

THE LOCATION OF NONCROSSED PRODUCTS IN BRAUER GROUPS OF LAURENT SERIES FIELDS OVER GLOBAL FIELDS

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ABSTRACT. Since Amitsur's discovery of noncrossed product division algebras more than 35 years ago, their existence over more familiar fields has been an object of investigation. Brussel's work was a culmination of this effort, exhibiting noncrossed products over the rational function field $k(\mathbf{t})$ and the Laurent series field $k((\mathbf{t}))$ over any global field k — the smallest possible centers of noncrossed products.

Witt's theorem gives a transparent description of the Brauer group of $k((\mathbf{t}))$ as the direct sum of the Brauer group of k and the character group of the absolute Galois group of k . We classify the Brauer classes over $k((\mathbf{t}))$ containing noncrossed products by analyzing the fiber over χ for each character χ in Witt's theorem. In this way, a picture of the partition of the Brauer group into crossed products/noncrossed products is obtained, which is in principle ruled by a relation between index and number of roots of unity.

As a side consequence of the result there are crossed products that have a noncrossed product primary component.

1. INTRODUCTION

A finite-dimensional division algebra is called a *crossed product* if one of its maximal commutative subfields is Galois over the center of the algebra, otherwise a *noncrossed product*. The existence of noncrossed products was for several decades the biggest open question in the theory

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of finite-dimensional division algebras before it was settled by Amitsur [2] in 1972.

The present paper is motivated by the work of Brussel [5,6] on the existence of noncrossed products over the fields $k(\mathbf{t})$ and $k((\mathbf{t}))$ for any global field k , which determined these fields to be the “smallest” possible centers of noncrossed products.

Brussel’s result is even more surprising over $k((\mathbf{t}))$ than over $k(\mathbf{t})$ because of the much simpler structure of its Brauer group. Due to *Witt’s theorem* [17], if k is perfect, there is a canonical isomorphism

$$\mathrm{Br}(k((\mathbf{t}))) \cong \mathrm{Br}(k) \oplus X(\mathfrak{G}_k) \quad (1.1)$$

(depending on \mathbf{t}), where $X(\mathfrak{G}_k) = \mathrm{Hom}_c(\mathfrak{G}_k, \mathbb{Q}/\mathbb{Z})$ denotes the group of continuous characters of the absolute Galois group of k . Each element χ of the torsion group $X(\mathfrak{G}_k)$ belongs to a finite cyclic extension field $k(\chi) \supseteq k$ with degree equal to the order $|\chi|$ of χ . ($k(\chi)$ is the fixed field of the kernel of χ .) Moreover, a canonical generator σ_χ of the Galois group of $k(\chi)$ over k is obtained by restricting to $k(\chi)$ any $\sigma \in \mathfrak{G}_k$ with $\chi(\sigma) = e^{2\pi i/|\chi|}$. Under (1.1), χ maps to the class of the cyclic algebra $(k(\chi)((\mathbf{t}))/k((\mathbf{t})), \sigma_\chi, \mathbf{t})$ over $k((\mathbf{t}))$.

If k is non-perfect then $\mathrm{Br}(k) \oplus X(\mathfrak{G}_k)$ is a subgroup of $\mathrm{Br}(k((\mathbf{t})))$, the so-called “tame” part. Throughout the paper we will work exclusively inside this subgroup, and the global field k may be perfect or not.

Let k be a global field, i.e. either a number field or a function field in one variable over a finite field. Brussel proves the existence of pairs of $\alpha \in \mathrm{Br}(k)$ and $\chi \in X(\mathfrak{G}_k)$ such that the $k((\mathbf{t}))$ -division algebra representing $\alpha + \chi$ is a noncrossed product. In this way, all indices for which noncrossed products are known to exist can be realized. However, it has not yet been determined for all classes $\alpha + \chi$ whether the representing division algebra is a noncrossed product. This task is the motivation for the present paper.

Our approach is to partition $\mathrm{Br}(k) \oplus X(\mathfrak{G}_k)$ into the union of fibers over χ where χ runs through $X(\mathfrak{G}_k)$ and to classify the noncrossed products in each fiber. (Division algebras are identified with their respective Brauer classes.)

Let $\chi \in X(\mathfrak{G}_k)$ and consider all division algebras of fixed index $N \in \mathbb{N}$ inside $\mathrm{Br}(k) + \chi$. We will show that one of the following two cases occurs:

- (I) All division algebras in the fiber over χ of index N are crossed products.
- (II) The fiber over χ contains infinitely many noncrossed products of index N .

Moreover, for fixed χ and varying N the cases (I) and (II) are separated by bounds on the prime powers dividing N in such a way that case (I) occurs “below” the bounds and case (II) “above”. The details are as follows. By a well-known formula (cf. (6.1) below) all elements in $\text{Br}(k) + \chi$ have index a multiple of $|\chi|$. Let $N = |\chi|m$ and let $\prod p^{n_p}$ be the prime factorization of m . There are simply defined bounds $b_p(\chi)$ where p runs through the prime factors of $|\chi|$ such that each $b_p(\chi)$ is a nonnegative integer or infinity and we are in case

- (I) if $n_p \leq b_p(\chi)$ for all prime factors p of $|\chi|$,
- (II) if $n_p > b_p(\chi)$ for some prime factor p of $|\chi|$.

The field $k((\mathbf{t}))$ can be replaced throughout the paper by any discrete rank one Henselian valued field with residue field k . Furthermore, analogous results hold over $k(\mathbf{t})$.

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2. NOTATION AND STATEMENT OF RESULTS

Let k be a field and k_{sep} its separable closure. We write \mathfrak{G}_k for $\text{Gal}(k_{sep}/k)$, the absolute Galois group of k . For each prime p define $s_p(k)$ to be the maximal nonnegative integer s such that k contains p^s distinct roots of unity of p -power order, or infinity if no maximal s exists. (If $p = \text{char } k$ then $s_p(k) = 0$. If k is a local or global field then $s_p(k)$ is finite.)

Denote by $X(\mathfrak{G}_k)$ the group $\text{Hom}_c(\mathfrak{G}_k, \mathbb{Q}/\mathbb{Z})$ of continuous characters. (\mathfrak{G}_k is a pro-finite group with the Krull topology and \mathbb{Q}/\mathbb{Z} is equipped with the discrete topology.) $X(\mathfrak{G}_k)$ is an abelian torsion group. For each $\chi \in X(\mathfrak{G}_k)$, the fixed field of $\ker \chi$ is a cyclic extension of k of degree $|\chi|$ that we shall denote by $k(\chi)$. We abbreviate $s_p(k(\chi))$ to $s_p(\chi)$.

Since $X(\mathfrak{G}_k)$ is an abelian torsion group, each of its p -primary components $X_p(\mathfrak{G}_k)$ is a direct summand. We let χ_p denote the projection of $\chi \in X(\mathfrak{G}_k)$ onto $X_p(\mathfrak{G}_k)$. As an element of the abelian p -group $X_p(\mathfrak{G}_k)$, χ_p has a *height*, which is defined to be the maximal nonnegative integer r such that χ_p is divisible by p^r , or infinity if no maximal r exists (cf. Kaplansky [10], §9, p.19). We denote the height of χ_p by $\text{ht}_p(\chi)$.

In terms of the corresponding cyclic field extensions, $k(\chi_p)$ is the unique subfield of $k(\chi)$ with maximal p -power degree over k . Moreover,

$\text{ht}_p(\chi)$ is the maximal nonnegative integer r such that the cyclic extension $k(\chi_p)/k$ embeds into a cyclic extension L/k with $[L : k(\chi_p)] = p^r$, or infinity if no maximal r exists. Since $\text{ht}_p(\chi)$ is an invariant of the field extension $k(\chi)/k$, the notation $\text{ht}_p(k(\chi)/k)$ is also valid.

Definition 2.1. Let k be a global field and let $\chi \in X(\mathfrak{G}_k)$.

a) χ is said to be **exceptional** if k is a number field, $i := \sqrt{-1} \in k(\chi)$ and

$$\text{ht}_2(k(\chi)/k(i)) > \text{ht}_2(k(\chi)/k) > 0.$$

b) For each prime p define

$$b_p(\chi) := \begin{cases} \text{ht}_p(\chi) + s_p(\chi) + 1 & \text{if } p = 2 \text{ and } \chi \text{ is exceptional,} \\ \text{ht}_p(\chi) + s_p(\chi) & \text{otherwise.} \end{cases}$$

The property of being exceptional as well as the numbers $b_p(\chi)$ are invariants of the field extension $k(\chi)/k$. Therefore, we can speak of an *exceptional extension* $k(\chi)/k$ and we can write $b_p(k(\chi)/k)$.

Clearly, χ is exceptional if and only if χ_2 is exceptional. Such characters exist (see the Remark at the end of the section).

Theorem 2.2 (Main Theorem). *Let k be a global field and let $\chi \in X(\mathfrak{G}_k)$ be a character. The indices of division algebras occurring in the fiber $\text{Br}(k) + \chi$ are precisely the numbers $|\chi|m$ where m runs through all positive integers. Let m be fixed and let $m = \prod p^{n_p}$ be its prime factorization.*

(i) *If $n_p \leq b_p(\chi)$ for all prime factors p of $|\chi|$ then all division algebras in the fiber over χ of index $|\chi|m$ are crossed products.*

(ii) *If $|\chi|$ has a prime divisor p with $n_p > b_p(\chi)$ then the fiber over χ contains infinitely many noncrossed product division algebras of index $|\chi|m$.*

For algebras of p -power index (p prime) the picture becomes particularly simple: if $|\chi|$ and m are both powers of p then we are in case

$$(I) \text{ if } n_p \leq b_p(\chi), \quad (II) \text{ if } n_p > b_p(\chi).$$

Since it is known that $\text{ht}_p(\chi) = \infty$ for $p = \text{char } k$, there are no noncrossed product p -algebras in $\text{Br}(k) \oplus X(\mathfrak{G}_k)$.

The proof of Theorem 2.2 for exceptional characters χ is more difficult than that for non-exceptional χ , so the additional pieces required for the exceptional characters are separated out and presented later, in the interest of greater readability, after the proof for non-exceptional characters is complete. On the other hand, there are interesting aspects to this part of the proof of the exceptional case, involving non-routine applications of the special case of the local-global principle for

the height of a cyclic extension of a number field, and of Neukirch's theorems on embedding problems with prescribed local solutions. The embedding problems arise for a family of metacyclic 2-groups which are exceptional in their own way (in the classification of metacyclic p -groups).

Remark. We have $b_p(\chi) = b_p(k(\chi)/k')$ where k'/k is the maximal intermediate prime-to- p extension of $k(\chi)/k$.

Proof. Since $[k' : k]$ is prime to p , a restriction-corestriction argument shows $\text{ht}_p(\chi) = \text{ht}_p(k(\chi_p)/k) = \text{ht}_p(k(\chi)/k')$. As a consequence, $k(\chi)/k$ is exceptional if and only if $k(\chi)/k'$ is exceptional. The claim follows because $s(\chi)$ is not changed. \square

Remark. The extension $k(i)/k$ is exceptional if and only if k is a number field, $i \notin k$ and $\infty > \text{ht}_2(k(i)/k) > 0$. For example, this is the case for all $k = \mathbb{Q}(\sqrt{-2a})$ with $a \in \mathbb{N}$ and $a \equiv 7 \pmod{8}$ (cf. Geyer-Jensen [6, p. 713]).

For any exceptional χ we have $\text{ht}_2(\chi) = s_2(\chi)$ (shown in §8), so that $b_2(\chi)$ is an odd number ≥ 5 . The existence of exceptional characters is discussed in detail in the separate paper [9]. As a result of that paper, all exceptional characters of order 2 (i.e. $k(\chi) = k(i)$) have $s_2(\chi) = 2$ and $b_2(\chi) = 5$. On the other hand, exceptional characters of order 4 occur with $s_2(\chi)$ taking any value ≥ 2 .¹

3. PRELIMINARIES

Cyclotomic fields. Let k be any field. We denote by $\mu(k), \mu_n(k)$ and $\mu_{p^\infty}(k)$ the group of all roots (resp. all n -th roots, all p -power roots) of unity contained in k . Given that k is fixed, $\mu, \mu_n, \mu_{p^\infty}$ are the respective groups of roots of unity contained in k_{sep} . Hence, $k(\mu_n)$ is the n -th cyclotomic field over k .

We choose once and for all a system of primitive roots of unity ζ_m such that $\zeta_m^n = \zeta_{m/n}$ for all $n|m$. We set $\eta_m := \zeta_m + \zeta_m^{-1}$ and $i := \sqrt{-1}$. The fields $k(\eta_m)$ do not depend on the choice of ζ_m .

For fields of characteristic zero we define $\tilde{k} := k \cap \mathbb{Q}(\mu_{2^\infty})$. Recall that (3.1) below is the full subfield lattice of $\mathbb{Q}(\mu_{2^\infty})$ in which all lines

¹Nigel Tufnel would pick the field $k = \mathbb{Q}(\sqrt{-33}, \zeta_5 + \zeta_5^{-1})$ and say: "I have a character χ of order 4 with $\text{ht}_2(\chi) = s_2(\chi) = 5$. However, the bound is not 10. It's one more! The crossed products go up to 11."

indicate quadratic extensions.

$$\begin{array}{ccccc}
 & & & & \uparrow \dots \\
 & & & & \mathbb{Q}(\zeta_{2^{s+1}}) \\
 & & & \swarrow & \downarrow \\
 & & & \mathbb{Q}(\eta_{2^{s+1}}) & \mathbb{Q}(i\eta_{2^{s+1}}) & \mathbb{Q}(\zeta_{2^s}) \\
 & & & \downarrow & \swarrow & \downarrow \\
 & & & \mathbb{Q}(\eta_{2^s}) & & \mathbb{Q}(\zeta_8) \\
 & & & \downarrow & \swarrow & \downarrow \\
 & & & \mathbb{Q}(\eta_8) & \mathbb{Q}(i\eta_8) & \mathbb{Q}(i) \\
 & & & \downarrow & \swarrow & \downarrow \\
 & & & \mathbb{Q} & & \mathbb{Q}
 \end{array} \tag{3.1}$$

Local fields. By a local field we mean a finite extension of \mathbb{Q}_p or $\mathbb{F}_p((t))$. Let K/k be a Galois extension of local fields. We write \bar{k} for the residue field. Let I denote the maximal unramified subextension of K/k , q the number of elements in the residue field of k , and e the ramification index of K/k . We suppose K/k is tamely ramified, i.e. $\text{char } \bar{k} \nmid e$. Then K/I is a cyclic Kummer extension of exponent e (cf. Lang [11, Chapter II, §5, Proposition 12, p. 52]). In particular, $\mu_e \subset I$ and $\text{Gal}(K/k)$ is metacyclic. Due to Kummer theory, the action of $\text{Gal}(I/k)$ on $\text{Gal}(K/I)$ is given by the action of $\text{Gal}(I/k)$ on μ_e . The latter lifts from the residue field, so the Frobenius element of $\text{Gal}(I/k)$ acts on $\text{Gal}(K/I)$ as the q -th power map. Thus, $\text{Gal}(K/k)$ has a presentation

$$\langle x, y \mid x^e = 1, y^f = x^t, x^y = x^q \rangle \tag{3.2}$$

for some $t|e$. Here, $y|_I$ is the Frobenius and x is a certain generator of $\text{Gal}(K/I)$. (Note that x has to be suitably chosen in order to achieve $t|e$.) The parameters are further subject to the relations $q^f \equiv 1 \pmod{e}$ and $q \equiv 1 \pmod{e/t}$ (because y^f commutes with x and x^t commutes with y respectively). Note that

$$K/k \text{ is abelian} \iff q \equiv 1 \pmod{e}, \tag{3.3}$$

in which case K/k is cyclic if and only if $t = 1$.

Global fields. By a global field we mean a finite extension of \mathbb{Q} or a finite extension of $\mathbb{F}_p(\mathbf{t})$. Let k be a global field. For any prime \mathfrak{p} of k — archimedean or non-archimedean — we write $k_{\mathfrak{p}}$ for the completion of k at \mathfrak{p} . If F/k is Galois then $F_{\mathfrak{p}}$ denotes the completion of F at any prime of F dividing \mathfrak{p} . ($F_{\mathfrak{p}}$ is unique up to k -isomorphism.)

Now, let \mathfrak{p} be a non-archimedean prime of k . We denote by $N(\mathfrak{p})$ the absolute norm of \mathfrak{p} , i.e. the number of elements in the residue field of \mathfrak{p} . Let F/k be a finite Galois extension and let \mathfrak{P} be a prime of F dividing \mathfrak{p} . We denote by $Z_{\mathfrak{P}}(F/k)$, $I_{\mathfrak{P}}(F/k)$, $e_{\mathfrak{P}}(F/k)$ and $\varphi_{\mathfrak{P}}(F/k)$ respectively the decomposition field, the inertia field, the ramification index and the Frobenius element for \mathfrak{P} ($e_{\mathfrak{P}}$ does not depend on the choice of \mathfrak{P} , for F/k is Galois). Recall that $\text{Gal}(I_{\mathfrak{P}}/Z_{\mathfrak{P}})$ is generated by $\varphi_{\mathfrak{P}}$. One says \mathfrak{p} splits completely in F if $Z_{\mathfrak{P}}(F/k) = F$ for all $\mathfrak{P}|\mathfrak{p}$, and \mathfrak{p} is unramified in F if $I_{\mathfrak{P}}(F/k) = F$ for all $\mathfrak{P}|\mathfrak{p}$. For instance, in $k(\mu_m)$ any \mathfrak{p} with $(N(\mathfrak{p}), m) = 1$ is unramified and

$$\mathfrak{p} \text{ splits completely in } k(\mu_m) \iff N(\mathfrak{p}) \equiv 1 \pmod{m}, \quad (3.4)$$

$$\mathfrak{p} \text{ splits completely in } k(\eta_m) \iff N(\mathfrak{p}) \equiv \pm 1 \pmod{m}. \quad (3.5)$$

Chebotarev's Density Theorem (cf. Weil [16], Chapter XIII, §12, Theorem 12, p.289) implies that any given $\sigma \in \text{Gal}(F/k)$ is the Frobenius element for infinitely many primes \mathfrak{P} of F . Applied to a compositum F_1F_2 of two Galois extensions $F_1, F_2 \supseteq k$ this means that for any $\sigma_1 \in \text{Gal}(F_1/k)$ and $\sigma_2 \in \text{Gal}(F_2/k)$ coinciding on $F_1 \cap F_2$, there are infinitely many non-archimedean primes \mathfrak{P} of F_1F_2 such that σ_i is the Frobenius element for $\mathfrak{P} \cap F_i$, $i = 1, 2$.

4. THE HEIGHT

Let K/k be a cyclic field extension. In the study of $\text{ht}_p(K/k)$ we assume $[K : k]$ to be a p -power, for $\text{ht}_p(K/k)$ depends by definition only on the maximal p -power subextension of K/k . However, all results of this section formally hold for arbitrary $[K : k]$. Before focusing on local and global fields we mention two important facts over arbitrary fields. The first one is known as

Albert's theorem. *If $\mu_{p^r} \subseteq k$ then*

$$\text{ht}_p(K/k) \geq r \iff \mu_{p^r} \subset N_{K/k}(K^*).$$

Proof. Albert [1, Chapter 9, Theorem 11]. □

For instance, any imaginary quadratic number field K has $\text{ht}_2(K/\mathbb{Q}) = 0$ because -1 is not a norm in K/\mathbb{Q} . The second fact is

Artin-Schreier theory. *If $\text{char } k = p$ and $p \mid [K : k]$ then*

$$\text{ht}_p(K/k) = \infty.$$

Proof. Albert [1, p. 194f]. □

For local and global fields we have $\text{ht}_p(k/k) = \infty$. Hence, $\text{ht}_p(K/k) = \infty$ for all p not dividing $[K : k]$. But infinite height is also possible in non-trivial cases and in characteristic zero: For instance, $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) = \mathbb{Q}(\eta_8) \subset \mathbb{Q}(\eta_{16}) \subset \dots$ is an infinite tower of cyclic number fields of 2-power degree, hence $\text{ht}_2(\mathbb{Q}(\eta_{2^l})/\mathbb{Q}) = \infty$ for all l .

Proposition 4.1. *Suppose K/k is a cyclic extension of local fields. Let e be the ramification index of K/k and let $v_p(e)$ denote the p -adic exponential value of e . Then*

$$\text{ht}_p(K/k) = \begin{cases} \infty & \text{if } p \nmid e, \\ s_p(k) - v_p(e) & \text{if } p \mid e \text{ and } p \neq \text{char } \bar{k}. \end{cases}$$

Proof. Assume $[K : k]$ is a p -power. If $p \nmid e$ then K/k is unramified, so $\text{ht}_p(K/k) = \infty$. Assume now $p \mid e$ and $p \neq \text{char } \bar{k}$. Let I denote the maximal unramified subfield of K . Write $s = s_p(k)$ and $v = v_p(e)$.

The inequality $\text{ht}_p(K/k) \geq s - v$ follows from Albert's Theorem. Indeed, since ζ_{p^s} is a norm in I/k and $[K : I] = e = p^v$, $\zeta_{p^{s-v}}$ is a norm in K/k . Conversely, let L/k be a cyclic extension containing K with $[L : K] = p^h$. We show $h \leq s - v$. The intermediate fields of L/k are linearly ordered. Therefore, since $I \subsetneq K$, the maximal unramified subfield of L/k equals I . Hence, the ramification index of K/k is p^{v+h} . By (3.3), $v + h \leq s$. □

Lemma 4.2. *Suppose K/k is a cyclic extension of global fields. Then $\text{ht}_p(K/k) \leq \text{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}})$ for any prime \mathfrak{p} in k .*

Proof. We may assume $[K : k]$ is a p -power. Suppose L/k is cyclic, $L \supseteq K$ and $[L : K] = p^r$. Let \mathfrak{p} be a prime in k and claim $\text{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}}) \geq r$. Let Z be the (unique) decomposition field of \mathfrak{p} in L/k . Since the subfields of L/k are linearly ordered we have $Z \supseteq K$ or $Z \subseteq K$. In the former case $K_{\mathfrak{p}} = k_{\mathfrak{p}}$, hence $\text{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}}) = \infty$. In the latter case $[L_{\mathfrak{p}} : K_{\mathfrak{p}}] = [L : K]$, hence $\text{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}}) \geq r$. □

Example 4.3. If p, q are primes with $q = ap^n + 1$ and $(a, p) = 1$ then the degree p^m subfield K of the q -th cyclotomic field $\mathbb{Q}(\mu_q)$, $1 < m \leq n$, has $\text{ht}_p(K/\mathbb{Q}) = n - m$.

Proof. Since $\mathbb{Q}(\mu_q)$ is cyclic of degree $q - 1$, it contains a cyclic extension of degree p^n . Thus, $\text{ht}_p(K/\mathbb{Q}) \geq n - m$. Conversely, the prime q is totally ramified in $\mathbb{Q}(\mu_q)$, in particular also in K . Hence, by Lemma

4.2 and Proposition 4.1, $\text{ht}_p(K/\mathbb{Q}) \leq \text{ht}_p(K_q/\mathbb{Q}_q) = s_p(\mathbb{Q}_q) - v_p(p^m) = n - m$. \square

If the global field k and a prime $p \neq \text{char } k$ are fixed then “with probability one” a cyclic extension K/k of p -power degree has $\text{ht}_p(K/k) = 0$. This (assuming we agree that an element in an infinite group is non-trivial “with probability one”) is an interpretation of

Proposition 4.4. *Let k be a global field. Then $X_p(\mathfrak{G}_k)/X_p(\mathfrak{G}_k)^p$ is infinite for any $p \neq \text{char } k$.*

(Note that $X_p(\mathfrak{G}_k)^p$ is the subgroup consisting of characters of height greater than zero.)

Proof. We write $X_p(\mathfrak{G}_k)$ additively (only in this proof) and show that $X_p(\mathfrak{G}_k)/pX_p(\mathfrak{G}_k)$ is infinite. Set $t := s_p(k(\mu_p))$. By Chebotarev’s density theorem there are infinitely many primes \mathfrak{p} of k which split completely in $k(\mu_p)$ and do not split completely in $k(\mu_{p^{t+1}})$. Let $S = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be any finite set of such \mathfrak{p} . By the Grunwald-Wang Theorem [3, Chapter X, Theorem 5], there exists a cyclic extension L_i/k of degree p^t in which \mathfrak{p}_i is totally ramified, and in which \mathfrak{p}_j splits completely for every $j \neq i$. Let χ_i be a character of \mathfrak{G}_k corresponding to L_i for $i = 1, \dots, n$. Let $\chi = \sum a_i \chi_i$ with some a_i , say a_1 , not divisible by p . Then if L is the cyclic extension corresponding to χ , the completion of L at \mathfrak{p}_1 coincides with the completion of L_1 at \mathfrak{p}_1 , which does not embed into a cyclic extension of degree p^{t+1} of $k_{\mathfrak{p}_1}$. Hence L/k likewise does not embed into a cyclic extension of degree p^{t+1} . Consequently $\chi \notin pX_p(\mathfrak{G}_k)$. It follows that χ_1, \dots, χ_n are linearly independent modulo $pX_p(\mathfrak{G}_k)$. Since n can be chosen arbitrarily large, the proof is complete. \square

Let $X_p(\mathfrak{G}_k)_{(p^l)}$ denote the p^l -torsion subgroup of $X_p(\mathfrak{G}_k)$. Observe that the proof of Proposition 4.4 actually shows that

$$X_p(\mathfrak{G}_k)_{(p^l)}/X_p(\mathfrak{G}_k)_{(p^l)} \cap X_p(\mathfrak{G}_k)^p = X_p(\mathfrak{G}_k)_{(p^l)}/X_p(\mathfrak{G}_k)_{(p^{l+1})}^p$$

is infinite for any $l \geq t$, where $t = s_p(k(\mu_p))$ as in the proof. Thus, a general character of order dividing p^l with $l \geq t$ has height zero. It remains to discuss the case $l < t$.

Of course if k contains the p^{l+1} th roots of unity, then every character of order dividing p^l has positive height. By a restriction-corestriction argument, this is still true if the p^{l+1} th roots of unity are merely contained in $k(\mu_p)$. Hence, every character of order dividing p^l has height at least $t - l$. In fact, by the following Proposition, a general character of order dividing p^l with $l < t$ has height equal to $t - l$. In particular, for any l , a general character of order p^l has bounded height.

Proposition 4.5. *If $l < t$ then $X_p(\mathfrak{G}_k)_{(p^l)}/X_p(\mathfrak{G}_k)_{(p^{t+1})}^{p^{t-l+1}}$ is infinite.*

Proof. Assume $l < t$. Let χ_1, \dots, χ_n be as in the proof of Proposition 4.4. Again using additive notation, consider $p^{t-l}\chi_1, \dots, p^{t-l}\chi_n$ and claim they are linearly independent mod $p^{t-l+1}X_p(\mathfrak{G}_k)_{(p^{t+1})}$. Suppose

$$\sum a_i p^{t-l}\chi_i \in p^{t-l+1}X_p(\mathfrak{G}_k)_{(p^{t+1})}$$

with say a_1 not divisible by p . Then $p^{t-l}\sum a_i\chi_i = p^{t-l+1}\psi$ where $p^{t+1}\psi = 0$. Setting $\chi = \sum a_i\chi_i$, we then have $p^{t-l}(\chi - p\psi) = 0$. Since $t-l < t$, it follows that the character $\chi - p\psi$ has positive height, hence so does χ . This is a contradiction, because χ has height zero by choice of the χ_i . \square

5. EXAMPLES

We determine fibers that contain noncrossed products of degree 8 and 9.

Example 5.1. a) Let $k = \mathbb{Q}$. To answer the question “For which quadratic number fields $\mathbb{Q}(\chi)$ are there noncrossed products of index 8 in the fiber over χ ?” we apply Theorem 2.2 with $m = 4$. By (i) and (ii), these are precisely the quadratic fields $\mathbb{Q}(\chi)$ with $\text{ht}_2(\chi) + s_2(\chi) < 2$, or, equivalently, $\text{ht}_2(\chi) = 0$ and $s_2(\chi) = 1$. This condition is satisfied, for instance, by the third cyclotomic field, $\mathbb{Q}(\sqrt{-3})$, according to Example 4.3. Moreover, $\mathbb{Q}(\sqrt{-3})$ is the field of this kind with smallest discriminant and smallest conductor. Example 7.9 in Section 7 discusses for which $\alpha \in \text{Br}(\mathbb{Q})$ the division algebra in $\alpha + \chi$ is a noncrossed product of index 8.

b) Similarly, the cubic number fields $\mathbb{Q}(\chi)$ that allow noncrossed products of index 9 in the fiber over χ are precisely the ones satisfying $\text{ht}_3(\chi) + s_3(\chi) < 1$, or, equivalently, $\text{ht}_3(\chi) = 0$ and $s_3(\chi) = 0$. The smallest such field is the cubic subfield of the 7-th cyclotomic field (cf. Example 4.3 with $p = 3$ and $q = 7$).

We shall also give two small examples in finite characteristic.

c) Let $k = \mathbb{F}_3(\mathbf{t})$ and let $k(\chi) = \mathbb{F}_3(\mathbf{t})(\sqrt{\mathbf{t}})$. The prime \mathbf{t} is tamely ramified in $k(\chi)$ and the residue field of k with respect to \mathbf{t} has 3 elements. So Proposition 4.1 shows $\text{ht}_2(\chi) \leq 1 - 1 = 0$. Since $s_2(\chi) = 1$, there are noncrossed products of index 8 in the fiber over χ .

d) Let $k = \mathbb{F}_2(\mathbf{t})$ and let $k(\chi)$ be generated by a root of $X^3 + p(\mathbf{t})X + p(\mathbf{t})$ where $p(\mathbf{t}) = \mathbf{t}^2 + \mathbf{t} + 1$. (Note that $k(\chi)/k$ is indeed a Galois extension because the discriminant of a polynomial of the form $X^3 + cX + c$ is always a square in characteristic 2.) Since $X^3 + p(\mathbf{t})X + p(\mathbf{t})$ is an Eisenstein polynomial for the prime $p(\mathbf{t})$, it is irreducible

over k and $p(\mathbf{t})$ is tamely ramified in $k(\chi)$. The residue field of k with respect to $p(\mathbf{t})$ has 4 elements, so Proposition 4.1 shows $\text{ht}_3(\chi) \leq 1 - 1 = 0$. Evidently, $s_3(\chi) = 0$ because $k(\chi)/k$ is cubic. Thus, there are noncrossed products of index 9 in the fiber over χ .

Remark. Let k be a global field. If $p \neq \text{char } k$ and $\mu_p \not\subset k$ then there are crossed products that have a noncrossed product p -primary component.

Proof. Let p be as described. Choose a character $\chi \in X(\mathfrak{G}_k)$ of p -power order with $s_p(\chi) = 0$ and $\text{ht}_p(\chi) < \infty$. By Theorem 2.2, there are noncrossed products $\alpha + \chi$ of index $|\chi|p^n$, where $n = \text{ht}_p(\chi) + 1$. (Note that χ is not exceptional, since $p \neq 2$.) Let $\chi' \in X(\mathfrak{G}_k)$ such that $k(\chi') = k(\mu_p)$. Since $|\chi'|$ is relatively prime to $|\chi|$, we have $\text{ht}_p(\chi + \chi') = \text{ht}_p(\chi)$ and $s_p(\chi + \chi') \geq s_p(\chi') > 0$. Thus, $n \leq \text{ht}_p(\chi + \chi') + s_p(\chi + \chi') = b_p(\chi + \chi')$. The product $(\alpha + \chi) \otimes (0 + \chi') = \alpha + (\chi + \chi')$ has index $|\chi|p^n \cdot |\chi'| = |\chi + \chi'|p^n$, so it is a crossed product by Theorem 2.2. \square

Example 5.2. Let $k = \mathbb{Q}$, $p = 3$ and let $k(\chi)$ be the cubic subfield of the 7-th cyclotomic field (cf. Example 5.1b). Then any noncrossed product $\alpha + \chi$ of index 9 (of which there are infinitely many) becomes a crossed product when tensored with the quaternion division algebra $(\frac{-3, \mathbf{t}}{\mathbb{Q}((\mathbf{t}))})$.

6. BRUSSEL'S LEMMA AND GALOIS COVERS

Let k be a global field. For any $\alpha \in \text{Br}(k)$ and $\chi \in X(\mathfrak{G}_k)$ we write $\alpha + \chi$ for the sum in $\text{Br}(k) \oplus X(\mathfrak{G}_k)$, regarded canonically (depending on \mathbf{t}) as a Brauer class over $k(\mathbf{t})$ or $k((\mathbf{t}))$. This means χ is identified with the class of the cyclic algebra $(k(\chi)(\mathbf{t})/k(\mathbf{t}), \sigma_\chi, \mathbf{t})$ resp. $(k(\chi)((\mathbf{t}))/k((\mathbf{t})), \sigma_\chi, \mathbf{t})$ defined by χ and α is identified with its restriction to $k(\mathbf{t})$ resp. $k((\mathbf{t}))$. Due to Nakayama [13] we have the index formula

$$\text{ind}(\alpha + \chi) = |\chi| \cdot \text{ind } \alpha^{k(\chi)}, \quad (6.1)$$

where $\alpha^{k(\chi)}$ denotes the restriction of α to the cyclic extension $k(\chi)$. In fact, one commonly proves this formula first over $k((\mathbf{t}))$ and then derives it over $k(\mathbf{t})$.

Brussel's Lemma. *Let $\alpha \in \text{Br}(k)$ and let $\chi \in X(\mathfrak{G}_k)$. Then $\alpha + \chi$ is a crossed product if and only if there is a Galois extension M/k containing $k(\chi)$ that splits α and has degree $[M : k(\chi)] = \text{ind } \alpha^{k(\chi)}$.*

In the case $\text{char } k = 0$ Brussel's Lemma is [4], Corollary on p.381, but also holds for arbitrary characteristic (cf. Hanke [7], Theorem 5.20).

We start our investigation into the existence of Galois extensions M/k as in Brussel's Lemma by introducing the following terminology. Let K/k be a cyclic extension of global fields and let m be a positive integer. A **Galois m -cover** of K/k is an extension field $M \supseteq K$ that is Galois over k and has degree m over K . Since non-Galois covers are not considered in this paper, we simply use the term *cover* and always mean Galois cover. We call a cover *cyclic* if M/k is cyclic. Using this terminology, $\text{ht}_p(K/k)$ is the maximal number r such that K/k has a cyclic p^r -cover, or infinity if no maximal r exists. Moreover, by taking field composita, for any positive integer m with prime factorization $m = \prod_p p^{n_p}$,

$$K/k \text{ has a cyclic } m\text{-cover} \iff n_p \leq \text{ht}_p(K/k) \text{ for all } p|m. \quad (6.2)$$

Let \mathfrak{P} be a prime of K . An m -cover M of K/k is said to have **full local degree at \mathfrak{P}** if \mathfrak{P} is non-archimedean and $[M_{\mathfrak{P}} : K_{\mathfrak{P}}] = m$, or if \mathfrak{P} is real and $[M_{\mathfrak{P}} : K_{\mathfrak{P}}] = (2, m)$, or if \mathfrak{P} is complex. Let S be a set of primes of K . An m -cover M of K/k is said to have *full local degree in S* if it has full local degree at each prime in S . Throughout the paper, S always denotes a finite set of primes while infinite sets of primes are written P .

Before establishing the connection between Brussel's Lemma and Galois covers with full local degree we recall a few essential facts about Hasse invariants. For an exposition of this material the reader is referred to Pierce [15]. The Brauer group of a global field K is given by the exact sequence

$$1 \longrightarrow \text{Br}(K) \xrightarrow{\text{inv}} \bigoplus_{\mathfrak{P} \text{ non-archim.}} \mathbb{Q}/\mathbb{Z} \oplus \bigoplus_{\mathfrak{P} \text{ real}} \frac{1}{2}\mathbb{Z}/\mathbb{Z} \xrightarrow{\Sigma} \mathbb{Q}/\mathbb{Z} \longrightarrow 1, \quad (6.3)$$

where \mathfrak{P} runs over all primes of K . We write $\text{inv}_{\mathfrak{P}} \alpha$ for the \mathfrak{P} -component of $\text{inv} \alpha$, which is called the Hasse invariant of α at \mathfrak{P} . For any $\alpha \in \text{Br}(K)$, we call the finite set

$$\text{supp}(\alpha) := \{\mathfrak{P} \mid \mathfrak{P} \text{ a prime of } K, \text{inv}_{\mathfrak{P}} \alpha \neq 0\}$$

the *support* of α . (In the literature $\text{supp}(\alpha)$ is sometimes written $\text{ram}(\alpha)$ for "ramified primes".) The index $\text{ind}_{\alpha_{\mathfrak{P}}}$ of the completion $\alpha_{\mathfrak{P}}$ is equal to the order of $\text{inv}_{\mathfrak{P}} \alpha$, and $\text{ind} \alpha$ is equal to the least common multiple of all $\text{ind}_{\alpha_{\mathfrak{P}}}$. We say α has *full local index at \mathfrak{P}* if \mathfrak{P} is non-archimedean and $\text{ind}_{\alpha_{\mathfrak{P}}} = \text{ind} \alpha$, or if \mathfrak{P} is real and $\text{ind}_{\alpha_{\mathfrak{P}}} = (2, \text{ind} \alpha)$, or if \mathfrak{P} is complex. We define the *restricted support* of α by (note the underline)

$$\underline{\text{supp}}(\alpha) := \{\mathfrak{P} \in \text{supp}(\alpha) \mid \alpha \text{ has full local index at } \mathfrak{P}\}.$$

Unless $\alpha = 1$, $\underline{\text{supp}}(\alpha)$ is finite and $\underline{\text{supp}}(\alpha) \subseteq \text{supp}(\alpha)$. Under extension of scalars to a finite Galois extension $M \supseteq K$,

$$\text{inv}_{\Omega} \alpha^M = [M_{\mathfrak{P}} : K_{\mathfrak{P}}] \cdot \text{inv}_{\mathfrak{P}} \alpha \quad (6.4)$$

holds for all primes Ω of M dividing \mathfrak{P} .

Lemma 6.1. *Let $\alpha \in \text{Br}(k)$, $\chi \in X(\mathfrak{G}_k)$ and $m = \text{ind} \alpha^{k(\chi)}$.*

- a) *If $\alpha + \chi$ is a crossed product then there is an m -cover of $k(\chi)/k$ with full local degree in $\underline{\text{supp}}(\alpha^{k(\chi)})$.*
- b) *If there is an m -cover of $k(\chi)/k$ with full local degree in $\text{supp}(\alpha^{k(\chi)})$ then $\alpha + \chi$ is a crossed product.*

Proof. If $\alpha + \chi$ is a crossed product then, by Brussel's Lemma, there is an m -cover M of $k(\chi)/k$ that splits $\alpha^{k(\chi)}$. Let $\mathfrak{P} \in \underline{\text{supp}}(\alpha^{k(\chi)})$ and let Ω be a prime of M with $\Omega | \mathfrak{P}$. By (6.3) and (6.4),

$$0 = \text{inv}_{\Omega} \alpha^M = [M_{\mathfrak{P}} : k(\chi)_{\mathfrak{P}}] \text{inv}_{\mathfrak{P}} \alpha^{k(\chi)}.$$

Since $\alpha^{k(\chi)}$ has full local index at \mathfrak{P} it follows that M has full local degree at \mathfrak{P} . Thus, a) is proved.

Conversely, any m -cover of $k(\chi)/k$ with full local degree in $\text{supp}(\alpha^{k(\chi)})$ splits $\alpha^{k(\chi)}$ by (6.3) and (6.4), hence b) also follows from Brussel's Lemma. \square

Quantification over all division algebras of fixed degree in a fiber yields

Proposition 6.2. *Let $\chi \in X(\mathfrak{G}_k)$. Then all division algebras in the fiber over χ of index $|\chi|m$ are crossed products if and only if for all finite sets S of primes of $k(\chi)$ there is an m -cover of $k(\chi)/k$ with full local degree in S .*

Proof. The “if”-part is immediate from Lemma 6.1 b). The “only if”-part follows from Lemma 6.1 a) and the fact that for any finite S there exists $\alpha \in \text{Br}(k)$ with $\text{ind} \alpha^{k(\chi)} = m$ and $\underline{\text{supp}}(\alpha^{k(\chi)}) \supseteq S$ (this is a consequence of (6.3)). \square

Remark. By negation of Proposition 6.2, the fiber over χ contains noncrossed products of index $|\chi|m$ if and only if there is a finite set S of primes of $k(\chi)$ such that no m -cover of $k(\chi)/k$ has full local degree in S .

We use Proposition 6.2 to reformulate Theorem 2.2 in terms of Galois covers. Let K/k be a cyclic extension of global fields. For each prime p , define $b_p := b_p(K/k)$ as in Definition 2.1. Let m be a positive integer with prime factorization $m = \prod_p p^{n_p}$. Then part (i) of Theorem 2.2 is equivalent to

Theorem 6.3. *If $n_p \leq b_p$ for all prime factors p of $[K : k]$ then for any finite set S of primes of K there is an m -cover with full local degree in S .*

Theorem 6.4 below implies part (ii) of Theorem 2.2.

Theorem 6.4. *If $n_p > b_p$ for some prime factor p of $[K : k]$ then there are non-archimedean primes $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ in K such that there is no m -cover of K/k with full local degree in $\{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2\}$. In fact, there are infinite sets P_0, P_1, P_2 of non-archimedean primes of K such that for any $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2) \in P_0 \times P_1 \times P_2$ there is no m -cover of K/k with full local degree in $\{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2\}$.*

7. PROOFS

We begin with Theorem 6.3. We can assume m to be a prime power because the compositum of p^{n_p} -covers with full local degree in S is clearly an m -cover with full local degree in S . So let $m = p^n$ (p prime). We can further assume that $[K : k]$ is a p -power, because any p^n -cover of K_p/k yields a p^n -cover of K/k . Under these assumptions Theorem 6.3 reads:

Theorem 7.1. *Let p be a prime and let $[K : k]$ be a p -power. For any $n \leq b_p$ and any finite set S of primes of K there is a p^n -cover of K/k with full local degree in S .*

Proposition 7.2. *Let p be a prime and let $[K : k]$ be a non-trivial p -power. For any $n \leq \text{ht}_p(K/k) + s_p(K)$ and any finite set S of primes of K there is a p^n -cover of K/k with full local degree in S .*

Proof. Write n as $n = n' + n''$ with $n' \leq \text{ht}_p(K/k)$ and $n'' \leq s_p(K)$. Since $n' \leq \text{ht}_p(K/k)$, there is a cyclic $p^{n'}$ -cover M' of K/k . We can assume M' to have full local degree in S (this is a standard argument, see e.g. [8, Proposition 2] for details). Now, we invoke the main theorem of [8]: since M' contains all $p^{n''}$ -th roots of unity there is a $p^{n''}$ -cover M of M'/k with full local degree in S (more precisely: in the set of primes of M' dividing a prime in S). The field M is clearly a p^n -cover of K/k with full local degree in S . \square

Proof of Theorem 7.1 if K/k is non-exceptional or $p \neq 2$.

Since we have $b_p = \text{ht}_p(K/k) + s_p(K)$ the assertion of Theorem 7.1 is exactly Proposition 7.2. \square

The proof of Theorem 7.1 if K/k is exceptional and $p = 2$ is postponed to Section 11. So far, we have completed the proof of Theorem 6.3 when K/k is non-exceptional or m is odd.

We turn to Theorem 6.4, starting with a reduction of the proof to prime powers m .

Lemma 7.3. *Let m be a positive integer. If $\text{char } k = p_0 > 0$ then write $m = p_0^{n_0} m_0$ with $p_0 \nmid m_0$; if $\text{char } k = 0$ then set $m_0 = m$.*

There are infinitely many primes \mathfrak{P}_0 of K such that any m -cover of K/k with full local degree at \mathfrak{P}_0 contains an m' -cover of K/k (with full local degree at \mathfrak{P}_0) that is abelian over K with $m_0 | m'$.

Proof. By Chebotarev's density theorem (cf. page 7) there are infinitely many non-archimedean primes \mathfrak{P}_0 of K that split completely in $K(\mu_{m_0})$. Since $\text{char } k \nmid m_0$, we can assume $(N(\mathfrak{P}_0), m_0) = 1$ for all of them, hence $N(\mathfrak{P}_0) \equiv 1 \pmod{m_0}$ by (3.4) on page 7. Let \mathfrak{P}_0 be such a prime and let M be an m -cover of K/k with full local degree at \mathfrak{P}_0 . Then $N := \text{Gal}(M/K) \cong \text{Gal}(M_{\mathfrak{P}_0}/K_{\mathfrak{P}_0})$. If $\text{char } k = p_0$ then let N_0 be the maximal normal p_0 -subgroup of N ; if $\text{char } k = 0$ then let $N_0 = 1$. Let W be the wild inertia group of \mathfrak{P}_0 in M/K . If $\text{char } k = 0$ then W is trivial; if $\text{char } k = p_0$ then W is normal in N . In any case, $W \subseteq N_0$. Let e_0 be the tame ramification index of \mathfrak{P}_0 in M/K . Then $e_0 | m_0$, so $N(\mathfrak{P}_0) \equiv 1 \pmod{m_0}$ implies $N(\mathfrak{P}_0) \equiv 1 \pmod{e_0}$. Hence, by (3.3) on page 6, N/W is abelian, so N/N_0 is abelian.

Let M_0 be the fixed field of N_0 . Since N_0 is a characteristic subgroup of N , N_0 is normal in $\text{Gal}(M/k)$. So M_0 is Galois over k and abelian over K . Clearly, m_0 divides $m' := |N : N_0| = [M_0 : K]$, so the proof is completed. \square

Corollary. *For any positive integer m there are infinitely many primes \mathfrak{P}_0 of K such that any m -cover M of K/k with full local degree at \mathfrak{P}_0 contains a p^n -cover M' of K/k (with full local degree at \mathfrak{P}_0) for each primary component p^n of m with $p \neq \text{char } k$.*

Proof. Let \mathfrak{P}_0 be a prime as in Lemma 7.3 and suppose M is an m -cover of K/k with full local degree at \mathfrak{P}_0 . Let $M_0 \subseteq M$ be an m' -cover of K/k as in Lemma 7.3. For any prime p , the prime-to- p part N' of the abelian group $\text{Gal}(M_0/K)$ is a characteristic subgroup. Since $\text{Gal}(M_0/K)$ is normal in $\text{Gal}(M_0/k)$, also N' is normal in $\text{Gal}(M_0/k)$. This proves that M_0 contains a p^n -cover of K/k (with full local degree at \mathfrak{P}_0), where p^n is the p -primary component of m' . Since the p -primary components of m and m' are equal for $p \neq \text{char } k$, we are done. \square

Let P_0 be the infinite set of primes \mathfrak{P}_0 in the Corollary. As we will explain, Theorem 6.4 reduces to

Theorem 7.4. *Let p be a prime different from $\text{char } k$. For any positive integer $n > b_p$ there are infinite sets P_1, P_2 of primes of K such that*

for any $\mathfrak{P}_1 \in P_1$ and any $\mathfrak{P}_2 \in P_2$ there is no p^n -cover of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$.

We show first that Theorem 6.4 follows from Theorem 7.4. Assume $n_p > b_p$ for some p dividing $[K : k]$. Then $p \neq \text{char } k$, for otherwise $b_p = \infty$. Let $n = n_p$ and let P_1, P_2 be as in Theorem 7.4. Suppose M is an m -cover of K/k with full local degree in $\{\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2\}$ for some $(\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2) \in P_0 \times P_1 \times P_2$. By choice of P_0 , the m -cover M contains a p^n -cover M' of K/k . This contradicts Theorem 7.4 because M' has full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$. Thus, M cannot exist and Theorem 6.4 is proved.

We now turn to the proof of Theorem 7.4. For the rest of this section, the prime p is fixed and different from $\text{char } k$. Define the cyclotomic part of K/k to be

$$T := K \cap k(\mu_{p^\infty}),$$

where μ_{p^∞} denotes the group of all p -power roots of unity. Of course, $s_p(T) = s_p(K)$.

Lemma 7.5. *For any nonnegative integer n there are infinitely many primes \mathfrak{P} of K such that any p^n -cover of K/k with full local degree at \mathfrak{P} is abelian over $T = K \cap k(\mu_{p^\infty})$.*

Proof. Since any p^n -cover of K/k with full local degree at \mathfrak{P} is also a p^n -cover of K/T with full local degree at \mathfrak{P} , it suffices to assume $T = k$. We apply Chebotarev's density theorem to the compositum $K \cdot k(\mu_{p^n})$. The extension K/k is cyclic and $K \cap k(\mu_{p^n}) = k$, so there are infinitely many non-archimedean primes \mathfrak{p} of k that are inert in K and split completely in $k(\mu_{p^n})$. Since $p \neq \text{char } k$, we can assume $p \nmid N(\mathfrak{p})$ for all of them, hence $N(\mathfrak{p}) \equiv 1 \pmod{p^n}$ by (3.4).

Suppose M is a p^n -cover of K/k with full local degree at a prime \mathfrak{P} of K dividing such a \mathfrak{p} . Since \mathfrak{P} is inert in K/k and M has full local degree at \mathfrak{P} , we have $\text{Gal}(M/k) \cong \text{Gal}(M_{\mathfrak{p}}/k_{\mathfrak{p}})$. Let e be the ramification index of \mathfrak{p} in M/k . Since \mathfrak{p} is unramified in K/k , we have $e|p^n$. Thus $N(\mathfrak{p}) \equiv 1 \pmod{e}$, and (3.3) shows that $M_{\mathfrak{p}}/k_{\mathfrak{p}}$ (hence also M/k) is abelian. \square

We distinguish two cases. Say K/k is **Case A** if $k(\mu_{p^{s_p(K)+1}})/k$ is cyclic and **Case B** if $k(\mu_{p^{s_p(K)+1}})/k$ is non-cyclic.

Proposition 7.6. *Suppose K/k is Case A. There are infinite sets P_1, P_2 of primes of K such that for any p^n -cover M of K/k with $n > s_p(K)$ and with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$, M contains a cyclic $p^{n-s_p(K)}$ -cover of K/k .*

Proof. Let $s = s_p(K)$. Let P_1 be an infinite set of primes of K as in Lemma 7.5. We apply Chebotarev's density theorem to the extensions K/k and $k(\mu_{p^{s+1}})/k$. Since they are both cyclic there are infinitely many non-archimedean primes \mathfrak{p} of k that are inert in K as well as in $k(\mu_{p^{s+1}})$. Let P_2 be an infinite set of primes \mathfrak{P} of K that divide such a \mathfrak{p} . Since $p \neq \text{char } k$, we can assume $p \nmid N(\mathfrak{P})$ for all of them.

Suppose M is a p^n -cover of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$. Then there is a unique prime \mathfrak{Q}_2 of M dividing \mathfrak{P}_2 . Let I be the inertia field and let p^e be the ramification index of \mathfrak{Q}_2 over k . Since \mathfrak{P}_2 is inert in K/k , the field I is clearly a cyclic p^{n-e} -cover of K/k . It remains to show $e \leq s$. Let $\mathfrak{p}_2 = \mathfrak{P}_2 \cap T$. By choice of \mathfrak{P}_1 , M/T is abelian, hence also $M_{\mathfrak{Q}_2}/T_{\mathfrak{p}_2}$ is abelian. Moreover, $M_{\mathfrak{Q}_2}/T_{\mathfrak{p}_2}$ is tame because $p \nmid N(\mathfrak{P}_2)$. Thus, $N(\mathfrak{p}_2) \equiv 1 \pmod{p^e}$ by (3.3). On the other hand, by (3.4), $N(\mathfrak{p}_2) \not\equiv 1 \pmod{p^{s+1}}$ because \mathfrak{p}_2 is inert in the nontrivial extension $T(\mu_{p^{s+1}})/T$. This proves $e \leq s$. \square

Lemma 7.7 (Case B). *If K/k is Case B then k is a number field, $p = 2$ and $s_2(K) > s_2(k) = 1$. Moreover,*

- a) $\tilde{k} = \mathbb{Q}(\eta_{2^s})$ where $s := s_2(K)$,
- b) $T = k(i)$.

Proof. Suppose K/k is Case B. Then k is a number field, $p = 2$ and $s_2(K) > 1$, for otherwise, $k(\mu_{p^{s_2(K)+1}})/k$ would be cyclic. Let $s = s_2(K)$. Since $\text{Gal}(k(\mu_{2^{s+1}})/k) \cong \text{Gal}(\tilde{k}(\mu_{2^{s+1}})/\tilde{k})$, we can conclude from the diagram (3.1) on page 6 that $\tilde{k} = \mathbb{Q}(\eta_{2^r})$ for some $r \leq s$. But $k(\mu_{2^s})/k$ is cyclic, so $r = s$. This proves $s(k) = 1$ and part a). Since $i \in \tilde{K}$ we have $\tilde{K} = \mathbb{Q}(\mu_{2^s})$. Hence $T = k \cdot \tilde{K} = k(i)$. Thus, b) is proved. \square

Proposition 7.8. *Suppose K/k is Case B. There are infinite sets P_1, P_2 of primes of K such that any 2^n -cover of K/k with $n > s_2(K)$ and with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$ contains a cyclic 2-cover of K/k .*

Proof. Let $s = s_2(K)$. By Lemma 7.7 a), k is a number field, $p = 2$, $s > 1 = s_2(k)$, $\eta_{2^s} \in k$ and $\eta_{2^{s+1}} \notin K$. Any non-archimedean prime \mathfrak{p} of k that is inert in K and does not divide 2 has $N(\mathfrak{p}) \equiv -1 \pmod{2^s}$. Indeed, by (3.5), $\eta_{2^s} \in k$ implies $N(\mathfrak{p}) \equiv \pm 1 \pmod{2^s}$, and by (3.4), $\mu_{2^s} \subset K$ and $s > s_2(k)$ imply $N(\mathfrak{p}) \not\equiv 1 \pmod{2^s}$. The extension $k(\eta_{2^{s+1}})/k$ is quadratic, hence $K \cap k(\eta_{2^{s+1}}) = k$. We apply Chebotarev's density theorem to the compositum $K \cdot k(\eta_{2^{s+1}})$. On the one hand, there are infinitely many non-archimedean primes \mathfrak{p}_1 of k not dividing 2 that are inert in K as well as in $k(\eta_{2^{s+1}})$. On the other hand, there are

infinitely many non-archimedean primes \mathfrak{p}_2 of k not dividing 2 that are inert in K and split completely in $k(\eta_{2^{s+1}})$. Owing to (3.5), we conclude $N(\mathfrak{p}_1) \equiv -1 + 2^s \pmod{2^{s+1}}$ for all \mathfrak{p}_1 and $N(\mathfrak{p}_2) \equiv -1 \pmod{2^{s+1}}$ for all \mathfrak{p}_2 . Hence,

$$N(\mathfrak{p}_1) \not\equiv N(\mathfrak{p}_2)^l \pmod{2^n} \quad \text{for any } n > s \text{ and any } l \in \mathbb{N}. \quad (7.1)$$

Let $n > s$ and let $\mathfrak{P}_1, \mathfrak{P}_2$ be the unique primes of K with $\mathfrak{P}_i | \mathfrak{p}_i$. Assume there is a 2^n -cover M of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ that does not contain a cyclic 2-cover of K/k . Then $\mathfrak{p}_1, \mathfrak{p}_2$ have unique prime divisors in M , so the notation $Z_{\mathfrak{p}_i}(M/k), I_{\mathfrak{p}_i}(M/k), \varphi_{\mathfrak{p}_i}(M/k)$ is unambiguous and $Z_{\mathfrak{p}_i} = k$. Moreover, $I_{\mathfrak{p}_i}(M/k) \supseteq K$ because \mathfrak{p}_i is inert in K . We conclude $I_{\mathfrak{p}_i}(M/k) = K$, for $I_{\mathfrak{p}_i}/Z_{\mathfrak{p}_i}$ is always cyclic and M was assumed not to contain a cyclic 2-cover of K/k . Thus, $\varphi_{\mathfrak{p}_1}$ and $\varphi_{\mathfrak{p}_2}$ are both generators of $\text{Gal}(K/k)$. Let $\varphi_{\mathfrak{p}_1} = \varphi_{\mathfrak{p}_2}^l$ with $l \in \mathbb{N}$. By (3.2), $\varphi_{\mathfrak{p}_i}$ acts on $\text{Gal}(F/K)$ as the $N(\mathfrak{p}_i)$ -th power map. This means that $\varphi_{\mathfrak{p}_1}$ acts at the same time as the $N(\mathfrak{p}_1)$ -th power map and as the $N(\mathfrak{p}_2)^l$ -th power map on a cyclic group of order 2^n , contradicting (7.1). The proof is thus completed by choosing, for each $i = 1, 2$, the set P_i to be the set of primes of K dividing one of the infinitely many \mathfrak{p}_i as above. \square

Proof of Theorem 7.4 if K/k is non-exceptional or $p \neq 2$.

Let K/k be non-exceptional or $p \neq 2$ and let $n > b_p = \text{ht}_p(K/k) + s_p(K)$. Theorem 7.4 claims the existence of infinite sets P_1, P_2 of primes of K such that for any $\mathfrak{P}_1 \in P_1$ and any $\mathfrak{P}_2 \in P_2$ there is no p^n -cover of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$. We choose P_1, P_2 depending on the case of K/k .

Case A : Choose P_1, P_2 as in Proposition 7.6. Then, if there exists a p^n -cover with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$, we can conclude $\text{ht}_p(K/k) \geq n - s_p(K)$. Since this contradicts the hypothesis $n > \text{ht}_p(K/k) + s_p(K)$, there is no p^n -cover of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for any $\mathfrak{P}_1 \in P_1, \mathfrak{P}_2 \in P_2$.

Case B : By Lemma 7.7, k is a number field, $p = 2$, -1 is a square in K and $T = k(i)$. Thus, K/k is non-exceptional by assumption, i.e. $\text{ht}_2(K/k) = 0$ or $\text{ht}_2(K/T) = \text{ht}_2(K/k)$.

Subcase $\text{ht}_2(K/k) = 0$: Choose P_1, P_2 as in Proposition 7.8. Then, if there is a 2^n -cover with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$, we can conclude $\text{ht}_2(K/k) > 0$, a contradiction.

Subcase $\text{ht}_2(K/T) = \text{ht}_2(K/k)$: Since every 2^n -cover of K/k is also a 2^n -cover of K/T , it suffices to prove the claim for K/T . But K/T is Case A because $s_2(T) > 1$, hence we are done. \square

The proof of Theorem 7.4 if K/k is exceptional and $p = 2$ is postponed to Section 11. So far, we have completed the proof of Theorem 6.4 when K/k is non-exceptional or m is odd. Altogether at this point, Theorem 2.2 is proved when K/k is non-exceptional or m is odd.

Example 7.9. Let $k = \mathbb{Q}$ and let $K = \mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-3})$ as in Example 5.1a. Then $s_2(\chi) = 1$ and $\text{ht}_2(\chi) = 0$ (cf. Example 4.3). Thus, Theorem 7.4 applies with $p = 2$ for all $n \geq 2$, and we shall determine explicitly the predicted sets P_1, P_2 . Note that K/\mathbb{Q} is case A, so the sets P_1, P_2 are described in the proof of Proposition 7.6. The odd primes that are inert in K are the primes $\equiv 2 \pmod{3}$. The set P_1 consists of the primes that are inert in K and split completely in $\mathbb{Q}(\mu_{2^n})$, so

$$\begin{aligned} P_1 &= \{q \in \mathbb{P} \mid q \equiv 1 + 2^n \pmod{3 \cdot 2^n}\} \quad \text{if } n \text{ is even,} \\ P_1 &= \{q \in \mathbb{P} \mid q \equiv 1 + 2^{n+1} \pmod{3 \cdot 2^n}\} \quad \text{if } n \text{ is odd.} \end{aligned}$$

The set P_2 consists of the primes that are inert in K as well as in $\mathbb{Q}(\mu_{2^2})$, so

$$P_2 = \{q \in \mathbb{P} \mid q \equiv -1 \pmod{12}\}.$$

The remainder of the paper is primarily devoted to the proof of Theorem 2.2 for exceptional K/k and even m .

8. THE LOCAL-GLOBAL PRINCIPLE FOR THE HEIGHT

Theorem 8.1. *Let K/k be a cyclic extension of local fields. For any prime-power p^r ,*

$$\text{ht}_p(K/k) \geq r \iff \mu_{p^r}(k) \subset N_{K/k}(K^*).$$

(In contrast to Albert's Theorem only those p^r -th roots that are actually contained in k are required to be norms here.)

Proof. This is a consequence of local class field theory and can be found in Neukirch [14, Satz (5.1), p. 83]. \square

Corollary 8.2. *Let k be a local field with $\text{char } k \neq 2$ and $i \notin k$. Then any finite cyclic extension K/k has $\text{ht}_2(K/k) \in \{0, \infty\}$.*

Proof. Since $\mu_{2^r}(k) = \{\pm 1\}$, the condition $\mu_{2^r}(k) \subset N_{K/k}(K^*)$ is independent of r for all $r \geq 1$. Thus, by Theorem 8.1, $\text{ht}_2(K/k) > 0$ if and only if $\text{ht}_2(K/k) = \infty$. \square

Now, let K/k be a cyclic extension of global fields. Recall that by Lemma 4.2, $\text{ht}_p(K/k) \leq \text{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}})$ for all primes \mathfrak{p} in k .

Definition 8.3. The global field k is called *special with index s* if k is a number field with $\tilde{k} = \mathbb{Q}(\eta_{2^s})$ for some $s \geq 2$ and $\tilde{k}_{\mathfrak{p}} \supsetneq \tilde{k}$ for all primes of k . (cf. Geyer-Jensen [6, Definition 2, p. 709])

Recall that \tilde{k} stands for the intersection $k \cap \mathbb{Q}(\mu_{2^\infty})$. The condition $\tilde{k}_{\mathfrak{p}} \neq \tilde{k}$ is in fact only required for even primes, for it always holds for odd primes.

Theorem 8.4. *Let K/k be a cyclic extension of global field and let p be a prime. For all $r \in \mathbb{N}$, if $p \neq 2$ or k is not special or k is special with index $\geq r$ then:*

$$\mathrm{ht}_p(K/k) \geq r \iff \mathrm{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}}) \geq r \text{ for all primes } \mathfrak{p} \text{ in } k. \quad (8.1)$$

If k is special with index s then there is an idèle class c of k , such that for all $r > s$:

$$\mathrm{ht}_p(K/k) \geq r \iff \begin{array}{l} \mathrm{ht}_p(K_{\mathfrak{p}}/k_{\mathfrak{p}}) \geq r \text{ for all primes } \mathfrak{p} \text{ in } k \text{ and} \\ c \text{ is the norm of an idèle class of } K. \end{array} \quad (8.2)$$

Proof. Artin-Tate [3], Chapter X, Theorem 6. (Beware of the misprint: “ c_p ” should be “ a_0 ” and “Lemma 2” should be “Theorem 2”. Furthermore, “in the special case” should be understood as “in the special case with $S_0 = \emptyset$ with $r > s$ ”.) \square

Corollary 8.5. *If K/k is exceptional then k is special with index $s = \mathrm{ht}_2(K/k)$.*

Proof. Let K/k be exceptional. By Definition 2.1, k is a number field, $i \in K$ and $\mathrm{ht}_2(K/k(i)) > \mathrm{ht}_2(K/k) > 0$. We prove the statement by contradiction, so assume k is not special or special with index $s \neq \mathrm{ht}_2(K/k)$. If k is not special or special with $s > \mathrm{ht}_2(K/k)$ then (8.1) holds for all $r \in \mathbb{N}$. If k is special with $s < \mathrm{ht}_2(K/k)$ then (8.2) applied with $r = \mathrm{ht}_2(K/k)$ shows that c is a norm. Since c is independent of r , (8.1) holds for $r := \mathrm{ht}_2(K/k) + 1$ in both cases. Hence, there is a prime \mathfrak{p} in k such that $\mathrm{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) < r$. Since $0 < \mathrm{ht}_2(K/k) \leq \mathrm{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) < \infty$, we conclude from Corollary 8.2 that $i \in k_{\mathfrak{p}}$. Therefore, $\mathrm{ht}_2(K/k(i)) \leq \mathrm{ht}_2(K_{\mathfrak{p}}/k(i)_{\mathfrak{p}}) = \mathrm{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) < r$. This means $\mathrm{ht}_2(K/k(i)) \leq r - 1 = \mathrm{ht}_2(K/k)$ in contradiction to K/k being exceptional, so the claim is proved. \square

Remark. Let k be special with index s . Then any finite cyclic extension K/k with $i \in K$ is Case B (for $p = 2$) and has $s_2(K) = s$.

Proof. If $i \in K$ then $s_2(K) + 1 \geq 3$. Since $\tilde{k} = \mathbb{Q}(\eta_{2^s})$, the extension $k(\mu_{2^{s_2(K)+1}})/k$ is non-cyclic, i.e. K/k is Case B. By Lemma 7.7, $s = s_2(K)$. \square

9. EMBEDDING PROBLEMS AND GALOIS COVERS

Let K/k be a Galois extension of fields and let $G = \text{Gal}(K/k)$. Suppose an embedding of K into the separable closure of k is fixed, and let $\varphi : \mathfrak{G}_k \rightarrow G$ be the canonical surjection where \mathfrak{G}_k denotes the absolute Galois group of k . An *embedding problem* for K/k is a diagram

$$\begin{array}{ccccccc} & & & \mathfrak{G}_k & & & \\ & & & \downarrow \varphi & & & \\ & & \psi \cdot \cdot & & & & \\ & & \blacktriangleright & & & & \\ 1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G \longrightarrow 1 \end{array} \quad (9.1)$$

where the bottom row is a group extension. By a *solution* to the embedding problem we mean either a homomorphism ψ that makes the diagram commute or the fixed field of the kernel of such ψ . We speak of a *proper solution* if ψ is surjective, or $A \subset \text{im } \psi$. Another embedding problem

$$\begin{array}{ccccccc} & & & \mathfrak{G}_k & & & \\ & & & \downarrow \varphi & & & \\ & & \psi \cdot \cdot & & & & \\ & & \blacktriangleright & & & & \\ 1 & \longrightarrow & B & \longrightarrow & \Gamma & \xrightarrow{\sigma} & G \longrightarrow 1 \end{array} \quad (9.2)$$

is said to *dominate* (9.1) if there is a commuting diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & B & \longrightarrow & \Gamma & \xrightarrow{\sigma} & G \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\pi} & G \longrightarrow 1 \end{array}$$

with surjective vertical arrows. In this case any proper solution to (9.2) yields a proper solution to (9.1).

Now let k be a global field. For each prime \mathfrak{p} in k , write $k_{\mathfrak{p}}$ for the completion and $\mathfrak{G}_{\mathfrak{p}}$ for the absolute Galois group of $k_{\mathfrak{p}}$. We regard $\mathfrak{G}_{\mathfrak{p}}$ as a subgroup of \mathfrak{G}_k . Let K/k be a finite Galois extension, $G = \text{Gal}(K/k)$, and let $\varphi : \mathfrak{G}_k \rightarrow G$ be the canonical surjection. We set $G_{\mathfrak{p}} := \varphi(\mathfrak{G}_{\mathfrak{p}})$ and define $\varphi_{\mathfrak{p}} : \mathfrak{G}_{\mathfrak{p}} \rightarrow G_{\mathfrak{p}}$ as a restriction of φ . In this way, we have associated to (9.1) for each prime \mathfrak{p} in k a corresponding local diagram

$$\begin{array}{ccccccc} & & & \mathfrak{G}_{\mathfrak{p}} & & & \\ & & & \downarrow \varphi_{\mathfrak{p}} & & & \\ & & \psi_{\mathfrak{p}} \cdot \cdot & & & & \\ & & \blacktriangleright & & & & \\ 1 & \longrightarrow & A & \longrightarrow & E_{\mathfrak{p}} & \xrightarrow{\pi} & G_{\mathfrak{p}} \longrightarrow 1 \end{array} \quad (9.3)$$

where $E_{\mathfrak{p}} = \pi^{-1}(G_{\mathfrak{p}}) \subseteq E$. For each prime \mathfrak{P} in K dividing \mathfrak{p} we can identify $\text{Gal}(K_{\mathfrak{P}}/k_{\mathfrak{p}})$ with $G_{\mathfrak{p}}$. (In fact, $G_{\mathfrak{p}}$ is one of the decomposition groups for \mathfrak{p} in $\text{Gal}(K/k)$.) Thus, (9.3) is regarded as an embedding

problem for $K_{\mathfrak{P}}/k_{\mathfrak{p}}$. Any global solution ψ gives local solutions $\psi_{\mathfrak{p}}$ at all \mathfrak{p} by restricting ψ to $\mathfrak{G}_{\mathfrak{p}}$, but they are not necessarily proper even if ψ is proper.

An m -cover of K/k is by definition the proper solution to an embedding problem of the form (9.1) with $|A| = m$ and arbitrary E . For a non-archimedean prime \mathfrak{P} in K , the m -cover given by ψ has full local degree at \mathfrak{P} if and only if the corresponding local solution $\psi_{\mathfrak{P}}$ is also proper, i.e. if and only if $A \subset \text{im}(\psi_{\mathfrak{P}})$. For a real prime \mathfrak{P} in K , the m -cover given by ψ has full local degree at \mathfrak{P} if and only if the corresponding local solution $\psi_{\mathfrak{P}}$ satisfies $|\text{im}(\psi_{\mathfrak{P}})| = (2, m)$.

A related well-studied topic is *embedding problems with local prescription*, which exactly prescribe local solutions $\psi_{\mathfrak{p}}$ at finitely many primes rather than just local degrees. Even though this problem is more restrictive than ours, we are required to make use of this theory in order to deal with exceptional case. The crucial result for this purpose is Theorem 9.1 below.

We regard A as a \mathfrak{G}_k -module through φ . Let $A' = \text{Hom}(A, \mu)$ denote the \mathfrak{G}_k -module dual to A , where μ is the natural \mathfrak{G}_k -module of all roots of unity over k . We write $k(A')$ for the fixed field of all $\sigma \in \mathfrak{G}_k$ that fix A' pointwise, and we set $G' := \text{Gal}(k(A')/k)$. Note that

$$k(A') \subseteq K(\mu_l), \quad \text{where } l = \exp A. \quad (9.4)$$

Theorem 9.1. *Let A be abelian. Suppose that for all subgroups $U \leq G'$ the map*

$$H^1(U, A') \xrightarrow{\text{res}} \prod_{\sigma \in U} H^1(\langle \sigma \rangle, A') \quad (9.5)$$

is injective. If the embedding problem (9.1) has local solutions everywhere then it has a global solution with any local prescription at finitely many primes.

Proof. We first recall two facts about embedding problems with abelian kernel from Neukirch [14]. If the map

$$H^2(\mathfrak{G}_k, A) \xrightarrow{\text{res}} \prod_{\mathfrak{p}} H^2(\mathfrak{G}_{\mathfrak{p}}, A) \quad (9.6)$$

is injective then we have a local-global principle saying: there is a global solution if and only if there are local solutions everywhere [14, (2.2)]. If there is a global solution and the map

$$H^1(\mathfrak{G}_k, A) \xrightarrow{\text{res}} \prod_{\mathfrak{p} \in S} H^1(\mathfrak{G}_{\mathfrak{p}}, A) \quad (9.7)$$

is surjective then any local prescription at S can be realized [14, (2.5)].

Next, we pass to the dual module. For each prime \mathfrak{p} in k choose in G' a decomposition group $G'_\mathfrak{p}$. The map (9.6) is injective if and only if

$$H^1(G', A') \xrightarrow{\text{res}} \prod_{\mathfrak{p}} H^1(G'_\mathfrak{p}, A') \quad (9.8)$$

is injective [14, (4.5)]. Furthermore, the map (9.7) is surjective if

$$H^1(G'_\mathfrak{p}, A') \xrightarrow{\text{res}} \prod_{\sigma \in G'_\mathfrak{p}} H^1(\langle \sigma \rangle, A') \quad (9.9)$$

is injective for all $\mathfrak{p} \in S$ [14, (6.4a)].

Now, (9.9) is injective for each \mathfrak{p} by hypothesis. Due to Chebotarev's density theorem, every $\sigma \in G'$ lies in $G'_\mathfrak{p}$ for some \mathfrak{p} . So the injectivity of (9.8) follows from the hypothesis for $U = G'$. This completes the proof. \square

Theorem 9.1 holds trivially if G' is cyclic. For the easiest non-trivial case we provide

Lemma 9.2. *For any bicyclic group $G = \langle \sigma \rangle \times \langle \tau \rangle$ and G -module M (multiplicatively written), the kernel of the map*

$$H^1(G, M) \longrightarrow H^1(\langle \sigma \rangle, M) \times H^1(\langle \tau \rangle, M) \times H^1(\langle \sigma\tau \rangle, M)$$

is isomorphic to the group Q defined as follows: For any $H \leq G$ let M^H denote the H -invariants. Consider the norm map $N_\sigma : M^{\langle \tau \rangle} \longrightarrow M^G$. Then

$$Q := (\ker(N_\sigma) \cap M^{\sigma^{-1}} \cap M^{\sigma\tau^{-1}}) / (M^{\langle \tau \rangle})^{\sigma^{-1}}.$$

Proof. Let $l : G \longrightarrow M$ be a 1-cocycle. We assume that l is normalized ($l_1 = 1$). Of course, l is fully determined by l_σ, l_τ and the cocycle conditions. If l is in the kernel then we can further assume w.l.o.g. that $l_\tau = 1$. Let $x = l_\sigma \in M$, the only remaining parameter. The cocycle conditions imply $x = l_{\sigma\tau} = l_{\tau\sigma} = x^\tau$, so $x \in M^{\langle \tau \rangle}$, as well as $N_\sigma(x) = 1$. Since the restrictions of l to $\langle \sigma \rangle$ and $\langle \sigma\tau \rangle$ also split, we have $x \in M^{\sigma^{-1}}$ and $x \in M^{\sigma\tau^{-1}}$. Conversely, any such x actually defines a 1-cocycle (with $l_\tau = 1$) because $N_\sigma(x) = 1$.

Finally, l is split if and only if there exists $a \in M$ with $a^{\tau^{-1}} = l_\tau = 1$, i.e. $a \in M^{\langle \tau \rangle}$, and $a^{\sigma^{-1}} = l_\sigma = x$. This is the assertion. \square

We point out that in the setup of the following section, K/k is always cyclic. By (9.4), $k(A') \subseteq K(\mu_l)$, $l = \exp A$. Hence, G' is cyclic or bicyclic unless $\text{char } k = 0, 8|l, \bar{k}$ is real and $i \notin K$. As we will not encounter this case, Lemma 9.2 is sufficient for our purposes.

10. METACYCLIC 2-GROUPS

A notable exception in the classification of metacyclic p -groups is the fact that the following two presentations give isomorphic groups (cf. [12, Theorem 22]):

$$\begin{aligned} & \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = a^{2^s}, a^c = a^{-1} \rangle \\ & \cong \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = a^{2^s}, a^c = a^{-1+2^s} \rangle, \end{aligned}$$

for all $s, t \geq 2$. Indeed, an isomorphism from left to right is established by mapping a to $ac^{2^{t-1}}$.

Let $1 \leq l < t$ and let C_l be a cyclic group of order 2^l with generator ρ . Then the isomorphism above is even an isomorphism between group extensions

$$\begin{array}{ccccccc} 1 & \rightarrow & \langle a, c^{2^l} \rangle & \rightarrow & \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = a^{2^s}, a^c = a^{-1} \rangle & \rightarrow & C_l \rightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \rightarrow & \langle a, c^{2^l} \rangle & \rightarrow & \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = a^{2^s}, a^c = a^{-1+2^s} \rangle & \rightarrow & C_l \rightarrow 1, \end{array}$$

where c maps to ρ in both rows. We denote this extension by

$$1 \rightarrow A_l \rightarrow E_{s,t} \rightarrow C_l \rightarrow 1. \quad (10.1)$$

and choose for $E_{s,t}$ in the sequel the presentation

$$E_{s,t} = \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = a^{2^s}, a^c = a^{-1} \rangle.$$

Clearly, $|E_{s,t}| = 2^{s+t+1}$. The center, commutator subgroup and socle are $\langle c^2 \rangle, \langle a^2 \rangle$ and $\langle a^{2^s} \rangle$ respectively. For each $1 \leq l < t$, the kernel A_l is abelian and decomposes as follows into cyclic factors. Case $t - l \leq s$: $A_l = \langle a \rangle \times \langle a^{2^{l-(t-s)}} c^{2^l} \rangle$, $\exp A_l = 2^{s+1}$. Case $t - l \geq s$: $A_l = \langle c^{2^l} \rangle \times \langle ac^{2^{t-s}} \rangle$, $\exp A_l = 2^{t+1-l}$.

We now turn to investigate the embedding problem defined by (10.1). Let K/k be any cyclic extension of order 2^l ($l \geq 1$) with $\text{Gal}(K/k) = \langle \rho \rangle$. We write C_l for $\text{Gal}(K/k)$ and consider the embedding problem

$$\begin{array}{c} \mathfrak{G}_k \\ \psi \cdot \cdot \downarrow \varphi \\ \bullet \end{array} \quad (10.2)$$

$$1 \longrightarrow A_l \longrightarrow E_{s,t} \longrightarrow C_l \longrightarrow 1$$

We first point out that (10.2) is dominated in two ways by a split embedding problem. Indeed, defining for all $s, t \geq 2$ the groups

$$\begin{aligned} \Gamma_{s,t} &= \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = 1, a^c = a^{-1} \rangle, \\ \Delta_{s,t} &= \langle a, c \mid a^{2^{s+1}} = 1, c^{2^t} = 1, a^c = a^{-1+2^s} \rangle, \end{aligned}$$

one obtains a dominating embedding problem for (10.2) by replacing the group extension by either

$$1 \rightarrow \langle a, c^{2^l} \rangle \rightarrow \Gamma_{s,t+1} \rightarrow C_l \rightarrow 1, \quad (10.3)$$

or

$$1 \rightarrow \langle a, c^{2^l} \rangle \rightarrow \Delta_{s,t+1} \rightarrow C_l \rightarrow 1. \quad (10.4)$$

(Here, c maps to ρ in all three group extensions.)

Lemma 10.1. *Let k be any field with $\text{char } k = 0$ and $i \notin k$. Let $\text{Gal}(K/k) = C_l$ with $l \geq 1$ and let $i \in K$. If k has more than one quadratic extension then (10.2) has a proper solution for all $s < s_2(k(i))$ and all $t < \text{ht}_2(K/k) + l$.*

Proof. Note that $\Gamma_{s,t+1}, \Delta_{s,t+1}$ are semi-direct products $\langle a \rangle \rtimes \langle c \rangle$ respectively with $\text{ord } a = 2^{s+1}$ and $\text{ord } c = 2^{t+1}$. Let $h = t + 1 - l$. Since $h \leq \text{ht}_2(K/k)$ there is a cyclic 2^h -cover L of K/k . We have $[L : k] = 2^{t+1}$. Choose any $a \in k$ with $\sqrt{a} \notin k(i)$. Then $M := L(\sqrt[2^{s+1}]{a})$ is a solution field for either (10.3) or (10.4). Indeed, since $\mu_{2^{s+1}} \in k(i) \subseteq L$, the extension M/k is Galois and M/L is a Kummer extension. The action of $\text{Gal}(L/k) = \langle c \rangle$ on $\text{Gal}(M/L) = \langle a \rangle$ is given by the action of $\text{Gal}(L/k)$ on $\mu_{2^{s+1}}$. Since $\mu_{2^{s+1}} \subset k(i)$, this action is of order 2. The possible actions of this order are $\zeta \mapsto \zeta^q$ for $q = -1, -1 + 2^s$ or $1 + 2^s$. Since i is not fixed, q is either -1 or $-1 + 2^s$. We have thus shown that M is a proper solution for one of the dominating embedding problems. \square

Lemma 10.2. *Let k be a number field with $\tilde{k} = \mathbb{Q}(\eta_{2^s}), s \geq 2$. Let $\text{Gal}(K/k) = C_l$ with $l \geq 1$ and let $i \in K$. For any $t > l$, Theorem 9.1 applies to the embedding problem (10.2).*

Proof. We give the proof only for $t - l \leq s$. The case $t - l \geq s$ is similar. Set $h := t - l > 0$. (The Lemma will be used only once, in the proof of Proposition 11.1, and this use is for $t - s = l$, i.e. $h = s$.) For $t \leq s + l$ we have

$$\begin{aligned} E_{s,t} &= \langle a, c \mid a^{2^{s+1}} = 1, c^{2^{t+h}} = a^{2^s}, a^c = a^{-1} \rangle, \\ A_l &= \langle a, c^{2^l} \rangle = \langle a \rangle \times \langle b \rangle \text{ with } b := a^{2^{s-h}} c^{2^l}, \end{aligned}$$

$\exp A_l = \text{ord } a = 2^{s+1}, \text{ord } b = 2^h$. Note $b^c = a^{-2^{s-h+1}} b$. The action of $\text{Gal}(K/k(i)) = \langle \rho \rangle$ on A is trivial because c^2 lies in the center of $E_{s,t}$. Thus, $k(A') \subseteq k(\mu_{2^{s+1}})$.

We continue to determine $k(A')$ and G' exactly. Let $\zeta = \zeta_{2^{s+1}}$. Then $A' = \langle a^* \rangle \times \langle b^* \rangle$ where

$$\begin{aligned} a^* : a &\mapsto \zeta, & b &\mapsto 1, \\ b^* : a &\mapsto 1, & b &\mapsto \zeta^{2^{s+1-h}}. \end{aligned}$$

The action of \mathfrak{G}_k on A' is through $G_0 := \text{Gal}(k(\mu_{2^{s+1}})/k)$. This group is the Klein 4-group, generated for instance by σ, τ with $\sigma : \zeta \mapsto \zeta^{-1}$ and $\tau : \zeta \mapsto \zeta^{2^{s+1}}$. Since σ restricts to a generator of $\text{Gal}(k(i)/k)$, it acts on A like ρ does. Hence:

$$\begin{aligned} \sigma(a^*)(a) &= \sigma(a^*(a^c)) = \sigma(a^*(a^{-1})) = \sigma(\zeta^{-1}) = \zeta, \\ \sigma(a^*)(b) &= \sigma(a^*(b^c)) = \sigma(a^*(a^{-2^{s+1-h}}b)) = \sigma(\zeta^{-2^{s+1-h}}) = \zeta^{2^{s+1-h}}, \\ \sigma(b^*)(a) &= \sigma(b^*(a^c)) = \sigma(b^*(a^{-1})) = 1, \\ \sigma(b^*)(b) &= \sigma(b^*(b^c)) = \sigma(b^*(a^{-2^{s+1-h}}b)) = \sigma(\zeta^{2^{s+1-h}}) = \zeta^{-2^{s+1-h}}, \end{aligned}$$

i.e. $\sigma(a^*) = a^*b^*, \sigma(b^*) = b^{*-1}$. Since τ restricts to the identity on $k(i)$, it acts trivially on A . Hence:

$$\begin{aligned} \tau(a^*)(a) &= \tau(a^*(a)) = \tau(\zeta) = \zeta^{2^{s+1}}, \\ \tau(a^*)(b) &= \tau(a^*(b)) = 1, \\ \tau(b^*)(a) &= \tau(b^*(a)) = 1, \\ \tau(b^*)(b) &= \tau(b^*(b)) = \tau(\zeta^{2^{s+1-h}}) = \zeta^{2^{s+1-h}}, \end{aligned}$$

i.e. $\tau(a^*) = (a^*)^{2^{s+1}}, \tau(b^*) = b^*$.

Furthermore: $\sigma\tau(a^*) = (a^*)^{2^{s+1}}b^*, \sigma\tau(b^*) = b^{*-1}$. Neither σ nor τ act trivially on A , hence $k(A') = k(\mu_{2^{s+1}})$ and $G' = G_0 = \langle \sigma, \tau \rangle$.

It suffices to check the injectivity of (9.5) for $U = G'$, the only non-cyclic subgroup of G' . Easy calculations yield $A'^{\sigma^{-1}} = \langle b^* \rangle$ and $A'^{\sigma\tau^{-1}} = \langle a^{*2^s}b^* \rangle$, hence $A'^{\sigma^{-1}} \cap A'^{\sigma\tau^{-1}} = \langle b^{*2} \rangle$. On the other hand, $b^* \in A'^{\langle \tau \rangle}$ and $(b^*)^{\sigma^{-1}} = b^{*-2}$. This shows that Q vanishes in Lemma 9.2 with A' for M and G' for G . \square

11. PROOFS (EXCEPTIONAL CASE)

This section completes the proof of Theorem 2.2 for exceptional K/k and even m . It remains to prove Theorems 7.1 and 7.4 for K/k exceptional and $p = 2$. So let K/k be a cyclic extension of global fields that is exceptional, i.e. k is a number field, $i \in K$ and $\text{ht}_2(K/k(i)) > \text{ht}_2(K/k) > 0$. We begin with Theorem 7.1.

Proposition 11.1. *Suppose K/k is exceptional and $[K : k]$ is a 2-power. Let $n = \text{ht}_2(K/k) + s_2(K) + 1$. For any finite set S of primes of K there is a 2^n -cover with full local degree in S .*

Proof. Let $\text{Gal}(K/k) = C_l = \langle \rho \rangle$. By Corollary 8.5 on page 20 and the Remark following it, k is special with index $s = s_2(K)$. By Lemma 10.2, Theorem 9.1 applies to the embedding problem (10.2) for all $t > l$. Setting $t := l + \text{ht}_2(K/k)$, any solution to (10.2) is a 2^n -cover of K/k because $|A_t| = 2^{s+t+1-l} = 2^{s+\text{ht}_2(K/k)+1} = 2^n$. Using Theorem 9.1, it remains to prove the existence of proper local solutions at all non-archimedean primes \mathfrak{p} in k . (All archimedean primes become complex in K since $i \in K$.) So let \mathfrak{p} be a non-archimedean prime in k and let $G_{\mathfrak{p}} = \langle \rho^{2^g} \rangle$, $0 \leq g \leq l$.

Case 1: $g = 0$. In this case, $G_{\mathfrak{p}} = C_l$ and $E_{\mathfrak{p}} = E_{s,t}$, so the local embedding problem is identical to (10.2) with K/k replaced by $K_{\mathfrak{p}}/k_{\mathfrak{p}}$. According to Lemma 10.1, it remains to verify $s < s_2(k_{\mathfrak{p}}(i))$ and $t < \text{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) + l$.

Since $g = 0$, \mathfrak{p} is not split in $k(i)$, i.e. $k_{\mathfrak{p}}(i)/k_{\mathfrak{p}}$ is quadratic. On the other hand, $k_{\mathfrak{p}}(\mu_{2^{s+1}})/k_{\mathfrak{p}}$ is at most quadratic, for k is special with index s . Thus, $\mu_{2^{s+1}} \subset k_{\mathfrak{p}}(i)$, i.e. $s_2(k_{\mathfrak{p}}(i)) > s$. By Lemma 4.2, we have $\text{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) \geq \text{ht}_2(K/k) > 0$. Since $i \notin k_{\mathfrak{p}}$, Corollary 8.2 implies $\text{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) = \infty$.

Case 2: $g > 0$. In this case, $E_{\mathfrak{p}} = \langle a, c^{2^g} \rangle$ is abelian, for c^2 lies in the center of $E_{s,t}$. By Corollary 8.5 we have $s = \text{ht}_2(K/k)$, i.e. $t = l + s$. Thus, $\text{ord } c^{2^g} = 2^{s+1+(l-g)} \geq 2^{s+1} = \text{ord } a$, so $E_{\mathfrak{p}} = \langle c^{2^g} \rangle \times \langle ac^{2^l} \rangle \cong C_{2^{s+1+(l-g)}} \times C_{2^s}$. A solution to

$$\begin{array}{ccccccc}
 & & & & \mathfrak{G}_{\mathfrak{p}} & & \\
 & & & & \psi_{\mathfrak{p}} \cdots \cdots & \downarrow \varphi_{\mathfrak{p}} & \\
 & & & & \swarrow & & \\
 1 & \longrightarrow & \langle a, c^{2^l} \rangle & \rightarrow & \langle c^{2^g} \rangle \times \langle ac^{2^l} \rangle & \rightarrow & \langle \rho^{2^g} \rangle \longrightarrow 1
 \end{array} \tag{11.1}$$

is then obtained by composing a cyclic 2^{s+1} -cover of $K_{\mathfrak{p}}/k_{\mathfrak{p}}$ with a Kummer extension of $k_{\mathfrak{p}}$ of degree 2^s . (Note that 2^{s+1} is the order of c^{2^l} .) It remains to verify $\text{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) \geq s + 1$ and $s_2(k_{\mathfrak{p}}) \geq s$.

Since $g > 0$, \mathfrak{p} splits in $k(i)$, i.e. $i \in k_{\mathfrak{p}}$. By Lemma 4.2 on page 8 and the fact that K/k is exceptional we have $\text{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}) = \text{ht}_2(K_{\mathfrak{p}}/k_{\mathfrak{p}}(i)) \geq \text{ht}_2(K/k(i)) > \text{ht}_2(K/k) = s$. Finally, $\eta_{2^s}, i \in k_{\mathfrak{p}}$ imply $s_2(k_{\mathfrak{p}}) \geq s$. \square

Proof of Theorem 7.1 for K/k exceptional and $p = 2$.

Since we have $n \leq b_2 = \text{ht}_2(K/k) + s_2(K) + 1$, the assertion of Theorem 7.1 follows from Propositions 7.2 and 11.1 together. \square

We now turn to Theorem 7.4.

Proposition 11.2. *Suppose K/k is exceptional. There are infinite sets P_1, P_2 of primes of K such that for any 2^n -cover M of K/k with*

$n > s_2(K)$ and with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$, M contains a cyclic $2^{n-s_2(K)-1}$ -cover of K/k .

Proof. The proof is a modification of the proof of Proposition 7.6. Let $s = s_2(K)$. By Proposition 7.6, we can assume K/k is Case B, since any cyclic 2^{n-s} -cover contains a cyclic 2^{n-s-1} -cover of K/k .

Let P_1 be an infinite set of primes of K as in Lemma 7.5. We apply Chebotarev's density theorem to the extensions K/k and $k(\eta_{2^{s+1}})/k$. Since they are both cyclic ($k(\eta_{2^{s+1}})/k$ is quadratic by Lemma 7.7), there are infinitely many non-archimedean primes \mathfrak{p} of k that are inert in K as well as in $k(\eta_{2^{s+1}})$. Let P_2 be an infinite set of primes \mathfrak{P} of K that divide such a \mathfrak{p} . Since $\text{char } k \neq 2$ (k is a number field), we can assume $N(\mathfrak{P})$ is odd for all of them.

Suppose M is a 2^n -cover of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$. Then there is a unique prime \mathfrak{Q}_2 of M dividing \mathfrak{P}_2 . Let I be the inertia field and let 2^e be the ramification index of \mathfrak{Q}_2 over k . Since \mathfrak{P}_2 is inert in K/k , the field I is clearly a cyclic 2^{n-e} -cover of K/k . It remains to show $e \leq s+1$. Let $\mathfrak{p}_2 = \mathfrak{P}_2 \cap T$. By choice of \mathfrak{P}_1 , M/T is abelian, hence also $M_{\mathfrak{Q}_2}/T_{\mathfrak{p}_2}$ is abelian. Moreover, $M_{\mathfrak{Q}_2}/T_{\mathfrak{p}_2}$ is tame because $N(\mathfrak{P}_2)$ is odd. Thus, $N(\mathfrak{p}_2) \equiv 1 \pmod{2^e}$ by (3.3) on page 6.

On the other hand, since $\eta_{2^s} \in k$ (Lemma 7.7) we have $N(\mathfrak{p}) \equiv \pm 1 \pmod{2^s}$ by (3.5). Furthermore, since \mathfrak{p} is inert in $k(\eta_{2^{s+1}})$, we have $N(\mathfrak{p}) \not\equiv \pm 1 \pmod{2^{s+1}}$ again by (3.5). It follows that $N(\mathfrak{p}) \equiv \pm 1 + 2^s \pmod{2^{s+1}}$. Since T/k is quadratic (Lemma 7.7) and \mathfrak{p} is inert in T , $N(\mathfrak{p}_2) = N(\mathfrak{p})^2 \equiv 1 + 2^{s+1} \not\equiv 1 \pmod{2^{s+2}}$. This together with $N(\mathfrak{p}_2) \equiv 1 \pmod{2^e}$ shows $e \leq s+1$. \square

Proof of Theorem 7.4 for K/k exceptional and $p = 2$.

Let $n > b_2(K/k) = \text{ht}_2(K/k) + s_2(K) + 1$. Choose P_1, P_2 as in Proposition 11.2. Then, if there exists a 2^n -cover with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for some $\mathfrak{P}_1 \in P_1$ and $\mathfrak{P}_2 \in P_2$, we can conclude $\text{ht}_2(K/k) \geq n - s_2(K) - 1$. Since this contradicts the hypothesis $n > \text{ht}_2(K/k) + s_2(K) + 1$, there is no 2^n -cover of K/k with full local degree in $\{\mathfrak{P}_1, \mathfrak{P}_2\}$ for any $\mathfrak{P}_1 \in P_1, \mathfrak{P}_2 \in P_2$. \square

This completes the proof of Theorem 2.2 also for exceptional K/k and even m .

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