also define free groups on pointed sets: We suppose given a pointed set X . A *free group* on X consists of a group F together with a pointed map $e: X \rightarrow F$ such that for every group G and every pointed map $f: X \rightarrow G$ there exists a unique group homomorphism $\varphi: F \rightarrow G$ with $f = e\varphi$.

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & G \\ e \uparrow & \nearrow f & \\ X & & \end{array}$$

By abuse of notation, we often refer to the free group as well as to its underlying group by F . The morphism e is said to be the (*ordered*) *basis* of the free group F . Given a free group F on X with basis e , we write $e = e^F := e$. The elements of $\text{Im } e^F$ are called *free generators* of F .

We suppose given a pointed set X . Roughly speaking, a free group on X is a free group on the set $X \setminus \{*\}$. More precisely: Given a free group F on the pointed set X , we obtain a free group F' on the set $X \setminus \{*\}$ with

underlying group F and basis $e^{F'} = e^F|_{X \setminus \{*\}}$. Conversely, given a free group F' on the set $X \setminus \{*\}$, we obtain a free group F on the X pointed set with underlying group F' and basis e^F defined by

$$xe^F = \begin{cases} xe^{F'} & \text{if } x \in X \setminus \{*\}, \\ 1 & \text{if } x = *. \end{cases}$$

Group actions

We suppose given a category \mathcal{C} and a group G . Recall that a (*group*) *action* of G on an object $X \in \text{Ob } \mathcal{C}$ is a group homomorphism $\alpha: G^{\text{op}} \rightarrow \text{Aut}_{\mathcal{C}} X$.

A G -*module* consists of a (not necessarily abelian) group M together with an action α of G on M , that is, a group homomorphism $\alpha: G^{\text{op}} \rightarrow \text{Aut}_{\mathbf{Grp}} M$. By abuse of notation, we often refer to the module over G as well as to its underlying group by M . The action α is called the G -*action* of the G -module M . Given a G -module M with G -action α , we often write ${}^g m := m(g\alpha)$ for $m \in M$, $g \in G$. A G -module M is said to be *abelian* if its underlying group is abelian. As usual, we often write M additively in this case, and we write $gm = g \cdot m := m(g\alpha)$ for $m \in M$, $g \in G$, where α denotes the G -action of M .

A G -module structure on G itself is provided by the conjugation homomorphism $G^{\text{op}} \rightarrow \text{Aut } G$, $g \mapsto {}^g(-)$, where ${}^g x = gxg^{-1}$ for $x, g \in G$.

Cohomology of groups

We suppose given an abelian G -module M . The *cochain complex* of G is the complex of abelian groups $\text{Ch}(G, M) = \text{Ch}_{\mathbf{Grp}}(G, M)$ with entries $\text{Ch}^n(G, M) := \text{Map}(G^{\times n}, M)$ and differentials given by

$$\begin{aligned} (g_j)_{j \in [n, 0]}(c\partial) &= (g_{j+1})_{j \in [n-1, 0]}c + \sum_{k \in [1, n]} (-1)^k ((g_{j+1})_{j \in [n-1, k]} \cup (g_k g_{k-1}) \cup (g_j)_{j \in [k-2, 0]})c \\ &\quad + (-1)^{n+1} g_n (g_j)_{j \in [n-1, 0]}c \end{aligned}$$

for $(g_j)_{j \in [n, 0]} \in G^{\times n}$, $c \in \text{Ch}^n(G, M)$, $n \in \mathbb{N}_0$. Moreover, we define the n -*th cocycle group* $Z^n(G, M) := Z^n \text{Ch}(G, M)$, the n -*th coboundary group* $B^n(G, M) := B^n \text{Ch}(G, M)$ and the n -*th cohomology group* $H^n(G, M) := H^n \text{Ch}(G, M) = Z^n(G, M)/B^n(G, M)$ of G with coefficients in M . An element $c \in \text{Ch}^n(G, M)$ resp. $z \in Z^n(G, M)$ resp. $b \in B^n(G, M)$ resp. $h \in H^n(G, M)$ is said to be an n -*cochain* resp. an n -*cocycle* resp. an n -*coboundary* resp. an n -*cohomology class* of G with coefficients in M .

Crossed modules

A *crossed module* consists of a group G , a G -module M and a group homomorphism $\mu: M \rightarrow G$ such that the following two axioms hold.

(Equi) *Equivariance*. We have $({}^g m)\mu = {}^g(m\mu)$ for all $m \in M$, $g \in G$.

(Peif) *Peiffer identity*. We have ${}^{n\mu} m = {}^n m$ for all $m, n \in M$.

Here, G acts on G via conjugation, and so does M on M . We call G the *group part* and M the *module part* of the crossed module. The group homomorphism $\mu: M \rightarrow G$ is said to be the *structure morphism* of the crossed module. Given a crossed module V with group part G , module part M and structure morphism μ , we write $\text{Gp } V := G$, $\text{Mp } V := M$ and $\mu = \mu^V := \mu$.

We let V and W be crossed modules. A *morphism of crossed modules* (or *crossed module morphism*) from V to W consists of group homomorphisms $\varphi_0: \text{Gp } V \rightarrow \text{Gp } W$ and $\varphi_1: \text{Mp } V \rightarrow \text{Mp } W$ such that $\varphi_1 \mu^V = \mu^W \varphi_0$ and such that $({}^g m)\varphi_1 = {}^{g\varphi_0}(m\varphi_1)$ holds for all $m \in \text{Mp } V$, $g \in \text{Gp } V$. The group homomorphisms φ_0 resp. φ_1 are said to be the *group part* resp. the *module part* of the morphism of crossed modules. Given a crossed module morphism φ from V to W with group part φ_0 and module part φ_1 , we write $\text{Gp } \varphi := \varphi_0$ and $\text{Mp } \varphi := \varphi_1$. Composition of morphisms of crossed modules is defined by composition on the group parts and on the module parts.

Let us consider two examples: Given a group G and a normal subgroup $N \trianglelefteq G$, the inclusion $\text{inc}: N \rightarrow G$ together with the conjugation action of G on N yields the crossed module $[N \trianglelefteq G]$, called *normal subgroup crossed module*. On the other hand, given a group G and an abelian G -module M , the trivial homomorphism $\text{triv}: M \rightarrow G$ yields the crossed module $[M \ G]$, called *trivial homomorphism crossed module*.

We let \mathfrak{U} be a Grothendieck universe. A crossed module V is said to be a \mathfrak{U} -crossed module if $\mathrm{Gp} V$ is a \mathfrak{U} -group and $\mathrm{Mp} V$ is a \mathfrak{U} - G -module. The category of \mathfrak{U} -crossed modules consisting of \mathfrak{U} -crossed modules as objects and morphisms of \mathfrak{U} -crossed modules as morphisms will be denoted by $\mathbf{CrMod} = \mathbf{CrMod}_{(\mathfrak{U})}$.

Given a crossed module V , the image $\mathrm{Im} \mu$ is a normal subgroup of $\mathrm{Gp} V$ and the kernel $\mathrm{Ker} \mu$ is a central subgroup of $\mathrm{Mp} V$. Moreover, the action of $\mathrm{Gp} V$ on $\mathrm{Mp} V$ restricts to a trivial action of $\mathrm{Im} \mu$ on $\mathrm{Ker} \mu$. See for example [15, prop. (5.3)].

The homotopy groups of V are defined by

$$\pi_n(V) := \begin{cases} \mathrm{Coker} \mu & \text{for } n = 0, \\ \mathrm{Ker} \mu & \text{for } n = 1, \\ \{1\} & \text{for } n \in \mathbb{N}_0 \setminus \{0, 1\}. \end{cases}$$

Thus $\pi_1(V)$ carries the structure of an abelian $\pi_0(V)$ -module, where the action of $\pi_0(V)$ on $\pi_1(V)$ is induced by the action of $\mathrm{Gp} V$ on $\mathrm{Mp} V$, that is, for $k \in \pi_1(V)$ and $p \in \pi_0(V)$ we have ${}^p k = {}^g k$ for any $g \in \mathrm{Gp} V$ with $g(\mathrm{Im} \mu) = p$.

Given crossed modules V and W , a crossed module morphism $\varphi: V \rightarrow W$ is said to be a *weak homotopy equivalence* if it induces isomorphisms $\pi_n(V) \rightarrow \pi_n(W)$ for all $n \in \mathbb{N}_0$.

Notation. Given a crossed module V , the module part $\mathrm{Mp} V$ resp. its opposite $(\mathrm{Mp} V)^{\mathrm{op}}$ act on (the underlying set of) the group part $\mathrm{Gp} V$ by $mg := (m\mu)g$ and $gm := g(m\mu)$ for $m \in \mathrm{Mp} V$, $g \in \mathrm{Gp} V$. Using this, we get for example

$$mg_n = (m\mu)g_n = m\mu(g_n) = m(g_n)$$

and

$$gm = g(m\mu) = {}^g(m\mu)g = ({}^g m)\mu g = ({}^g m)g$$

for $m, n \in \mathrm{Mp} V$, $g \in \mathrm{Gp} V$. Also note that $(mg)n = m(gn)$ for $m, n \in \mathrm{Mp} V$, $g \in \mathrm{Gp} V$.

Given a set X and a map $f: \mathrm{Gp} V \rightarrow X$, we usually write $mf := m\mu f$ for $m \in \mathrm{Mp} V$. Similarly for maps $\mathrm{Gp} V \times \mathrm{Gp} V \rightarrow X$, etc.

Moreover, given crossed modules V and W and a morphism of crossed modules $\varphi: V \rightarrow W$, we may write $m\varphi$ and $g\varphi$ instead of $m(\mathrm{Mp} \varphi)$ and $g(\mathrm{Gp} \varphi)$. Using this, we have

$$(mg)\varphi = ((m\mu^V)g)(\mathrm{Gp} \varphi) = (m\mu^V(\mathrm{Gp} \varphi))(g(\mathrm{Gp} \varphi)) = (m(\mathrm{Mp} \varphi)\mu^W)(g(\mathrm{Gp} \varphi)) = (m\varphi)(g\varphi)$$

for $m \in \mathrm{Mp} V$, $g \in \mathrm{Gp} V$.

3 Componentwise pointed cochains

We suppose given a group G , an abelian G -module M and a non-negative integer $n \in \mathbb{N}_0$. Then G resp. M can naturally be considered as pointed sets with 1 resp. 0 as distinguished points. We want to make use of those cochains of G with coefficients in M that preserve these distinguished points.

This section follows [4, ch. II, sec. 6].

(3.1) Definition (componentwise pointed maps). We suppose given pointed sets X_i for $i \in I$ and Y , where I is an index set. A map $f: \times_{i \in I} X_i \rightarrow Y$ is said to be *componentwise pointed* if $(x_i)_{i \in I} f = *$ for all $(x_i)_{i \in I} \in \times_{i \in I} X_i$ with $x_i = *$ for some $i \in I$.

(3.2) Definition (componentwise pointed cochains). The subset of $\mathrm{Ch}^n(G, M)$ consisting of all componentwise pointed n -cochains of G with coefficients in M will be denoted by

$$\mathrm{Ch}_{\mathrm{cpt}}^n(G, M) = \mathrm{Ch}_{\mathbf{Grp}, \mathrm{cpt}}^n(G, M) := \{c \in \mathrm{Ch}^n(G, M) \mid c \text{ componentwise pointed}\}.$$

Moreover, we set

$$Z_{\mathrm{cpt}}^n(G, M) = Z_{\mathbf{Grp}, \mathrm{cpt}}^n(G, M) := \mathrm{Ch}_{\mathrm{cpt}}^n(G, M) \cap Z^n(G, M)$$

for the set of componentwise pointed n -cocycles and

$$B_{\text{cpt}}^n(G, M) = B_{\mathbf{Grp}, \text{cpt}}^n(G, M) := \text{Ch}_{\text{cpt}}^n(G, M) \cap B^n(G, M)$$

for the set of componentwise pointed n -coboundaries and

$$H_{\text{cpt}}^n(G, M) = H_{\mathbf{Grp}, \text{cpt}}^n(G, M) := Z_{\text{cpt}}^n(G, M) / B_{\text{cpt}}^n(G, M)$$

for the set of componentwise pointed n -cohomology classes of G with coefficients in M .

The next definition is for technical purposes.

(3.3) Definition (k -pointed cochains). We suppose given $k \in [0, n]$. An n -cochain $c \in \text{Ch}^n(G, M)$ is said to be k -pointed if $(g_j)_{j \in [n-1, 0]}c = 0$ for all $(g_j)_{j \in [n-1, 0]} \in G^{\times n}$ with $g_j = 1$ for some $j \in [k-1, 0]$.

By definition, a 0-pointed n -cochain is just an arbitrary n -cochain, while an n -pointed n -cochain is actually a componentwise pointed n -cochain.

(3.4) Remark. We suppose given $k \in [0, n]$ and an n -cochain $c \in \text{Ch}^n(G, M)$.

- (a) If c is k -pointed, then $c\partial$ is k -pointed.
- (b) If c is componentwise pointed, then $c\partial$ is componentwise pointed.

Proof.

- (a) We suppose that c is k -pointed, and we let $g_j \in G$ for $j \in [n, 0]$ be given with $g_l = 1$ for some $l \in [k-1, 0]$. For $l = 0$, we have

$$\begin{aligned} (g_j)_{j \in [n, 0]}(c\partial) &= (g_{j+1})_{j \in [n-1, 0]}c + \sum_{i \in [1, n]} (-1)^i ((g_{j+1})_{j \in [n-1, i]} \cup (g_i g_{i-1}) \cup (g_j)_{j \in [i-2, 0]})c \\ &\quad + (-1)^{n+1} g_n (g_j)_{j \in [n-1, 0]}c \\ &= (g_{j+1})_{j \in [n-1, 0]}c - ((g_{j+1})_{j \in [n-1, 1]} \cup (g_1))c = 0, \end{aligned}$$

and for $l \in [k-1, 1]$, we have

$$\begin{aligned} (g_j)_{j \in [n, 0]}(c\partial) &= (g_{j+1})_{j \in [n-1, 0]}c + \sum_{i \in [1, n]} (-1)^i ((g_{j+1})_{j \in [n-1, i]} \cup (g_i g_{i-1}) \cup (g_j)_{j \in [i-2, 0]})c \\ &\quad + (-1)^{n+1} g_n (g_j)_{j \in [n-1, 0]}c \\ &= (-1)^l ((g_{j+1})_{j \in [n-1, l]} \cup (g_{l-1}) \cup (g_j)_{j \in [l-2, 0]})c \\ &\quad + (-1)^{l+1} ((g_{j+1})_{j \in [n-1, l+1]} \cup (g_{l+1}) \cup (g_j)_{j \in [l-1, 0]})c \\ &= (-1)^l ((g_{j+1})_{j \in [n-1, l]} \cup (g_j)_{j \in [l-1, 0]})c + (-1)^{l+1} ((g_{j+1})_{j \in [n-1, l]} \cup (g_j)_{j \in [l-1, 0]})c = 0. \end{aligned}$$

Hence $c\partial$ is also k -pointed.

- (b) We suppose that c is componentwise pointed. Then $c\partial$ is n -pointed by (a). Moreover, given $g_j \in G$ for $j \in [n-1, 0]$, we have

$$\begin{aligned} ((1) \cup (g_j)_{j \in [n-1, 0]})(c\partial) &= ((1) \cup (g_{j+1})_{j \in [n-2, 0]})c + \sum_{i \in [1, n-1]} (-1)^i ((1) \cup (g_{j+1})_{j \in [n-2, i]} \cup (g_i g_{i-1}) \cup (g_j)_{j \in [i-2, 0]})c \\ &\quad + (-1)^n ((g_{n-1}) \cup (g_j)_{j \in [n-2, 0]})c + (-1)^{n+1} (g_j)_{j \in [n-1, 0]}c \\ &= (-1)^n (g_j)_{j \in [n-1, 0]}c + (-1)^{n+1} (g_j)_{j \in [n-1, 0]}c = 0. \end{aligned} \quad \square$$

(3.5) Definition (componentwise pointisation of n -cocycles). Given an n -cochain $c \in \text{Ch}^n(G, M)$, the k -pointisation $c^{\text{pt},k} \in \text{Ch}^n(G, M)$ of c for $k \in [0, n]$ is given recursively by

$$c^{\text{pt},k} := \begin{cases} c & \text{if } k = 0, \\ c^{\text{pt},k-1} - p_c^k \partial & \text{if } k \in [1, n], \end{cases}$$

where the k -pointiser of c for $k \in [1, n]$ is defined to be the $(n-1)$ -cochain $p_c^k \in \text{Ch}^{n-1}(G, M)$ given by

$$(g_j)_{j \in [n-2,0]} p_c^k := (-1)^k ((g_{j-1})_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1}$$

for $g_j \in G_j$, $j \in [n-2, 0]$.

(3.6) Proposition. We suppose given an n -cochain $c \in \text{Ch}^n(G, M)$ such that $c\partial$ is componentwise pointed. Then $c^{\text{pt},k}$ is k -pointed for all $k \in [0, n]$.

Proof. We proceed by induction on $k \in [0, n]$, where for $k = 0$ there is nothing to do. So let us suppose given $k \in [1, n]$ and let us suppose that $c^{\text{pt},k-1}$ is $(k-1)$ -pointed. Then p_c^k is $(k-1)$ -pointed by definition and hence $c^{\text{pt},k} = c^{\text{pt},k-1} - p_c^k \partial$ is $(k-1)$ -pointed by remark (3.4)(a). It remains to show that

$$((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k} = 0$$

for $g_j \in G_j$, $j \in [n-1, k] \cup [k-2, 0]$. Indeed, if $k \in [1, n-1]$, then we have, since $c^{\text{pt},k-1}$ and p_c^k are $(k-1)$ -pointed,

$$\begin{aligned} & ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k} \\ &= ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} - ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) (p_c^k \partial) \\ &= ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad - \sum_{i \in [k+1, n-1]} (-1)^i ((g_{j+1})_{j \in [n-2,i]} \cup (g_i g_{i-1}) \cup (g_j)_{j \in [i-2,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) p_c^k \\ &\quad - (-1)^n g_{n-1} ((g_j)_{j \in [n-2,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) p_c^k \\ &= ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad - \sum_{i \in [k+1, n-1]} (-1)^{i+k} ((g_j)_{j \in [n-1, i+1]} \cup (g_i g_{i-1}) \cup (g_{j-1})_{j \in [i-1, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad - (-1)^{n+k-1} g_{n-1} ((g_{j-1})_{j \in [n-1, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &= ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad + \sum_{i \in [k+2, n]} (-1)^{i+k-1} ((g_j)_{j \in [n-1, i]} \cup (g_{i-1} g_{i-2}) \cup (g_{j-1})_{j \in [i-2, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad + (-1)^{n+k} g_{n-1} ((g_{j-1})_{j \in [n-1, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &= (-1)^{k-1} ((-1)^{k+1} ((g_j)_{j \in [n-1,k]} \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad + \sum_{i \in [k+2, n]} (-1)^i ((g_j)_{j \in [n-1, i]} \cup (g_{i-1} g_{i-2}) \cup (g_{j-1})_{j \in [i-2, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1} \\ &\quad + (-1)^{n+1} g_{n-1} ((g_{j-1})_{j \in [n-1, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) c^{\text{pt},k-1}) \\ &= (-1)^{k-1} ((g_{j-1})_{j \in [n, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) (c^{\text{pt},k-1} \partial) \\ &= (-1)^{k-1} ((g_{j-1})_{j \in [n, k+1]} \cup (1) \cup (1) \cup (g_j)_{j \in [k-2,0]}) (c\partial) = 0. \end{aligned}$$

Moreover, for $k = n$, we have $n \geq 1$ and obtain, since $c^{\text{pt},n-1}$ and p_c^n are $(n-1)$ -pointed,

$$\begin{aligned} ((1) \cup (g_j)_{j \in [n-2,0]}) c^{\text{pt},n} &= ((1) \cup (g_j)_{j \in [n-2,0]}) c^{\text{pt},n-1} - ((1) \cup (g_j)_{j \in [n-2,0]}) (p_c^n \partial) \\ &= ((1) \cup (g_j)_{j \in [n-2,0]}) c^{\text{pt},n-1} = (-1)^{n-1} ((1) \cup (1) \cup (g_j)_{j \in [n-2,0]}) (c^{\text{pt},n-1} \partial) \\ &= (-1)^{n-1} ((1) \cup (1) \cup (g_j)_{j \in [n-2,0]}) (c\partial) = 0. \end{aligned} \quad \square$$

(3.7) Corollary (cf. [4, lem. 6.1, lem. 6.2]).

(a) We have

$$Z_{\text{cpt}}^n(G, M) = \{z \in Z^n(G, M) \mid z^{\text{pt},k} = z^{\text{pt},k-1} \text{ for } k \in [1, n]\} = \{z \in Z^n(G, M) \mid z^{\text{pt},n} = z\}.$$

(b) If $n \in \mathbb{N}$, then we have

$$B_{\text{cpt}}^n(G, M) = (\text{Ch}_{\text{cpt}}^{n-1}(G, M))\partial.$$

(c) The embedding $Z_{\text{cpt}}^n(G, M) \rightarrow Z^n(G, M)$ and the n -pointisation homomorphism $Z^n(G, M) \rightarrow Z_{\text{cpt}}^n(G, M)$, $z \mapsto z^{\text{pt},n}$ induce mutually inverse isomorphisms between $H_{\text{cpt}}^n(G, M)$ and $H^n(G, M)$. In particular,

$$H^n(G, M) \cong H_{\text{cpt}}^n(G, M).$$

Proof.

(a) We suppose given an n -cocycle $z \in Z^n(G, M)$. If z is componentwise pointed, we inductively have $p_z^k = 0$ and hence $z^{\text{pt},k} = z^{\text{pt},k-1}$ for $k \in [1, n]$. If $z^{\text{pt},k} = z^{\text{pt},k-1}$ for $k \in [1, n]$, it follows inductively that $z^{\text{pt},n} = z^{\text{pt},0} = z$. Finally, if $z^{\text{pt},n} = z$, it follows from the componentwise pointedness of $z^{\text{pt},n}\partial = z\partial = 0$ that $z = z^{\text{pt},n}$ is n -pointed by proposition (3.6), that is, z is componentwise pointed.

(b) By remark (3.4)(b), we have $(\text{Ch}_{\text{cpt}}^{n-1}(G, M))\partial \subseteq B_{\text{cpt}}^n(G, M)$. Conversely, we suppose given an n -coboundary $b \in B^n(G, M)$ and we choose an $(n-1)$ -cochain $c \in \text{Ch}^{n-1}(G, M)$ with $b = c\partial$. Then we also have $b = c^{\text{pt},n-1}\partial$, and by proposition (3.6) it follows that if b is componentwise pointed, then so is $c^{\text{pt},n-1}$. Thus we also have $B_{\text{cpt}}^n(G, M) = (\text{Ch}_{\text{cpt}}^{n-1}(G, M))\partial$.

(c) By definition of the n -pointisation, we have $z = z^{\text{pt},n} + (\sum_{k \in [1, n]} p_z^k)\partial$ for every n -cocycle $z \in Z^n(G, M)$ and since the n -pointisation $z^{\text{pt},n}$ is componentwise pointed by (a), it follows that

$$H^n(G, M) = Z^n(G, M)/B^n(G, M) = (Z_{\text{cpt}}^n(G, M) + B^n(G, M))/B^n(G, M).$$

Moreover,

$$H_{\text{cpt}}^n(G, M) = Z_{\text{cpt}}^n(G, M)/B_{\text{cpt}}^n(G, M) = Z_{\text{cpt}}^n(G, M)/(Z_{\text{cpt}}^n(G, M) \cap B^n(G, M)),$$

and thus Noether's first law of isomorphism provides the asserted isomorphisms

$$\begin{aligned} H_{\text{cpt}}^n(G, M) &\rightarrow H^n(G, M), z + B_{\text{cpt}}^n(G, M) \mapsto z + B^n(G, M) \text{ and} \\ H^n(G, M) &\rightarrow H_{\text{cpt}}^n(G, M), z + B^n(G, M) \mapsto z^{\text{pt},n} + B_{\text{cpt}}^n(G, M). \end{aligned} \quad \square$$

4 Crossed module extensions and their equivalence classes

In this section, we suppose given a group G and an abelian G -module M .

(4.1) Definition (crossed module extension).

(a) A *crossed module extension* (or *2-extension*) of G with M consists of a crossed module E together with a group monomorphism $\iota: M \rightarrow \text{Mp } E$ and a group epimorphism $\pi: \text{Gp } E \rightarrow G$ such that

$$M \xrightarrow{\iota} \text{Mp } E \xrightarrow{\mu} \text{Gp } E \xrightarrow{\pi} G$$

is an exact sequence of groups and such that the induced action of G on M caused by the action of the crossed module E coincides with the a priori given action of G on M , that is, such that ${}^e(m\iota) = ((e\pi)m)\iota$ for $e \in \text{Gp } E$ and $m \in M$.

By abuse of notation, we often refer to the crossed module extension as well as to its underlying crossed module by E . The morphism ι is said to be the *canonical monomorphism* and the morphism π is said to be the *canonical epimorphism* of the crossed module extension E .

Given a crossed module extension E of G with M with canonical monomorphism ι and canonical epimorphism π , we write $\iota = \iota^E := \iota$ and $\pi = \pi^E := \pi$.

- (b) We suppose given a Grothendieck universe \mathfrak{U} such that G and M are in \mathfrak{U} . A crossed module extension is said to be a \mathfrak{U} -crossed module extension if its underlying crossed module is a \mathfrak{U} -crossed module. The set of \mathfrak{U} -crossed module extensions of G with M will be denoted by $\underline{\text{Ext}}^2(G, M) = \underline{\text{Ext}}_{\mathfrak{U}}^2(G, M)$.

(4.2) Remark.

- (a) We have $\pi_0(E) \cong G$ and $\pi_1(E) \cong M$ for every crossed module extension E of G with M .
- (b) Conversely, given an arbitrary crossed module V , we get a crossed module extension of $\pi_0(V)$ with $\pi_1(V)$, where $\iota = \text{inc}^{\pi_1(V)}$ and $\pi = \text{quo}^{\pi_0(V)}$.

$$\pi_1(V) \xrightarrow{\text{inc}} \text{Mp } V \xrightarrow{\mu} \text{Gp } V \xrightarrow{\text{quo}} \pi_0(V)$$

(4.3) Example.

- (a) The trivial homomorphism crossed module $[M \ G]$ provides a crossed module extension together with id_M as canonical monomorphism and id_G as canonical epimorphism, the *trivial crossed module extension* of G with M .

$$M \xrightarrow{\text{id}_M} M \xrightarrow{\text{triv}} G \xrightarrow{\text{id}_G} G$$

- (b) We suppose given a group E_0 and a group epimorphism $\pi: E_0 \rightarrow G$. Then the normal subgroup crossed module $[\text{Ker } \pi \trianglelefteq E_0]$ yields a crossed module extension of G with 0 , where the canonical monomorphism is trivial and the canonical epimorphism is π .

$$0 \longrightarrow \text{Ker } \pi \xrightarrow{\text{inc}} E_0 \xrightarrow{\pi} G$$

(4.4) Definition (equivalence of crossed module extensions).

- (a) We let E and \tilde{E} be crossed module extensions of G with M . An (*extension*) *equivalence* from E to \tilde{E} is a morphism of crossed modules $\varphi: E \rightarrow \tilde{E}$ such that $\iota^{\tilde{E}} = \iota^E(\text{Mp } \varphi)$ and $\pi^{\tilde{E}} = (\text{Gp } \varphi)\pi^E$.

$$\begin{array}{ccccccc} M & \xrightarrow{\iota^E} & \text{Mp } E & \xrightarrow{\mu^E} & \text{Gp } E & \xrightarrow{\pi^E} & G \\ \parallel & & \downarrow \text{Mp } \varphi & & \downarrow \text{Gp } \varphi & & \parallel \\ M & \xrightarrow{\iota^{\tilde{E}}} & \text{Mp } \tilde{E} & \xrightarrow{\mu^{\tilde{E}}} & \text{Gp } \tilde{E} & \xrightarrow{\pi^{\tilde{E}}} & G \end{array}$$

- (b) We suppose given a Grothendieck universe \mathfrak{U} such that G is a \mathfrak{U} -group and M is a \mathfrak{U} - G -module. We let $\approx = \approx_{\mathfrak{U}}$ be the equivalence relation on $\underline{\text{Ext}}_{\mathfrak{U}}^2(G, M)$ generated by the following relation: Given extensions $E, \tilde{E} \in \underline{\text{Ext}}_{\mathfrak{U}}^2(G, M)$, the extension E is in relation to the extension \tilde{E} if there exists an extension equivalence $E \rightarrow \tilde{E}$. Given crossed module extensions E and \tilde{E} with $E \approx \tilde{E}$, we say that E and \tilde{E} are (*extension*) *equivalent*. The set of equivalence classes of \mathfrak{U} -crossed module extensions of G with M with respect to $\approx_{\mathfrak{U}}$ is denoted by $\text{Ext}^2(G, M) = \text{Ext}_{\mathfrak{U}}^2(G, M) := \underline{\text{Ext}}_{\mathfrak{U}}^2(G, M)/\approx_{\mathfrak{U}}$, and an element of $\text{Ext}^2(G, M)$ is said to be a \mathfrak{U} -crossed module extension class of G with M .

(4.5) Remark.

- (a) Every extension equivalence $\varphi: E \rightarrow \tilde{E}$ between crossed module extensions E and \tilde{E} of G with M is a weak homotopy equivalence between the underlying crossed modules of E and \tilde{E} .
- (b) Given a weak homotopy equivalence $\varphi: V \rightarrow W$ between crossed modules V and W , there exist structures of crossed module extensions on V and W such that φ is an extension equivalence.

(4.6) Example. We suppose given a group E_0 and a group epimorphism $\pi: E_0 \rightarrow G$. Then

$$[\text{Ker } \pi \trianglelefteq E_0] \approx [0 \ G]$$

since π induces an extension equivalence.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ker } \pi & \xrightarrow{\text{inc}} & E_0 & \xrightarrow{\pi} & G \\ \parallel & & \downarrow & & \downarrow \pi & & \parallel \\ 0 & \xlongequal{\quad} & 0 & \longrightarrow & G & \xlongequal{\quad} & G \end{array}$$

5 The associated cohomology class

During this section, we suppose given a group G and an abelian G -module M .

The aim of this manuscript is to show that there is a bijection between the set of crossed module extension classes $\text{Ext}^2(G, M)$ and the third cohomology group $H^3(G, M)$, see theorem (6.11). Since $H^3(G, M) \cong H_{\text{cpt}}^3(G, M)$ by corollary (3.7)(c), we are able to work with componentwise pointed cocycles and coboundaries. Most steps of the construction can be done with “unpointed” data, but it seems to the author that componentwise pointedness cannot be avoided in the proofs of proposition (5.19) and proposition (6.6). So for convenience, we will work with pointed sets and componentwise pointed maps throughout the whole procedure.

We start by constructing for a given crossed module extension class of G with M a cohomology class in $H_{\text{cpt}}^3(G, M)$. The arguments used here are adapted from [5, sec. 7].

(5.1) Remark. We suppose given a group E_0 and an epimorphism $\pi: E_0 \rightarrow G$. For every section s^0 of the underlying pointed map of π , the map

$$z^2 = z_{E_0, s^0}^2: G \times G \rightarrow \text{Ker } \pi, (h, g) \mapsto (hs^0)(gs^0)((hg)s^0)^{-1}$$

is well-defined, componentwise pointed and fulfills

$$(k, h)z^2(kh, g)z^2 = {}^{ks^0}((h, g)z^2)(k, hg)z^2$$

for $g, h, k \in G$.

Proof. We suppose given a section $s^0: G \rightarrow E_0$ of the underlying pointed map of π . Since π is a group homomorphism, we have $(hs^0)(gs^0)((hg)s^0)^{-1} \in \text{Ker } \pi$ for $g, h \in G$. That is, we obtain a well-defined map $z^2: G \times G \rightarrow \text{Ker } \pi$ given by $(h, g)z^2 := (hs^0)(gs^0)((hg)s^0)^{-1}$, that is, such that

$$(hs^0)(gs^0) = (h, g)z^2(hg)s^0$$

for $g, h \in G$. Since s^0 is pointed, we have

$$\begin{aligned} (g, 1)z^2 &= (gs^0)(1s^0)((g1)s^0)^{-1} = 1 \text{ and} \\ (1, g)z^2 &= (1s^0)(gs^0)((1g)s^0)^{-1} = 1 \end{aligned}$$

for all $g \in G$, that is, z^2 is componentwise pointed. By computing the product $(ks^0)(hs^0)(gs^0)$ in E_0 for $g, h, k \in G$ in two different ways, we get on the one hand

$$((ks^0)(hs^0))(gs^0) = (k, h)z^2(kh)s^0(gs^0) = (k, h)z^2(kh, g)z^2(khg)s^0,$$

and on the other hand

$$(ks^0)((hs^0)(gs^0)) = (ks^0)(h, g)z^2(hg)s^0 = {}^{ks^0}((h, g)z^2)(ks^0)(hg)s^0 = {}^{ks^0}((h, g)z^2)(k, hg)z^2(khg)s^0.$$

Hence z^2 fulfills

$$((k, h)z^2)((kh, g)z^2) = {}^{ks^0}((h, g)z^2)((k, hg)z^2)$$

for $g, h, k \in G$. □

(5.2) Definition (non-abelian 2-cocycle of a crossed module extension).

- (a) We suppose given a group E_0 and an epimorphism $\pi: E_0 \rightarrow G$. Given a section s^0 of the underlying pointed map of π , we call

$$z^2 = z_{E_0, s^0}^2: G \times G \rightarrow \text{Ker } \pi, (h, g) \mapsto (hs^0)(gs^0)((hg)s^0)^{-1}$$

the *non-abelian 2-cocycle* of E_0 with respect to s^0 . ⁽¹⁾

¹The G -module $\text{Ker } \pi$ is non-abelian in general. However, if $\text{Ker } \pi$ is abelian, then z^2 is the well-known 2-cocycle in $Z^2(G, \text{Ker } \pi)$ of the group extension E_0 of G with $\text{Ker } \pi$.

- (b) Given a crossed module extension E of G with M and a section s^0 of the underlying pointed map of π , the non-abelian 2-cocycle $z_{\text{Gp } E, s^0}^2$ of $\text{Gp } E$ with respect to s^0 is also said to be the *non-abelian 2-cocycle* of E with respect to s^0 and is also denoted by $z^2 = z_{E, s^0}^2 := z_{\text{Gp } E, s^0}^2$.

In the situation of definition (5.2)(a), the identity $z^1 := \text{id}_G: G \rightarrow G$ fulfills the “non-abelian 1-cocycle condition” ${}^h((g)z^1)((h)z^1) = (hg)z^1$ for $g, h \in G$. Setting $Z^1 := z^1 s^0 = s^0$, we have $(h, g)z^2 = {}^{hZ^1}((g)Z^1)((h)Z^1)((hg)Z^1)^{-1}$ for $g, h \in G$. Cf. definition (5.6) and definition (5.8).

(5.3) Definition (lifting and section systems for crossed module extensions). We suppose given a crossed module extension E of G with M .

- (a) A *lifting system* for E is a pair (Z^2, Z^1) consisting of a lift $Z^1: G \rightarrow \text{Gp } E$ of id_G along the underlying pointed map of π and a lift $Z^2: G \times G \rightarrow \text{Mp } E$ of z_{E, Z^1}^2 along the underlying pointed map of $\mu|^{\text{Im } \mu}$ such that Z^2 is componentwise pointed.
- (b) A *section system* for E is a pair (s^1, s^0) consisting of a section $s^0: G \rightarrow \text{Gp } E$ of the underlying pointed map of π and a section $s^1: \text{Im } \mu \rightarrow \text{Mp } E$ of the underlying pointed map of $\mu|^{\text{Im } \mu}$.

(5.4) Example. The unique section system for the trivial crossed module extension $[M \ G]$ of G with M is given by $(\text{triv}, \text{id}_G)$.

(5.5) Remark. We suppose given a crossed module extension E of G with M . Every section system (s^1, s^0) for E provides a lifting system (Z^2, Z^1) for E , where $Z^1 := s^0$ and $Z^2 := z^2 s^1$.

Proof. We suppose given a section system (s^1, s^0) for E . Then $Z^1 := s^0$ is a section of π and hence a lift of id_G along the underlying pointed map of π . Further, $Z^2 := z^2 s^1$ is a lift of z^2 along the underlying pointed map of $\mu|^{\text{Im } \mu}$. It is componentwise pointed since z^2 is componentwise pointed by remark (5.1) and s^1 is pointed by assumption. \square

(5.6) Definition (lifting systems coming from section systems). Given a crossed module extension E of G with M and a section system (s^1, s^0) for E , we say that a lifting system (Z^2, Z^1) for E *comes from* (s^1, s^0) if $Z^1 = s^0$ and $Z^2 = z^2 s^1$.

(5.7) Remark. We suppose given a crossed module extension E of G with M . For every lifting system (Z^2, Z^1) for E , the map

$$z^3 = z_{E, (Z^2, Z^1)}^3: G \times G \times G \rightarrow M,$$

$$(k, h, g) \mapsto ((k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1}({}^{kZ^1}((h, g)Z^2))^{-1})(\iota|^{\text{Im } \iota})^{-1}$$

is a well-defined componentwise pointed 3-cocycle of G with values in M .

Proof. We suppose given a lifting system (Z^2, Z^1) for E . By remark (5.1), we have

$$(k, h)z^2(kh, g)z^2 = {}^{kZ^1}((h, g)z^2)(k, hg)z^2$$

for $g, h, k \in G$. Hence it follows that $(k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1}({}^{kZ^1}((h, g)Z^2))^{-1} \in \text{Ker } \mu = \text{Im } \iota$ for $g, h, k \in G$. Since ι is injective, we obtain a well-defined map $z^3: G \times G \times G \rightarrow M$ given by $(k, h, g)z^3 := ((k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1}({}^{kZ^1}((h, g)Z^2))^{-1})(\iota|^{\text{Im } \iota})^{-1}$, that is, such that

$$(k, h)Z^2(kh, g)Z^2 = (k, h, g)z^3 \iota^{kZ^1}((h, g)Z^2)(k, hg)Z^2.$$

Since Z^1 and Z^2 are componentwise pointed, we have

$$(h, g, 1)z^3 = ((h, g)Z^2(hg, 1)Z^2((h, g)Z^2)^{-1}({}^{hZ^1}((g, 1)Z^2))^{-1})(\iota|^{\text{Im } \iota})^{-1} = 0,$$

$$(h, 1, g)z^3 = ((h, 1)Z^2(h, g)Z^2((h, g)Z^2)^{-1}({}^{hZ^1}((1, g)Z^2))^{-1})(\iota|^{\text{Im } \iota})^{-1} = 0,$$

$$(1, h, g)z^3 = ((1, h)Z^2(h, g)Z^2((1, hg)Z^2)^{-1}({}^{1Z^1}((h, g)Z^2))^{-1})(\iota|^{\text{Im } \iota})^{-1} = 0$$

for all $g, h \in G$, that is, z^3 is also componentwise pointed. To show that $z^3 \in Z_{\text{cpt}}^3(G, M)$, we compute $(l, k)Z^2(lk, h)Z^2(lkh, g)Z^2$ for $g, h, k, l \in G$ in two different ways. On the one hand, we have

$$(l, k)Z^2(lk, h)Z^2(lkh, g)Z^2 = (l, k)Z^2(lk, h, g)z^3 \iota^{(lk)Z^1}((h, g)Z^2)(lk, hg)Z^2$$

$$\begin{aligned}
&= (lk, h, g)z^3 \iota(l, k)Z^2 \iota^{(lk)Z^1}((h, g)Z^2)(lk, hg)Z^2 \\
&= (lk, h, g)z^3 \iota^{(l, k)Z^2} \iota^{(lk)Z^1}((h, g)Z^2)(l, k)Z^2(lk, hg)Z^2 \\
&= (lk, h, g)z^3 \iota^{(lZ^1)(kZ^1)}((h, g)Z^2)(l, k, hg)z^3 \iota^{lZ^1}((k, hg)Z^2)(l, khg)Z^2 \\
&= (lk, h, g)z^3 \iota(l, k, hg)z^3 \iota^{(lZ^1)(kZ^1)}((h, g)Z^2) \iota^{lZ^1}((k, hg)Z^2)(l, khg)Z^2 \\
&= ((lk, h, g)z^3 + (l, k, hg)z^3) \iota^{(lZ^1)(kZ^1)}((h, g)Z^2) \iota^{lZ^1}((k, hg)Z^2)(l, khg)Z^2,
\end{aligned}$$

and on the other hand, we get

$$\begin{aligned}
(l, k)Z^2(lk, h)Z^2(lkh, g)Z^2 &= (l, k, h)z^3 \iota^{lZ^1}((k, h)Z^2)(l, kh)Z^2(lkh, g)Z^2 \\
&= (l, k, h)z^3 \iota^{lZ^1}((k, h)Z^2)(l, kh, g)z^3 \iota^{lZ^1}((kh, g)Z^2)(l, khg)Z^2 \\
&= (l, k, h)z^3 \iota(l, kh, g)z^3 \iota^{lZ^1}((k, h)Z^2) \iota^{lZ^1}((kh, g)Z^2)(l, khg)Z^2 \\
&= (l, k, h)z^3 \iota(l, kh, g)z^3 \iota^{lZ^1}((k, h)Z^2(kh, g)Z^2)(l, khg)Z^2 \\
&= (l, k, h)z^3 \iota(l, kh, g)z^3 \iota^{lZ^1}((k, h, g)z^3 \iota^{kZ^1}((h, g)Z^2)(k, hg)Z^2)(l, khg)Z^2 \\
&= (l, k, h)z^3 \iota(l, kh, g)z^3 \iota^{lZ^1}((k, h, g)z^3 \iota)^{(lZ^1)(kZ^1)}((h, g)Z^2) \iota^{lZ^1}((k, hg)Z^2)(l, khg)Z^2 \\
&= ((l, k, h)z^3 + (l, kh, g)z^3 + l \cdot (k, h, g)z^3) \iota^{(lZ^1)(kZ^1)}((h, g)Z^2) \iota^{lZ^1}((k, hg)Z^2)(l, khg)Z^2.
\end{aligned}$$

By the injectivity of ι , we conclude that

$$(lk, h, g)z^3 + (l, k, hg)z^3 = (l, k, h)z^3 + (l, kh, g)z^3 + l \cdot (k, h, g)z^3$$

for $g, h, k, l \in G$, that is, $z^3 \in \mathbf{Z}_{\text{cpt}}^3(G, M)$. \square

(5.8) Definition (3-cocycle of a crossed module extension with respect to a lifting system). We suppose given a crossed module extension E of G with M .

(a) Given a lifting system (Z^2, Z^1) for E , we call

$$\begin{aligned}
z^3 &= z_{E, (Z^2, Z^1)}^3: G \times G \times G \rightarrow M, \\
(k, h, g) &\mapsto ((k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1}(kZ^1((h, g)Z^2))^{-1})(\iota|_{\text{Im } \iota})^{-1}
\end{aligned}$$

the 3-cocycle of E with respect to (Z^2, Z^1) .

(b) Given a section system (s^1, s^0) , the 3-cocycle of E with respect to the lifting system (Z^2, Z^1) coming from (s^1, s^0) is also called the 3-cocycle of E with respect to (s^1, s^0) and denoted by $z^3 = z_{E, (s^1, s^0)}^3 := z_{E, (Z^2, Z^1)}^3$.

(5.9) Example. As we have seen in example (5.4), the unique section system for $[M \ G]$ is given by $(\text{triv}, \text{id}_G)$. The 3-cocycle of $[M \ G]$ with respect to $(\text{triv}, \text{id}_G)$ is the trivial 3-cocycle $0 \in \mathbf{Z}_{\text{cpt}}^3(G, M)$.

(5.10) Proposition. We suppose given a crossed module extension E of G with M and a lifting system (Z^2, Z^1) for E .

(a) The maps $\tilde{Z}^2: G \times G \rightarrow \text{Mp } E$ such that (\tilde{Z}^2, Z^1) is a lifting system for E are exactly the maps of the form $G \times G \rightarrow \text{Mp } E, (h, g) \mapsto (h, g)c^2 \iota(h, g)Z^2$ for some componentwise pointed 2-cochain $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$.

(b) For every 2-cochain $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$, the 3-cocycle $z_{E, (\tilde{Z}^2, Z^1)}^3$ of E with respect to the lifting system (\tilde{Z}^2, Z^1) , where $(h, g)\tilde{Z}^2 := (h, g)c^2 \iota(h, g)Z^2$ for $g, h \in G$, is given by $z_{E, (\tilde{Z}^2, Z^1)}^3 = c^2 \partial + z_{E, (Z^2, Z^1)}^3$.

Proof.

(a) First, we suppose given a componentwise pointed 2-cochain $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$. Then the map $\tilde{Z}^2: G \times G \rightarrow \text{Mp } E, (h, g) \mapsto (h, g)c^2 \iota(h, g)Z^2$ is componentwise pointed since c^2, Z^2 and ι are componentwise pointed. Moreover, we have $\tilde{Z}^2 \mu|_{\text{Im } \mu} = Z^2 \mu|_{\text{Im } \mu} = z^2$, that is, \tilde{Z}^2 is a lift of z^2 along the underlying pointed map of $\mu|_{\text{Im } \mu}$. Hence (\tilde{Z}^2, Z^1) is a lifting system for E .

Conversely, we suppose given a lifting system (\tilde{Z}^2, Z^1) for E . Then Z^2 and \tilde{Z}^2 are componentwise pointed and lifts of z^2 along the underlying pointed map of $\mu|_{\text{Im } \mu}$, that is, we have $(h, g)Z^2 \mu = (h, g)\tilde{Z}^2 \mu = (h, g)z^2$

for $g, h \in G$. It follows that $(h, g)\tilde{Z}^2((h, g)Z^2)^{-1} \in \text{Ker } \mu = \text{Im } \iota$ for $g, h \in G$. Hence we obtain a map $c^2: G \times G \rightarrow M$, $(h, g) \mapsto ((h, g)\tilde{Z}^2((h, g)Z^2)^{-1})(\iota|_{\text{Im } \iota})^{-1}$, that is, such that

$$(h, g)\tilde{Z}^2 = (h, g)c^2\iota(h, g)Z^2$$

for $g, h \in G$. Finally, c^2 is componentwise pointed since Z^2 , \tilde{Z}^2 and ι are componentwise pointed.

(b) We suppose given $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$ and we define $\tilde{Z}^2: G \times G \rightarrow \text{Mp } E$, $(h, g) \mapsto (h, g)c^2\iota(h, g)Z^2$. By (a), (\tilde{Z}^2, Z^1) is a lifting system for E . We get

$$\begin{aligned} (k, h)\tilde{Z}^2(kh, g)\tilde{Z}^2 &= (k, h)c^2\iota(k, h)Z^2(kh, g)c^2\iota(kh, g)Z^2 = (k, h)c^2\iota(kh, g)c^2\iota(k, h)Z^2(kh, g)Z^2 \\ &= (k, h)c^2\iota(kh, g)c^2\iota(k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{kZ^1}((h, g)Z^2)(k, hg)Z^2 \\ &= (k, h)c^2\iota(kh, g)c^2\iota(k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{kZ^1}(((h, g)c^2\iota)^{-1}(h, g)\tilde{Z}^2)((k, hg)c^2\iota)^{-1}(k, hg)\tilde{Z}^2 \\ &= (k, h)c^2\iota(kh, g)c^2\iota(k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{kZ^1}(((h, g)c^2\iota)^{-1})^{kZ^1}((h, g)\tilde{Z}^2)((k, hg)c^2\iota)^{-1}(k, hg)\tilde{Z}^2 \\ &= (k, h)c^2\iota(kh, g)c^2\iota((k, hg)c^2\iota)^{-1}kZ^1(((h, g)c^2\iota)^{-1})(k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{kZ^1}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2 \\ &= ((k, h)c^2 + (kh, g)c^2 - (k, hg)c^2 - k \cdot (h, g)c^2 + (k, h, g)z_{E, (Z^2, Z^1)}^3)\iota^{kZ^1}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2 \\ &= ((k, h, g)(c^2\partial) + (k, h, g)z_{E, (Z^2, Z^1)}^3)\iota^{kZ^1}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2 \end{aligned}$$

and thus $(k, h, g)z_{E, (\tilde{Z}^2, Z^1)}^3 = (k, h, g)(c^2\partial) + (k, h, g)z_{E, (Z^2, Z^1)}^3$ for $g, h, k \in G$, that is,

$$z_{E, (\tilde{Z}^2, Z^1)}^3 = c^2\partial + z_{E, (Z^2, Z^1)}^3. \quad \square$$

(5.11) Proposition. We have a map

$$\text{cl}: \underline{\text{Ext}}^2(G, M) \rightarrow \text{H}_{\text{cpt}}^3(G, M)$$

that assigns to every crossed module extension E of G with M the cohomology class of the 3-cocycle of E with respect to an arbitrarily chosen lifting system. The map cl is independent from the chosen lifting system.

Proof. We suppose given a crossed module extension E of G with M and we choose a lifting system (Z^2, Z^1) for E . By proposition (5.10), the cohomology class of $z_{E, (Z^2, Z^1)}^3$ is independent from the choice of Z^2 . Thus it remains to show that the cohomology class of $z_{E, (Z^2, Z^1)}^3$ is independent from the choice of Z^1 . To this end, we let \tilde{Z}^1 be an alternative to Z^1 , that is, a section of the underlying pointed map of π . Then $(g\tilde{Z}^1)(gZ^1)^{-1} \in \text{Ker } \pi = \text{Im } \mu$ for $g \in G$, and we obtain a well-defined pointed map $c^1: G \rightarrow \text{Im } \mu$, $g \mapsto (g\tilde{Z}^1)(gZ^1)^{-1}$, that is, such that

$$g\tilde{Z}^1 = (gc^1)(gZ^1)$$

for $g \in G$. This implies

$$\begin{aligned} (h\tilde{Z}^1)(g\tilde{Z}^1) &= (hc^1)(hZ^1)(gc^1)(gZ^1) = (hc^1)^{hZ^1}(gc^1)(hZ^1)(gZ^1) = (hc^1)^{hZ^1}(gc^1)(h, g)z_{E, Z^1}^2(hg)Z^1 \\ &= (hc^1)^{hZ^1}(gc^1)(h, g)z_{E, Z^1}^2((hg)c^1)^{-1}(hg)\tilde{Z}^1 \end{aligned}$$

and hence

$$(h, g)z_{E, \tilde{Z}^1}^2 = (hc^1)^{hZ^1}(gc^1)(h, g)z_{E, Z^1}^2((hg)c^1)^{-1}$$

for $g, h \in G$. We let $C^1: G \rightarrow \text{Mp } E$ be a lift of c^1 along the underlying pointed map of $\mu|_{\text{Im } \mu}$, that is, a pointed map $C^1: G \rightarrow \text{Mp } E$ such that $C^1(\mu|_{\text{Im } \mu}) = c^1$. Moreover, we define a lift $\tilde{Z}^2: G \times G \rightarrow \text{Mp } E$ of z_{E, \tilde{Z}^1}^2 along the underlying pointed map of $\mu|_{\text{Im } \mu}$ by $(h, g)\tilde{Z}^2 := (hC^1)^{hZ^1}(gC^1)(h, g)Z^2((hg)C^1)^{-1}$ ⁽²⁾, that is, such that

$$(h, g)\tilde{Z}^2(hg)C^1 = (hC^1)^{hZ^1}(gC^1)(h, g)Z^2$$

²This is possible since the independence from the choice of this lift has already been shown.

for $g, h \in G$. Since Z^1 , C^1 and Z^2 are componentwise pointed, we have

$$(g, 1)\tilde{Z}^2 = (gC^1)^{gZ^1}(1C^1)(g, 1)Z^2(gC^1)^{-1} = 1 \text{ and}$$

$$(1, g)\tilde{Z}^2 = (1C^1)^{1Z^1}(gC^1)(1, g)Z^2(gC^1)^{-1} = 1$$

for all $g \in G$, whence \tilde{Z}^2 is also componentwise pointed. Finally, we compute

$$\begin{aligned} (k, h)\tilde{Z}^2(kh, g)\tilde{Z}^2(khg)C^1 &= (k, h)\tilde{Z}^2(kh)C^1{}^{(kh)Z^1}(gC^1)(kh, g)Z^2 \\ &= (kC^1)^{kZ^1}(hC^1)(k, h)Z^2{}^{(kh)Z^1}(gC^1)(kh, g)Z^2 \\ &= (kC^1)^{kZ^1}(hC^1)^{(k, h)Z^2(kh)Z^1}(gC^1)(k, h)Z^2(kh, g)Z^2 \\ &= (kC^1)^{kZ^1}(hC^1)^{(kZ^1)(hZ^1)}(gC^1)(k, h, g)z_{E, (Z^2, Z^1)}^3 \iota^{kZ^1}((h, g)Z^2)(k, hg)Z^2 \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota(kC^1)^{kZ^1}(hC^1)^{(kZ^1)(hZ^1)}(gC^1)^{kZ^1}((h, g)Z^2)(k, hg)Z^2 \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota(kC^1)^{kZ^1}((hC^1)^{hZ^1}(gC^1)(h, g)Z^2)(k, hg)Z^2 \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota(kC^1)^{kZ^1}((h, g)\tilde{Z}^2(hg)C^1)(k, hg)Z^2 \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota^{(kC^1)(kZ^1)}((h, g)\tilde{Z}^2)(kC^1)^{kZ^1}((hg)C^1)(k, hg)Z^2 \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota^{(kc^1)(kZ^1)}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2(khg)C^1 \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota^{k\tilde{Z}^1}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2(khg)C^1 \end{aligned}$$

for $g, h, k \in G$. It follows that

$$(k, h)\tilde{Z}^2(kh, g)\tilde{Z}^2 = (k, h, g)z_{E, (Z^2, Z^1)}^3 \iota^{k\tilde{Z}^1}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2$$

for $g, h, k \in G$, that is, $z_{E, (\tilde{Z}^2, \tilde{Z}^1)}^3 = z_{E, (Z^2, Z^1)}^3$. \square

(5.12) Definition (cohomology class associated to a crossed module extension). Given a crossed module extension E of G with M , the cohomology class $\text{cl}(E) := z_{E, (Z^2, Z^1)}^3 + \text{B}_{\text{cpt}}^3(G, M) \in \text{H}_{\text{cpt}}^3(G, M)$ for an arbitrarily chosen lifting system (Z^2, Z^1) for E is called the *cohomology class associated to E* .

(5.13) Example. Following example (5.9), we have $\text{cl}([M \ G]) = 0$.

Our next aim is to show that the cohomology class associated to a crossed module extension is independent from a chosen representative in its crossed module extension class.

(5.14) Proposition. We let E and \tilde{E} be crossed module extensions of G with M and we let $\varphi: E \rightarrow \tilde{E}$ be an extension equivalence.

$$\begin{array}{ccccccc} M & \xrightarrow{\iota^E} & \text{Mp } E & \xrightarrow{\mu^E} & \text{Gp } E & \xrightarrow{\pi^E} & G \\ \parallel & & \downarrow \text{Mp } \varphi & & \downarrow \text{Gp } \varphi & & \parallel \\ M & \xrightarrow{\iota^{\tilde{E}}} & \text{Mp } \tilde{E} & \xrightarrow{\mu^{\tilde{E}}} & \text{Gp } \tilde{E} & \xrightarrow{\pi^{\tilde{E}}} & G \end{array}$$

- We suppose given a section s^0 of the underlying pointed map of π^E and a section \tilde{s}^0 of the underlying pointed map of $\pi^{\tilde{E}}$ with $\tilde{s}^0 = s^0(\text{Gp } \varphi)$. Then we have $z_{\tilde{E}, \tilde{s}^0}^2 = z_{E, s^0}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$.
- We suppose given a lifting system (Z^2, Z^1) for E and a lifting system $(\tilde{Z}^2, \tilde{Z}^1)$ for \tilde{E} with $\tilde{Z}^1 = Z^1(\text{Gp } \varphi)$ and $\tilde{Z}^2 = Z^2(\text{Mp } \varphi)$. Then we have $z_{\tilde{E}, \tilde{Z}^1}^2 = z_{E, Z^1}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$ and $z_{\tilde{E}, (\tilde{Z}^2, \tilde{Z}^1)}^3 = z_{E, (Z^2, Z^1)}^3$.

In particular, $\text{cl}(\tilde{E}) = \text{cl}(E)$.

- We suppose given a section system (s^1, s^0) for E and a section system $(\tilde{s}^1, \tilde{s}^0)$ for \tilde{E} with $\tilde{s}^0 = s^0(\text{Gp } \varphi)$ and $s^1(\text{Mp } \varphi) = (\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}} \tilde{s}^1$. Moreover, we let (Z^2, Z^1) be the lifting system coming from (s^1, s^0) and $(\tilde{Z}^2, \tilde{Z}^1)$ be the section system coming from $(\tilde{s}^1, \tilde{s}^0)$. Then we have $\tilde{Z}^1 = Z^1(\text{Gp } \varphi)$ and $\tilde{Z}^2 = Z^2(\text{Mp } \varphi)$ as well as $z_{\tilde{E}, \tilde{s}^0}^2 = z_{E, s^0}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$ and $z_{\tilde{E}, (\tilde{s}^1, \tilde{s}^0)}^3 = z_{E, (s^1, s^0)}^3$.

Proof.

(a) We have

$$\begin{aligned} (h\tilde{s}^0)(g\tilde{s}^0) &= (hs^0\varphi)(gs^0\varphi) = ((hs^0)(gs^0))\varphi = ((h, g)z_{E, s^0}^2(hg)s^0)\varphi = (h, g)z_{E, s^0}^2\varphi(hg)s^0\varphi \\ &= (h, g)z_{E, s^0}^2\varphi(hg)\tilde{s}^0 \end{aligned}$$

for $g, h \in G$ and thus $z_{\tilde{E}, \tilde{s}^0}^2 = z_{E, s^0}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$.

(b) By (a), we have $z_{\tilde{E}, \tilde{Z}^1}^2 = z_{E, Z^1}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$. Further, we obtain

$$\begin{aligned} (k, h)\tilde{Z}^2(kh, g)\tilde{Z}^2 &= (k, h)Z^2\varphi(kh, g)Z^2\varphi = ((k, h)Z^2(kh, g)Z^2)\varphi \\ &= ((k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{E, kZ^1}((h, g)Z^2)(k, hg)Z^2)\varphi \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{E, kZ^1}\varphi((h, g)Z^2\varphi)(k, hg)Z^2\varphi \\ &= (k, h, g)z_{E, (Z^2, Z^1)}^3\iota^{\tilde{E}, k\tilde{Z}^1}((h, g)\tilde{Z}^2)(k, hg)\tilde{Z}^2 \end{aligned}$$

for $g, h, k \in G$, that is, $z_{\tilde{E}, (\tilde{Z}^2, \tilde{Z}^1)}^3 = z_{E, (Z^2, Z^1)}^3$. In particular, it follows that

$$\text{cl}(\tilde{E}) = z_{\tilde{E}, (\tilde{Z}^2, \tilde{Z}^1)}^3 + \text{B}_{\text{cpt}}^3(G, M) = z_{E, (Z^2, Z^1)}^3 + \text{B}_{\text{cpt}}^3(G, M) = \text{cl}(E).$$

(c) First of all, we get $z_{\tilde{E}, \tilde{s}^0}^2 = z_{E, s^0}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$ by (a). Since the lifting systems (Z^2, Z^1) resp. $(\tilde{Z}^2, \tilde{Z}^1)$ come from the section systems (s^1, s^0) resp. $(\tilde{s}^1, \tilde{s}^0)$, we have $(Z^2, Z^1) = (z_{E, s^0}^2 s^1, s^0)$ and $(\tilde{Z}^2, \tilde{Z}^1) = (z_{\tilde{E}, \tilde{s}^0}^2 \tilde{s}^1, \tilde{s}^0)$. This implies

$$\tilde{Z}^1 = \tilde{s}^0 = s^0(\text{Gp } \varphi) = Z^1(\text{Gp } \varphi)$$

as well as

$$\tilde{Z}^2 = z_{\tilde{E}, \tilde{s}^0}^2 \tilde{s}^1 = z_{E, s^0}^2(\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}} \tilde{s}^1 = z_{E, s^0}^2 s^1(\text{Mp } \varphi) = Z^2(\text{Mp } \varphi).$$

Now (b) yields

$$z_{\tilde{E}, (\tilde{s}^1, \tilde{s}^0)}^3 = z_{\tilde{E}, (\tilde{Z}^2, \tilde{Z}^1)}^3 = z_{E, (Z^2, Z^1)}^3 = z_{E, (s^1, s^0)}^3. \quad \square$$

(5.15) Proposition. We let E and \tilde{E} be crossed module extensions of G with M and we let $\varphi: E \rightarrow \tilde{E}$ be an extension equivalence.

- (a) For every section s^0 of the underlying pointed map of π^E , the pointed map $\tilde{s}_0 := s_0(\text{Gp } \varphi)$ is a section of the underlying pointed map of $\pi^{\tilde{E}}$.
- (b) We suppose given a lifting system (Z^2, Z^1) for E . Setting $\tilde{Z}^1 := Z^1(\text{Gp } \varphi)$ and $\tilde{Z}^2 := Z^2(\text{Mp } \varphi)$, we obtain a lifting system $(\tilde{Z}^2, \tilde{Z}^1)$ for \tilde{E} .
- (c) For every section \tilde{s}^1 of the underlying pointed map of $\mu^{\tilde{E}}|_{\text{Im } \mu^{\tilde{E}}}$, there exists exactly one section s^1 of the underlying pointed map of $\mu^E|_{\text{Im } \mu^E}$ with $s^1(\text{Mp } \varphi) = (\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}} \tilde{s}^1$. It is constructed as follows: For an arbitrarily chosen section s'^1 of the underlying pointed map of $\mu^E|_{\text{Im } \mu^E}$, we have $gs^1 = ((g\varphi\tilde{s}^1)(gs'^1\varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1}\iota^E(gs'^1)$ for $g \in \text{Im } \mu^E$.

Proof.

(a) We suppose given a section s^0 of the underlying pointed map of π^E . Setting $\tilde{s}^0 := s^0(\text{Gp } \varphi)$, we get

$$\tilde{s}^0\pi^{\tilde{E}} = s^0(\text{Gp } \varphi)\pi^{\tilde{E}} = s^0\pi^E = \text{id}_G,$$

that is, \tilde{s}^0 is a section of the underlying pointed map of $\pi^{\tilde{E}}$.

- (b) We have $\tilde{Z}^1 \pi^{\tilde{E}} = \text{id}_G$ by (a) and therefore $z_{\tilde{E}, \tilde{Z}^1}^2 = z_{E, Z^1}^2 (\text{Gp } \varphi)|_{\text{Im } \mu^{\tilde{E}}}$ by proposition (5.14)(a). Further, \tilde{Z}^2 is a lift of $z_{\tilde{E}, \tilde{Z}^1}^2$ along the underlying pointed map of $\mu^{\tilde{E}}|_{\text{Im } \mu^{\tilde{E}}}$ since

$$\tilde{Z}^2(\mu^{\tilde{E}}|_{\text{Im } \mu^{\tilde{E}}}) = Z^2(\text{Mp } \varphi)(\mu^{\tilde{E}}|_{\text{Im } \mu^{\tilde{E}}}) = Z^2(\mu^E|_{\text{Im } \mu^E})((\text{Gp } \varphi)|_{\text{Im } \mu^E}) = z_{E, Z^1}^2((\text{Gp } \varphi)|_{\text{Im } \mu^E}) = z_{\tilde{E}, \tilde{Z}^1}^2,$$

and it is componentwise pointed since Z^2 and $\text{Mp } \varphi$ are componentwise pointed. Thus $(\tilde{Z}^2, \tilde{Z}^1)$ is a lifting system for E .

- (c) We suppose given a section \tilde{s}^1 of the underlying pointed map of $\mu^{\tilde{E}}|_{\text{Im } \mu^{\tilde{E}}}$ and we choose a section s'^1 of the underlying pointed map of $\mu^E|_{\text{Im } \mu^E}$. Then

$$gs'^1 \varphi \mu^{\tilde{E}} = gs'^1 \mu^E \varphi = g\varphi = g\varphi \tilde{s}^1 \mu^{\tilde{E}}$$

and hence $(g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1} \in \text{Ker } \mu^{\tilde{E}} = \text{Im } \iota^{\tilde{E}}$ for $g \in \text{Im } \mu^E$. Thus we obtain a well-defined pointed map

$$s^1: \text{Im } \mu^E \rightarrow \text{Mp } E, g \mapsto ((g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E (gs'^1)$$

with

$$gs^1 \mu^E = (((g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E (gs'^1)) \mu^E = gs'^1 \mu^E = g$$

for all $g \in \text{Im } \mu^E$, that is, s^1 is a section of the underlying pointed map of $\mu^E|_{\text{Im } \mu^E}$. Moreover, we have

$$\begin{aligned} gs^1 \varphi &= (((g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E (gs'^1)) \varphi = ((g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E \varphi (gs'^1) \\ &= ((g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^{\tilde{E}} (gs'^1 \varphi) = (g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1} (gs'^1 \varphi) = g\varphi \tilde{s}^1 \end{aligned}$$

for $g \in \text{Im } \mu^E$, that is, $s^1(\text{Mp } \varphi) = (\text{Gp } \varphi)|_{\text{Im } \mu^E} \tilde{s}^1$.

Conversely, given arbitrary sections s^1 and s'^1 of the underlying pointed map of $\mu^E|_{\text{Im } \mu^E}$ such that s^1 fulfills $s^1(\text{Mp } \varphi) = (\text{Gp } \varphi)|_{\text{Im } \mu^E} \tilde{s}^1$, it follows that $gs^1 \mu^E = g = gs'^1 \mu^E$ for all $g \in \text{Im } \mu^E$, that is, $(gs^1)(gs'^1)^{-1} \in \text{Ker } \mu^E = \text{Im } \iota^E$ and hence

$$\begin{aligned} gs^1 &= (gs^1)(gs'^1)^{-1}(gs'^1) = ((gs^1)(gs'^1)^{-1})(\iota^E|_{\text{Im } \iota^E})^{-1} \iota^E (gs'^1) \\ &= ((gs^1)(gs'^1)^{-1}) \varphi (\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E (gs'^1) = ((gs^1 \varphi)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E (gs'^1) \\ &= ((g\varphi \tilde{s}^1)(gs'^1 \varphi)^{-1})(\iota^{\tilde{E}}|_{\text{Im } \iota^{\tilde{E}}})^{-1} \iota^E (gs'^1) \end{aligned}$$

for all $g \in \text{Im } \mu^E$. □

(5.16) Corollary. We let E and \tilde{E} be crossed module extensions of G with M and we let $\varphi: E \rightarrow \tilde{E}$ be an extension equivalence.

- (a) There exists a lifting system (Z^2, Z^1) for E and a lifting system $(\tilde{Z}^2, \tilde{Z}^1)$ for \tilde{E} with $\tilde{Z}^1 = Z^1(\text{Gp } \varphi)$ and $\tilde{Z}^2 = Z^2(\text{Mp } \varphi)$.
- (b) There exists a section system (s^1, s^0) for E and a section system $(\tilde{s}^1, \tilde{s}^0)$ for \tilde{E} with $\tilde{s}^0 = s^0(\text{Gp } \varphi)$ and $s^1(\text{Mp } \varphi) = (\text{Gp } \varphi)|_{\text{Im } \mu^E} \tilde{s}^1$.

Proof.

- (a) This follows from proposition (5.15)(b).
- (b) We choose a section s^0 of the underlying pointed map of π^E and a section \tilde{s}^1 of the underlying pointed map of $\mu^{\tilde{E}}|_{\text{Im } \mu^{\tilde{E}}}$. By proposition (5.15)(a), $\tilde{s}^0 := s^0(\text{Gp } \varphi)$ is a section of the underlying pointed map of $\pi^{\tilde{E}}$, and by proposition (c), there exists a unique section s^1 of the underlying pointed map of $\mu^E|_{\text{Im } \mu^E}$ such that $s^1(\text{Mp } \varphi) = (\text{Gp } \varphi)|_{\text{Im } \mu^E} \tilde{s}^1$. □

We remark that corollary (5.16)(b) will be used in [16, prop. (4.11)].

(5.17) Corollary. The map $\text{cl}: \underline{\text{Ext}}^2(G, M) \rightarrow \text{H}_{\text{cpt}}^3(G, M)$ induces a well-defined map

$$\text{cl}: \text{Ext}^2(G, M) \rightarrow \text{H}_{\text{cpt}}^3(G, M), [E]_{\approx} \mapsto \text{cl}(E).$$

Proof. We suppose given crossed module extensions E and \tilde{E} of G with M and an extension equivalence $\varphi: E \rightarrow \tilde{E}$. Then by corollary (5.16)(a), there exists a lifting system (Z^2, Z^1) for E and a lifting system $(\tilde{Z}^2, \tilde{Z}^1)$ for \tilde{E} with $\tilde{Z}^1 = Z^1(\text{Gp } \varphi)$ and $\tilde{Z}^2 = Z^2(\text{Mp } \varphi)$ and proposition (5.14)(b) implies $\text{cl}(\tilde{E}) = \text{cl}(E)$. \square

(5.18) Definition (cohomology class associated to a crossed module extension class). Given a crossed module extension E of G with M , the 3-cohomology class $\text{cl}([E]_{\approx}) = \text{cl}(E) \in \text{H}_{\text{cpt}}^3(G, M)$ is also called the *cohomology class associated to* $[E]_{\approx}$.

(5.19) Proposition. We suppose given a crossed module extension E of G with M and a 3-cocycle $z^3 \in \text{Z}_{\text{cpt}}^3(G, M)$ with $\text{cl}(E) = z^3 + \text{B}_{\text{cpt}}^3(G, M)$. For every lift Z^1 of id_G along the underlying pointed map of π , there exists a map $Z^2: G \times G \rightarrow \text{Mp } E$ such that (Z^2, Z^1) is a lifting system for E with $z_{E, (Z^2, Z^1)}^3 = z^3$.

Proof. We suppose given a lift Z^1 of id_G along the underlying pointed map of π and we choose an arbitrary componentwise pointed map $\tilde{Z}^2: G \times G \rightarrow \text{Mp } E$ such that (\tilde{Z}^2, Z^1) is a lifting system for E , that is, with $\tilde{Z}^2(\mu^E |_{\text{Im } \mu^E}) = z_{E, Z^1}^2$. Then we have

$$z_{E, (\tilde{Z}^2, Z^1)}^3 + \text{B}_{\text{cpt}}^3(G, M) = \text{cl}(E) = z^3 + \text{B}_{\text{cpt}}^3(G, M),$$

that is, there exists a componentwise pointed 2-cochain $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$ with $z_{E, (\tilde{Z}^2, Z^1)}^3 = (c^2 \partial) + z^3$. Setting $(h, g)Z^2 := ((h, g)c^2 \iota^E)^{-1}(h, g)\tilde{Z}^2$, proposition (5.10) implies that (Z^2, Z^1) is a lifting system for E with $z_{E, (Z^2, Z^1)}^3 = z^3$. \square

We let E be a crossed module extension of G with M and we let $z^3 \in \text{Z}_{\text{cpt}}^3(G, M)$ be given with $\text{cl}(E) = z^3 + \text{B}_{\text{cpt}}^3(G, M)$. By the preceding proposition, every lift Z^1 of id_G along the underlying pointed map of π can be completed to a lifting system (Z^2, Z^1) for E such that z^3 is the 3-cocycle of E with respect to (Z^2, Z^1) . A lift of id_G along the underlying pointed map of π is nothing but a section of the underlying pointed map of π . The following example shows that in general not every section s^0 of the underlying pointed map of π can be completed to a section system (s^1, s^0) for E such that z^3 is the 3-cocycle of E with respect to (s^1, s^0) .

(5.20) Example. We suppose that G , M and the action of G on M are non-trivial. Then the coboundary group $\text{B}_{\text{cpt}}^3(G, M)$ has also at least one non-trivial element: We choose $g_0 \in G \setminus \{1\}$ and $m_0 \in M \setminus \{1\}$ such that $g_0 m_0 \neq m_0$, and we define $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$ by $(h, g)c^2 := \delta_{(h, g), (g_0, g_0)} m$ for $g, h \in G$. This leads to

$$(g_0, g_0, g_0)(c^2 \partial) = (g_0, g_0)c^2 - (g_0, g_0^2)c^2 + (g_0^2, g_0)c^2 - g_0 \cdot (g_0, g_0)c^2 = m_0 - g_0 m_0 \neq 0.$$

By example (5.9), the 3-cocycle of the trivial crossed module extension $[M \ G]$ with respect to the unique section system for $[M \ G]$ is the trivial cocycle $0 \in \text{Z}_{\text{cpt}}^3(G, M)$. Thus there cannot be a section system for $[M \ G]$ leading to $c^2 \partial$.

6 The standard extension

In this section, we construct for a given cocycle $z^3 \in \text{Z}_{\text{cpt}}^3(G, M)$, where G is a group and M is an abelian G -module, a crossed module extension of G with M whose associated cohomology class is $z^3 + \text{B}_{\text{cpt}}^3(G, M)$. The arguments are adapted from [12, sec. 7], [5, sec. 9]. At the very end, this leads to the classification theorem (6.11) due to HOLT [7, th. 4.5], HUEBSCHMANN [8, p. 310] and RATCLIFFE [14, th. 9.4].

Throughout the whole section, we suppose given a group G and an abelian G -module M .

Before we begin with the construction of the standard extension, we recall two facts from group theory.

(6.1) Remark. We suppose given group homomorphisms $\varphi_1: G_1 \rightarrow H$ and $\varphi_2: G_2 \rightarrow H$. The map $\varphi: G_1 \times G_2 \rightarrow H, (g_1, g_2) \rightarrow (g_1 \varphi_1)(g_2 \varphi_2)$ is a group homomorphism if and only if

$$(g_1 \varphi_1)(g_2 \varphi_2) = (g_2 \varphi_2)(g_1 \varphi_1)$$

for all $g_1 \in G_1, g_2 \in G_2$.

Proof. If φ is a group homomorphism, then we necessarily have

$$(g_1\varphi_1)(g_2\varphi_2) = (g_1, g_2)\varphi = ((1, g_2)(g_1, 1))\varphi = (1, g_2)\varphi(g_1, 1)\varphi = (1\varphi_1)(g_2\varphi_2)(g_1\varphi_1)(1\varphi_2) = (g_2\varphi_2)(g_1\varphi_1)$$

for all $g_1 \in G_1, g_2 \in G_2$. Conversely, if $(g_1\varphi_1)(g_2\varphi_2) = (g_2\varphi_2)(g_1\varphi_1)$ for all $g_1 \in G_1, g_2 \in G_2$, then we obtain

$$\begin{aligned} ((g_1, g_2)(g'_1, g'_2))\varphi &= (g_1g'_1, g_2g'_2)\varphi = (g_1g'_1)\varphi_1(g_2g'_2)\varphi_2 = (g_1\varphi_1)(g'_1\varphi_1)(g_2\varphi_2)(g'_2\varphi_2) \\ &= (g_1\varphi_1)(g_2\varphi_2)(g'_1\varphi_1)(g'_2\varphi_2) = (g_1, g_2)\varphi(g'_1, g'_2)\varphi \end{aligned}$$

for all $g_1, g'_1 \in G_1, g_2, g'_2 \in G_2$, that is, φ is a group homomorphism. \square

(6.2) Remark. We let F be a free group on the underlying pointed set of G with basis $s^0: G \rightarrow F$, and we let π be the extension of $\text{id}_G: G \rightarrow G$ to F .

- (a) The kernel $\text{Ker } \pi$ is a free group on the pointed set $\text{Im } z_{F, s^0}^2$.
- (b) Given $g, h \in G$, we have $(h, g)z_{F, s^0}^2 = 1$ if and only if $g = 1$ or $h = 1$.
- (c) Given $g, g', h, h' \in G \setminus \{1\}$ with $(h, g)z_{F, s^0}^2 = (h', g')z_{F, s^0}^2$, it follows that $(h, g) = (h', g')$.

Proof.

- (a) This follows from the Nielsen-Schreier theorem, see for example [9, §36, p. 36] (cf. also [12, sec. 7, p. 747]).
- (b) For $g, h \in G$, we have $(h, g)z^2 = 1$ if and only if $(hs^0)(gs^0) = (hg)s^0$. But since F is freely generated on the underlying pointed set of G and the length of $(hg)s^0$ for $g, h \in G$ is less than 2, this is equivalent to $g = 1$ or $h = 1$.
- (c) We suppose given $g, g', h, h' \in G \setminus \{1\}$ with $(h, g)z^2 = (h', g')z^2$. It follows that

$$(g's^0)^{-1}(h's^0)^{-1}(hs^0)(gs^0) = ((h'g')s^0)^{-1}(hg)s^0$$

and hence the length of this element is less or equal than 2. But since $g, g', h, h' \neq 1$, this implies that $(h's^0)^{-1}(hs^0) = 1$, that is, $hs^0 = h's^0$ and therefore $h = h'$. Thus we have

$$(hg')s^0(g's^0)^{-1}(gs^0) = (hg)s^0$$

and hence the length of $(hg')s^0(g's^0)^{-1}(gs^0)$ is less or equal than 1. But now $g, g' \neq 1$ and $hg' \neq g'$ implies $g' = g$. \square

(6.3) Proposition. We suppose given a 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$.

We let F be a free group on the underlying pointed set of G with basis $s^0 = Z^1: G \rightarrow F$. We let π be the extension of $\text{id}_G: G \rightarrow G$ to F . The basis s^0 is a section of the underlying pointed map of π . We let $\iota: M \rightarrow M \times \text{Ker } \pi, m \mapsto (m, 1)$ and $\mu: M \times \text{Ker } \pi \rightarrow F, (m, r) \mapsto r$. We let $s^1: \text{Ker } \pi \rightarrow M \times \text{Ker } \pi, r \mapsto (1, r)$ and we let $Z^2: G \times G \rightarrow M \times \text{Ker } \pi$ be given by $Z^2 := z_{F, s^0}^2 s^1$. The direct product $M \times \text{Ker } \pi$ is generated by $\text{Im } \iota \cup \text{Im } Z^2$ and carries the structure of an F -module uniquely determined on this set of generators by ${}^{kZ^1}(m\iota) := (km)\iota$ for $m \in M, k \in G$, and ${}^{kZ^1}((h, g)Z^2) := ((k, h, g)z^3\iota)^{-1}({}^{kZ^1}((h, g)z^2))s^1 = ((k, h, g)z^3\iota)^{-1}(k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1}$ for $g, h, k \in G$.

These data define a crossed module extension $E(z^3)$ and a section system $(s_{z^3}^1, s_{z^3}^0)$ for $E(z^3)$ as follows. The group part of $E(z^3)$ is given by $\text{Gp } E(z^3) := F$, the module part is given by $\text{Mp } E(z^3) := M \times \text{Ker } \pi$ and the structure morphism is given by $\mu^{E(z^3)} := \mu$. We have the canonical monomorphism $\iota^{E(z^3)} := \iota$ and the canonical epimorphism $\pi^{E(z^3)} := \pi$. The section system $(s_{z^3}^1, s_{z^3}^0)$ is defined by $s_{z^3}^0 := s^0$ and $s_{z^3}^1 := s^1$. By construction, the 3-cocycle of $E(z^3)$ with respect to the section system $(s_{z^3}^1, s_{z^3}^0)$ is

$$z_{E(z^3), (s_{z^3}^1, s_{z^3}^0)}^3 = z^3.$$

In particular, $\text{cl}(E(z^3)) = z^3 + B_{\text{cpt}}^3(G, M)$.

Proof. Since the kernel $\text{Ker } \pi$ is a free group on the pointed set $\text{Im } z^2 = \{(hZ^1)(gZ^1)((hg)Z^1)^{-1} \mid g, h \in G\}$ by remark (6.2)(a), the direct product $M \times \text{Ker } \pi$ is generated by $\text{Im } \iota \cup \text{Im } Z^2$. By definition of ι , μ and π , we have an exact sequence of groups

$$M \xrightarrow{\iota} M \times \text{Ker } \pi \xrightarrow{\mu} F \xrightarrow{\pi} G$$

with ι injective and π surjective. The pointed map s^1 is a section of the underlying pointed map of $\mu|_{\text{Im } \mu}$ and hence Z^2 is a lift of z^2 along the underlying pointed map of $\mu|_{\text{Im } \mu}$.

We want to define an action of F on $M \times \text{Ker } \pi$, that is, a group homomorphism $\alpha: F^{\text{op}} \rightarrow \text{Aut}(M \times \text{Ker } \pi)$. Since F is a free group on the underlying pointed set of G , it suffices to define a pointed map $a: G \rightarrow \text{Aut}(M \times \text{Ker } \pi)$. So we suppose given a group element $k \in G$. Since M is a G -module, we have a group automorphism $M \rightarrow M$, $m \mapsto km$ and hence a group homomorphism $M \rightarrow M \times \text{Ker } \pi$, $m \mapsto (km)\iota$. Moreover, since $\text{Ker } \pi$ is a free group on the pointed set $\text{Im } z^2$, the pointed map $\text{Ker } \pi \rightarrow M \times \text{Ker } \pi$, $(h, g)z^2 \mapsto ((k, h, g)z^3\iota)^{-1}(kZ^1((h, g)z^2))s^1$, which is well-defined by remark (6.2), extends to a group homomorphism $\text{Ker } \pi \rightarrow M \times \text{Ker } \pi$. By remark (6.1), we obtain a group homomorphism $ka: M \times \text{Ker } \pi \rightarrow M \times \text{Ker } \pi$, which is given on the elements of $\text{Im } \iota \cup \text{Im } Z^2$ by $m\iota(ka) = (km)\iota$ for $m \in M$ resp. by $(h, g)Z^2(ka) = ((k, h, g)z^3\iota)^{-1}(kZ^1((h, g)z^2))s^1$ for $g, h \in G$. Now remark (5.1) implies that

$$\begin{aligned} (h, g)Z^2(ka) &= ((k, h, g)z^3\iota)^{-1}(kZ^1((h, g)z^2))s^1 = ((k, h, g)z^3\iota)^{-1}((k, h)z^2(kh, g)z^2((k, hg)z^2)^{-1})s^1 \\ &= ((k, h, g)z^3\iota)^{-1}(k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1} \end{aligned}$$

for all $g, h \in G$. To show that ka is an automorphism on $M \times \text{Ker } \pi$, we show that the group homomorphism $kb: M \times \text{Ker } \pi \rightarrow M \times \text{Ker } \pi$, which is defined on the elements of $\text{Im } \iota \cup \text{Im } Z^2$ by $m\iota(kb) := (k^{-1}m)\iota$ for $m \in M$ resp. $(h, g)Z^2(kb) := ((k^{-1}, h, g)z^3\iota)^{-1}(kZ^1((h, g)z^2))s^1$, is inverse to ka . (The proof that kb exists and is uniquely determined on $\text{Im } \iota \cup \text{Im } Z^2$ is done in the same way as that of ka .) Indeed, we have

$$m\iota(ka)(kb) = (km)\iota(kb) = (k^{-1}km)\iota = m\iota$$

and

$$m\iota(kb)(ka) = (k^{-1}m)\iota(ka) = (kk^{-1}m)\iota = m\iota$$

for all $m \in M$ as well as

$$\begin{aligned} (h, g)Z^2(ka)(kb) &= (((k, h, g)z^3\iota)^{-1}(k, h)Z^2(kh, g)Z^2((k, hg)Z^2)^{-1})(kb) \\ &= ((k, h, g)z^3\iota(kb))^{-1}((k, h)Z^2(kb))((kh, g)Z^2(kb))((k, hg)Z^2(kb))^{-1} \\ &= ((k^{-1}(k, h, g)z^3\iota)^{-1}(((k^{-1}, k, h)z^3\iota)^{-1}(kZ^1((k, h)z^2))s^1))(((k^{-1}, kh, g)z^3\iota)^{-1}(kZ^1((kh, g)z^2))s^1) \\ &\quad (((k^{-1}, k, hg)z^3\iota)^{-1}(kZ^1((k, hg)z^2))s^1)^{-1} \\ &= ((k^{-1}, k, h)z^3\iota)^{-1}((k^{-1}, k, hg)z^3\iota)((k^{-1}, kh, g)z^3\iota)^{-1}((k^{-1}(k, h, g)z^3\iota)^{-1}(kZ^1((k, h)z^2))s^1) \\ &\quad (((kZ^1)^{-1}((kh, g)z^2))s^1)((kZ^1)^{-1}((k, hg)z^2))s^1)^{-1} \\ &= (- (k^{-1}, k, h)z^3 + (k^{-1}, k, hg)z^3 - (k^{-1}, kh, g)z^3 - k^{-1}(k, h, g)z^3)\iota \\ &\quad ((kZ^1)^{-1}((k, h)z^2(kh, g)z^2((k, hg)z^2)^{-1}))s^1 \\ &= (kZ^1)^{-1}(kZ^1((h, g)z^2))s^1 = (h, g)z^2s^1 = (h, g)Z^2 \end{aligned}$$

and

$$\begin{aligned} (h, g)Z^2(kb)(ka) &= (((k^{-1}, h, g)z^3\iota)^{-1}(kZ^1((h, g)z^2))s^1)(ka) \\ &= (((k^{-1}, h, g)z^3\iota)^{-1}(kZ^1)^{-1}(k^{-1}Z^1)^{-1}(k^{-1}Z^1((h, g)z^2))s^1)(ka) \\ &= (((k^{-1}, h, g)z^3\iota)^{-1}(((k^{-1}, k)z^2)^{-1}((k^{-1}, h)z^2(k^{-1}h, g)z^2((k^{-1}, hg)z^2)^{-1}))s^1)(ka) \\ &= (((k^{-1}, h, g)z^3\iota)^{-1}((k^{-1}, k)Z^2)^{-1}((k^{-1}, h)Z^2(k^{-1}h, g)Z^2((k^{-1}, hg)Z^2)^{-1}))(ka) \\ &= ((k^{-1}, h, g)z^3\iota(ka))^{-1}((k^{-1}, k)Z^2(ka))^{-1}(((k^{-1}, h)Z^2(ka))((k^{-1}h, g)Z^2(ka))((k^{-1}, hg)Z^2(ka))^{-1}) \end{aligned}$$

$$\begin{aligned}
&= ((k(k^{-1}, h, g)z^3\iota)^{-1}(((k, k^{-1}, k)z^3\iota)^{-1}({}^{kZ^1}((k^{-1}, k)z^2))s^1)^{-1}(((k, k^{-1}, h)z^3\iota)^{-1}({}^{kZ^1}((k^{-1}, h)z^2))s^1)) \\
&\quad (((k, k^{-1}h, g)z^3\iota)^{-1}({}^{kZ^1}((k^{-1}h, g)z^2))s^1)((k, k^{-1}, hg)z^3\iota)^{-1}({}^{kZ^1}((k^{-1}, hg)z^2))s^1)^{-1}) \\
&= ((k, k^{-1}, h)z^3\iota)^{-1}((k, k^{-1}, hg)z^3\iota)((k, k^{-1}h, g)z^3\iota)^{-1}((k(k^{-1}, h, g)z^3)\iota)^{-1} \\
&\quad ((k, k^{-1})z^2s^1)^{-1}({}^{kZ^1}((k^{-1}, h)z^2))s^1({}^{kZ^1}((k^{-1}h, g)z^2))s^1({}^{kZ^1}((k^{-1}, hg)z^2))s^1)^{-1}) \\
&= (- (k, k^{-1}, h)z^3 + (k, k^{-1}, hg)z^3 - (k, k^{-1}h, g)z^3 - (k(k^{-1}, h, g)z^3)\iota \\
&\quad ((k, k^{-1})z^2)^{-1}({}^{kZ^1}(((k^{-1}, h)z^2)((k^{-1}h, g)z^2)((k^{-1}, hg)z^2)^{-1}))s^1) \\
&= ({}^{k^{-1}Z^1})^{-1}({}^{k^{-1}Z^1}((h, g)z^2))s^1 = (h, g)z^2s^1 = (h, g)Z^2
\end{aligned}$$

for all $g, h \in G$, that is, $(ka)(kb) = (kb)(ka) = \text{id}_{M \times \text{Ker } \pi}$ and hence $ka \in \text{Aut}(M \times \text{Ker } \pi)$. Altogether, we have defined a map $a: G \rightarrow \text{Aut}(M \times \text{Ker } \pi)$. This map is pointed since

$$m\iota(1a) = (1m)\iota = m\iota$$

for all $m \in M$ and

$$(h, g)Z^2(1a) = ((1, h, g)z^3\iota)^{-1}({}^{1Z^1}((h, g)z^2))s^1 = (h, g)z^2s^1 = (h, g)Z^2$$

for all $g, h \in G$. Therefore, we get a group homomorphism $\alpha: F^{\text{op}} \rightarrow \text{Aut}(M \times \text{Ker } \pi)$ with $a = Z^1\alpha$, turning $M \times \text{Ker } \pi$ into an F -module with

$${}^{kZ^1}(m\iota) = m\iota(k\alpha) = m\iota(ka) = (km)\iota$$

for $m \in M, k \in G$, and

$$\begin{aligned}
{}^{kZ^1}((h, g)Z^2) &= (h, g)Z^2(k\alpha) = (h, g)Z^2(ka) = ((k, h, g)z^3\iota)^{-1}({}^{kZ^1}((h, g)z^2))s^1 \\
&= ((k, h, g)z^3\iota)^{-1}((k, h)Z^2)((kh, g)Z^2)((k, hg)Z^2)^{-1}
\end{aligned}$$

for $g, h, k \in G$. So we have

$$((k, h)Z^2)((kh, g)Z^2) = ((k, h, g)z^3\iota)^{kZ^1}((h, g)Z^2)((k, hg)Z^2)$$

for $g, h, k \in G$.

We want to show that the group F , the F -module $M \times \text{Ker } \pi$ and the homomorphism $\mu: M \times \text{Ker } \pi \rightarrow F$ define a crossed module.

(Equi) We have

$$({}^{kZ^1}(m\iota))\mu = (km)\iota\mu = 1 = {}^{kZ^1}(m\mu)$$

for $m \in M$ and

$$({}^{kZ^1}((h, g)Z^2))\mu = (((k, h, g)z^3\iota)^{-1}({}^{kZ^1}((h, g)z^2))s^1)\mu = {}^{kZ^1}((h, g)z^2) = {}^{kZ^1}((h, g)Z^2)\mu$$

for $g, h \in G$.

(Peif) We have

$${}^{n\mu}(m\iota) = m\iota = (n + m + (-n))\iota = {}^{n\mu}(m\iota)$$

for $m, n \in M$ as well as

$${}^{n\mu}((h, g)Z^2) = (h, g)Z^2 = {}^{n\mu}((h, g)Z^2)$$

for $g, h \in G, n \in M$. Moreover, we have

$$({}^{lZ^1})({}^{kZ^1})(m\iota) = {}^{lZ^1}({}^{kZ^1}(m\iota)) = {}^{lZ^1}((km)\iota) = (l(km))\iota = ((lk)m)\iota = ({}^{lk}Z^1)(m\iota)$$

for $m \in M$, $k, l \in G$, and therefore

$${}^{(l,k)Z^2}\mu(mu) = {}^{(l,k)z^2}(mu) = ({}^{lZ^1}({}^{kZ^1}({}^{(lk)Z^1})^{-1}(mu))) = mu = {}^{(l,k)Z^2}(mu)$$

for $m \in M$, $k, l \in G$. Finally, we have

$$\begin{aligned} ({}^{lZ^1}({}^{kZ^1})((h, g)Z^2)) &= {}^{lZ^1}({}^{kZ^1}((h, g)Z^2)) = {}^{lZ^1}(((k, h, g)z^3\iota)^{-1}((k, h)Z^2)((kh, g)Z^2)((k, hg)Z^2)^{-1}) \\ &= {}^{lZ^1}((k, h, g)z^3\iota)^{-1} {}^{lZ^1}((k, h)Z^2) {}^{lZ^1}((kh, g)Z^2) ({}^{lZ^1}((k, hg)Z^2))^{-1} \\ &= ((l(k, h, g)z^3\iota)^{-1}(((l, k, h)z^3\iota)^{-1}({}^{lZ^1}((k, h)z^2))s^1)((l, kh, g)z^3\iota)^{-1}({}^{lZ^1}((kh, g)z^2))s^1) \\ &\quad ((l, k, hg)z^3\iota)^{-1}({}^{lZ^1}((k, hg)z^2))s^1)^{-1} \\ &= ((l(k, h, g)z^3\iota)^{-1}((l, k, h)z^3\iota)^{-1}((l, kh, g)z^3\iota)^{-1}(l, k, hg)z^3\iota ({}^{lZ^1}((k, h)z^2))s^1 ({}^{lZ^1}((kh, g)z^2))s^1 \\ &\quad ({}^{lZ^1}((k, hg)z^2))s^1)^{-1} \\ &= (- (l, k, h)z^3 + (l, k, hg)z^3 - (l, kh, g)z^3 - l(k, h, g)z^3)\iota ({}^{lZ^1}((k, h)z^2)(kh, g)z^2((k, hg)z^2)^{-1})s^1 \\ &= (- (lk, h, g)z^3)\iota ({}^{lZ^1}({}^{kZ^1}((h, g)z^2)))s^1 = ((lk, h, g)z^3\iota)^{-1} ({}^{(lZ^1)({}^{kZ^1})}((h, g)z^2))s^1 \\ &= ((lk, h, g)z^3\iota)^{-1} ({}^{(l,k)z^2}({}^{(lk)Z^1}((h, g)z^2))s^1) = ((lk, h, g)z^3\iota)^{-1} ({}^{(l,k)Z^2}({}^{(lk)Z^1}((h, g)z^2))s^1) \\ &= {}^{(l,k)Z^2}(((lk, h, g)z^3\iota)^{-1} ({}^{(lk)Z^1}((h, g)z^2))s^1) = {}^{(l,k)Z^2}({}^{(lk)Z^1}((h, g)Z^2)) \end{aligned}$$

for $g, h, k, l \in G$, and hence we get

$$\begin{aligned} {}^{(l,k)Z^2}\mu((h, g)Z^2) &= {}^{(l,k)z^2}((h, g)Z^2) = ({}^{lZ^1}({}^{kZ^1}({}^{(lk)Z^1})^{-1}((h, g)Z^2))) \\ &= ({}^{lZ^1}({}^{kZ^1})({}^{(lk)Z^1})^{-1}((h, g)Z^2)) = {}^{(l,k)Z^2}((h, g)Z^2) \end{aligned}$$

for $g, h, k, l \in G$.

Altogether, we have constructed a crossed module $E(z^3)$ with $\text{Mp}E(z^3) = M \times \text{Ker } \pi$, $\text{Gp}E(z^3) = F$ and $\mu^{E(z^3)} = \mu$. Since the induced action of G on M is by definition given by the a priori given action of G on M , we even have a crossed module extension $E(z^3)$ of G with M , where $\iota^{E(z^3)} = \iota$ and $\pi^{E(z^3)} = \pi$. \square

(6.4) Corollary. We suppose given a Grothendieck universe \mathfrak{U} that contains an infinite set and we suppose G to be a \mathfrak{U} -group and M to be an abelian \mathfrak{U} - G -module. The map $\text{cl}: \underline{\text{Ext}}_{\mathfrak{U}}^2(G, M) \rightarrow \text{H}_{\text{cpt}}^3(G, M)$ is surjective.

Proof. This follows since the free group construction can be done in \mathfrak{U} , provided \mathfrak{U} contains an infinite set. \square

(6.5) Definition (standard extension with respect to a 3-cocycle). We suppose given a 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$. The crossed module extension $E(z^3)$ as constructed in proposition (6.3) is called the *standard extension* of G with M with respect to z^3 . The section system $(s_{z^3}^1, s_{z^3}^0)$ as also defined in loc. cit. is said to be the *standard section system* for $E(z^3)$ resp. z^3 .

Roughly said, the next proposition states that every 3-cocycle of the standard extension $E(z^3)$ with respect to a given 3-cocycle z^3 comes from a section system. This is a particular feature of the standard extension, cf. example (5.20).

(6.6) Proposition. We suppose given a 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$. For every componentwise pointed 2-cochain $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$, the group homomorphism $\tilde{s}^1: \text{Im } \mu^{E(z^3)} \rightarrow \text{Mp}E(z^3)$ uniquely determined by $(h, g)z_{E(z^3), s_{z^3}^0}^2 \tilde{s}^1 := (h, g)c^2 \iota^{E(z^3)}(h, g)z_{E(z^3), s_{z^3}^0}^2 s_{z^3}^1$ for $g, h \in G$ is a section of the underlying pointed map of $\mu^{E(z^3)}|_{\text{Im } \mu^{E(z^3)}}$ such that

$$z_{E(z^3), (\tilde{s}^1, s_{z^3}^0)}^3 = c^2 \partial + z^3.$$

Proof. By remark (6.2), we have a well-defined pointed map $\text{Im } z^2 \rightarrow \text{Mp}E(z^3)$, $(h, g)z^2 \mapsto (h, g)c^2(h, g)z_{z^3}^2 s_{z^3}^1$. Since $\text{Im } \mu$ is freely generated by the pointed set $\text{Im } z^2$, it follows that there exists a unique group homomorphism $\tilde{s}^1: \text{Im } \mu \rightarrow \text{Mp}E(z^3)$ with $(h, g)z^2 \tilde{s}^1 = (h, g)c^2 \iota(h, g)z_{z^3}^2 s_{z^3}^1$ for $g, h \in G$. This group homomorphism \tilde{s}^1 is a section of the underlying pointed map of μ and thus $(\tilde{s}^1, s_{z^3}^0)$ is a section system for $E(z^3)$. We denote the lifting

system of $E(z^3)$ coming from $(s_{z^3}^1, s_{z^3}^0)$ by (Z^2, Z^1) , and the one coming from $(\tilde{s}^1, s_{z^3}^0)$ by (\tilde{Z}^2, Z^1) . Then we have

$$(h, g)\tilde{Z}^2 = (h, g)z^2\tilde{s}^1 = (h, g)c^2\iota(h, g)z^2s_{z^3}^1 = (h, g)c^2\iota(h, g)Z^2,$$

and hence

$$z_{E(z^3), (\tilde{s}^1, s_{z^3}^0)}^3 = c^2\partial + z^3$$

by proposition (5.10)(b). \square

(6.7) Proposition. We suppose given a crossed module extension E of G with M , and we choose $z^3 \in Z_{\text{cpt}}^3(G, M)$ with $\text{cl}(E) = z^3 + B_{\text{cpt}}^3(G, M)$. Moreover, we suppose given a lifting system (Z^2, Z^1) for E with $z_{E, (Z^2, Z^1)}^3 = z^3$ and we denote by $(\tilde{Z}^2, \tilde{Z}^1)$ the lifting system of $E(z^3)$ coming from the standard section system $(s_{z^3}^1, s_{z^3}^0)$. There exists a unique extension equivalence $\omega: E(z^3) \rightarrow E$ with $g\tilde{Z}^1\omega = gZ^1$ for $g \in G$ and $(h, g)\tilde{Z}^2\omega = (h, g)Z^2$ for $g, h \in G$.

$$\begin{array}{ccccccc} M & \xrightarrow{\iota^{E(z^3)}} & \text{Mp } E(z^3) & \xrightarrow{\mu^{E(z^3)}} & \text{Gp } E(z^3) & \xrightarrow{\pi^{E(z^3)}} & G \\ \parallel & & \downarrow \text{Mp } \omega & & \downarrow \text{Gp } \omega & & \parallel \\ M & \xrightarrow{\iota^E} & \text{Mp } E & \xrightarrow{\mu^E} & \text{Gp } E & \xrightarrow{\pi^E} & G \end{array}$$

In particular,

$$E(z^3) \approx E.$$

Proof. Since $\text{Gp } E(z^3)$ is a free group on the underlying pointed set of G , the pointed map $Z^1: G \rightarrow \text{Gp } E$ extends uniquely to a group homomorphism $\omega_0: \text{Gp } E(z^3) \rightarrow \text{Gp } E$ with $(g\tilde{Z}^1)\omega_0 = gZ^1$ for $g \in G$. Further, by remark (6.1) there exists a unique group homomorphism $\omega_1: \text{Mp } E(z^3) \rightarrow \text{Mp } E$ given by $m\iota^{E(z^3)}\omega_1 = m\iota^E$ for $m \in M$ and $((h, g)\tilde{Z}^2)\omega_1 = (h, g)Z^2$ for $g, h \in G$. We get

$$m\iota^{E(z^3)}\omega_1\mu^E = m\iota^E\mu^E = 1 = 1\omega_0 = m\iota^{E(z^3)}\mu^{E(z^3)}\omega_0$$

for all $m \in M$ and

$$(h, g)\tilde{Z}^2\omega_1\mu^E = (h, g)Z^2\mu^E = (h, g)z_{E, Z^1}^2 = (h, g)z_{E(z^3), \tilde{Z}^1}^2\omega_0 = (h, g)\tilde{Z}^2\mu^{E(z^3)}\omega_0$$

for all $g, h \in G$, that is, $\omega_1\mu^E = \mu^{E(z^3)}\omega_0$. Moreover, we have

$$(k\tilde{Z}^1(m\iota^{E(z^3)}))\omega_1 = (km)\iota^{E(z^3)}\omega_1 = (km)\iota^E = kZ^1(m\iota^E) = k\tilde{Z}^1\omega_0(m\iota^{E(z^3)}\omega_1)$$

for all $m \in M, k \in G$, and

$$\begin{aligned} (k\tilde{Z}^1(h, g)\tilde{Z}^2)\omega_1 &= (((k, h, g)z^3\iota^{E(z^3)})^{-1}(k, h)\tilde{Z}^2(kh, g)\tilde{Z}^2((k, gh)\tilde{Z}^2)^{-1})\omega_1 \\ &= ((k, h, g)z^3\iota^{E(z^3)}\omega_1)^{-1}(k, h)\tilde{Z}^2\omega_1(kh, g)\tilde{Z}^2\omega_1((k, gh)\tilde{Z}^2\omega_1)^{-1} \\ &= ((k, h, g)z^3\iota^E)^{-1}(k, h)Z^2(kh, g)Z^2((k, gh)Z^2)^{-1} = kZ^1((h, g)Z^2) \\ &= (k\tilde{Z}^1)\omega_0((h, g)\tilde{Z}^2\omega_1) \end{aligned}$$

for all $g, h, k \in G$, that is, we have a morphism of crossed modules $\omega: E(z^3) \rightarrow E$ with $\text{Mp } \omega = \omega_1$ and $\text{Gp } \omega = \omega_0$. Since $\iota^{E(z^3)}(\text{Mp } \omega) = \iota^E$ and $(\text{Gp } \omega)\pi^E = \pi^{E(z^3)}$ by construction, we even have an extension equivalence $\omega: E(z^3) \rightarrow E$. \square

(6.8) Corollary. We suppose given crossed module extensions E and \tilde{E} of G with M , and we choose $z^3 \in Z_{\text{cpt}}^3(G, M)$ such that $\text{cl}(E) = z^3 + B_{\text{cpt}}^3(G, M)$. Then $E \approx \tilde{E}$ if and only if there exist extension equivalences $\omega: E(z^3) \rightarrow E$ and $\tilde{\omega}: E(z^3) \rightarrow \tilde{E}$.

$$\begin{array}{ccccccc}
M & \xrightarrow{\iota^E} & \text{Mp } E & \xrightarrow{\mu^E} & \text{Gp } E & \xrightarrow{\pi^E} & G \\
\parallel & & \uparrow \text{Mp } \omega & & \uparrow \text{Gp } \omega & & \parallel \\
M & \xrightarrow{\iota^{E(z^3)}} & \text{Mp } E(z^3) & \xrightarrow{\mu^{E(z^3)}} & \text{Gp } E(z^3) & \xrightarrow{\pi^{E(z^3)}} & G \\
\parallel & & \downarrow \text{Mp } \tilde{\omega} & & \downarrow \text{Gp } \tilde{\omega} & & \parallel \\
M & \xrightarrow{\iota^{\tilde{E}}} & \text{Mp } \tilde{E} & \xrightarrow{\mu^{\tilde{E}}} & \text{Gp } \tilde{E} & \xrightarrow{\pi^{\tilde{E}}} & G
\end{array}$$

Proof. If there exist extension equivalences $\omega: E(z^3) \rightarrow E$ and $\tilde{\omega}: E(z^3) \rightarrow \tilde{E}$, then $E \approx \tilde{E}$ by definition. So we suppose conversely that $E \approx \tilde{E}$. By corollary (5.17), this implies $\text{cl}(E) = \text{cl}(\tilde{E})$. We choose a componentwise pointed 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$ with $\text{cl}(E) = \text{cl}(\tilde{E}) = z^3 + B_{\text{cpt}}^3(G, M)$. By proposition (5.19), there exist lifting systems (Z^2, Z^1) for E and $(\tilde{Z}^2, \tilde{Z}^1)$ for \tilde{E} with $z_{E, (Z^2, Z^1)}^3 = z_{\tilde{E}, (\tilde{Z}^2, \tilde{Z}^1)}^3 = z^3$, and we are done with proposition (6.7). \square

As a further corollary, we obtain the fact that the standard extensions with respect to cohomologous cocycles are extension equivalent.

(6.9) Corollary. Given cocycles $z^3, \tilde{z}^3 \in Z_{\text{cpt}}^3(G, M)$ with $z^3 + B_{\text{cpt}}^3(G, M) = \tilde{z}^3 + B_{\text{cpt}}^3(G, M)$, we have $E(z^3) \approx E(\tilde{z}^3)$. That is, given a Grothendieck universe \mathfrak{U} that contains an infinite set and supposed that G is a \mathfrak{U} -group and M is a \mathfrak{U} - G -module, we obtain an induced map

$$e: H_{\text{cpt}}^3(G, M) \rightarrow \text{Ext}_{\mathfrak{U}}^2(G, M), z^3 + B_{\text{cpt}}^3(G, M) \mapsto [E(z^3)]_{\approx}.$$

Proof. We suppose given cocycles $z^3, \tilde{z}^3 \in Z_{\text{cpt}}^3(G, M)$ such that $z^3 + B_{\text{cpt}}^3(G, M) = \tilde{z}^3 + B_{\text{cpt}}^3(G, M)$, and we let $c^2 \in \text{Ch}_{\text{cpt}}^2(G, M)$ be given such that $\tilde{z}^3 = c^2 \partial + z^3$. By proposition (6.6), there exists a section system $(\tilde{s}^1, s_{z^3}^0)$ for $E(z^3)$ such that \tilde{z}^3 is the 3-cocycle of $E(z^3)$ with respect to $(\tilde{s}^1, s_{z^3}^0)$. Hence $E(\tilde{z}^3) \approx E(z^3)$ by proposition (6.7). \square

(6.10) Definition (crossed module extension class associated to a componentwise pointed 3-cohomology class). For every 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$, the crossed module extension class $e(z^3 + B_{\text{cpt}}^3(G, M)) = [E(z^3)]_{\approx}$ is called the *crossed module extension class* associated to $z^3 + B_{\text{cpt}}^3(G, M)$.

Finally, we can deduce the desired bijection.

(6.11) Theorem (cf. [7, th. 4.5], [8, p. 310], [13], [14, th. 9.4]). We suppose given a Grothendieck universe \mathfrak{U} that contains an infinite set and we suppose G to be a \mathfrak{U} -group and M to be an abelian \mathfrak{U} - G -module. The maps

$$\begin{aligned}
&\text{cl}: \text{Ext}_{\mathfrak{U}}^2(G, M) \rightarrow H_{\text{cpt}}^3(G, M) \text{ and} \\
&e: H_{\text{cpt}}^3(G, M) \rightarrow \text{Ext}_{\mathfrak{U}}^2(G, M)
\end{aligned}$$

are mutually inverse bijections, where the extension class of the trivial crossed module extension $[M \ G]$ corresponds to the trivial cohomology class $0 \in H_{\text{cpt}}^3(G, M)$. In particular,

$$\text{Ext}_{\mathfrak{U}}^2(G, M) \cong H^3(G, M).$$

Proof. By proposition (6.3), we have

$$\text{cl}(e(z^3 + B_{\text{cpt}}^3(G, M))) = \text{cl}([E(z^3)]_{\approx}) = \text{cl}(E(z^3)) = z^3 + B_{\text{cpt}}^3(G, M)$$

for every 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$, that is, $\text{cl} \circ e = \text{id}_{H_{\text{cpt}}^3(G, M)}$. Thus it remains to show that $e \circ \text{cl} = \text{id}_{\text{Ext}_{\mathfrak{U}}^2(G, M)}$. To this end, we suppose given a crossed module extension E and we choose $z^3 \in Z_{\text{cpt}}^3(G, M)$

with $\text{cl}(E) = z^3 + B_{\text{cpt}}^3(G, M)$. By proposition (5.19), there exists a lifting system (Z^2, Z^1) for E such that $z_{E, (Z^2, Z^1)}^3 = z^3$. Hence proposition (6.7) implies that $E(z^3) \approx E$ and thus

$$e(\text{cl}([E]_{\approx})) = e(\text{cl}(E)) = e(z^3 + B_{\text{cpt}}^3(G, M)) = [E(z^3)]_{\approx} = [E]_{\approx}.$$

The assertion on the trivial crossed module extension has been shown in example (5.13). Finally, corollary (3.7)(c) yields

$$\text{Ext}_{\mathbb{U}}^2(G, M) \cong H_{\text{cpt}}^3(G, M) \cong H^3(G, M). \quad \square$$

The standard extension $E(z^3)$ with respect to a given 3-cocycle $z^3 \in Z_{\text{cpt}}^3(G, M)$ involves free groups. So even if G and M are both finite, the module part and the group part of $E(z^3)$ are both infinite. The question occurs whether there exists a crossed module extension E with $\text{cl}(E) = z^3 + B_{\text{cpt}}^3(G, M)$ and with $\text{Gp } E$ and $\text{Mp } E$ finite. Such an extension has been constructed explicitly by ELLIS [6, proof for $c = 2$, p. 502].

References

- [1] ARTIN, MICHAEL; GROTHENDIECK, ALEXANDER; VERDIER, JEAN-LOUIS. *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, vol. 269. Springer-Verlag Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA4). With the collaboration of N. BOURBAKI, P. DELIGNE and B. SAINT-DONAT.
- [2] BROWN, KENNETH S. *Cohomology of Groups*. Graduate Texts in Mathematics, vol. 87. Springer-Verlag, New York-Berlin, 1982.
- [3] CONDUCHÉ, DANIEL. *Modules croisés généralisés de longueur 2*. Journal of Pure and Applied Algebra **34**(2–3) (1984), pp. 155–178.
- [4] EILENBERG, SAMUEL; MAC LANE, SAUNDERS. *Cohomology theory in abstract groups. I*. Annals of Mathematics **48**(1) (1947), pp. 51–78.
- [5] EILENBERG, SAMUEL; MAC LANE, SAUNDERS. *Cohomology theory in abstract groups. II. Group Extensions with a non-Abelian Kernel*. Annals of Mathematics **48**(2) (1947), pp. 326–341.
- [6] ELLIS, GRAHAM J. *Spaces with finitely many non-trivial homotopy groups all of which are finite*. Topology **36**(2) (1997), pp. 501–504.
- [7] HOLT, DEREK F. *An interpretation of the cohomology groups $H^n(G, M)$* . Journal of Algebra **60**(2) (1979), pp. 307–320.
- [8] HUEBSCHMANN, JOHANNES. *Crossed n -fold extensions of groups and cohomology*. Commentarii Mathematici Helvetici **55**(2) (1980), pp. 302–313.
- [9] KUROSH, A. G. *The theory of groups. Vol. II*. Second English edition. Chelsea Publishing Company, New York (NY), 1960.
- [10] LANG, SERGE. *Algebra*. Graduate Texts in Mathematics, vol. 211, third edition. Springer-Verlag, New York, 2002.
- [11] LODAY, JEAN-LOUIS. *Spaces with finitely many non-trivial homotopy groups*. Journal of Pure and Applied Algebra **24**(2) (1982), pp. 179–202.
- [12] MAC LANE, SAUNDERS. *Cohomology theory in abstract groups. III. Operator Homomorphisms of Kernels*. Annals of Mathematics **50**(3) (1949), pp. 736–761.
- [13] MAC LANE, SAUNDERS. *Historical Note*. Journal of Algebra **60**(2) (1979), pp. 319–320. Appendix in [7].
- [14] RATCLIFFE, JOHN G. *Crossed extensions*. Transactions of the American Mathematical Society **257**(1) (1980), pp. 73–89.
- [15] THOMAS, SEBASTIAN. *(Co)homology of crossed modules*. Diploma thesis, RWTH Aachen University, 2007. <http://www.math.rwth-aachen.de/~Sebastian.Thomas/publications/>

- [16] THOMAS, SEBASTIAN. *On the second cohomology group of a simplicial group*. Homology, Homotopy and Applications **12**(2) (2010), pp. 167-210.

Sebastian Thomas
Lehrstuhl D für Mathematik
RWTH Aachen University
Templergraben 64
52062 Aachen
Germany
sebastian.thomas@math.rwth-aachen.de
<http://www.math.rwth-aachen.de/~Sebastian.Thomas/>