Invariants and time-reversibility in polynomial systems of ODEs

- The theory of invariants of ordinary differential equations has been developed by K.S. Sibirski and his coworkers in 1960-70th:
  K. S. Sibirsky. *Introduction to the Algebraic Theory of Invariants of Differential Equations.* Nonlinear Science: Theory and Applications. Manchester: Manchester University Press, 1988.
- Generalization to complex systems:
  Chapter 5 of V. G. Romanovski and D. S. Shafer, *The Center and Cyclicity Problems: A Computational Algebra Approach*, Birkhüser, Boston, 2009.

Definition

Let k be a field, G be a group of $n \times n$ matrices with elements in k, $A \in G$ and $\mathbf{x} \in k^n$. A polynomial $f \in k[x_1, \ldots, x_n]$ is *invariant under* G if $f(\mathbf{x}) = f(A \cdot \mathbf{x})$ for every $A \in G$. An invariant is *irreducible* if it does not factor as a product of polynomials that are themselves invariants.

Example. Let $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let I_2 denote the 2×2 identity matrix. The set $C_4 = \{I_2, B, B^2, B^3\}$ is a group under multiplication, and for the polynomial $f(\mathbf{x}) = f(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ we have $f(\mathbf{x}) = f(B \cdot \mathbf{x})$, $f(\mathbf{x}) = f(B^2 \cdot \mathbf{x})$, and $f(\mathbf{x}) = f(B^3 \cdot \mathbf{x})$. Thus, f is an invariant of the group C_4. When $k = \mathbb{R}$, B is simply the group of rotations by multiples of $\frac{\pi}{2}$ radians (mod 2π) about the origin in $\mathbb{R}^2$, and f is an invariant because its level sets are circles centered at the origin, which are unchanged by such rotations.

Consider the system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{1}$$

Let $Q = GL_2(\mathbb{R})$ be the group of all linear invertible transformations of $\mathbb{R}^2$:

$$\mathbf{y} = C\mathbf{x},$$

where

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det\ C \neq 0.$$

Then,

$$\frac{d\mathbf{y}}{dt} = B\mathbf{y}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = CAC^{-1} = \frac{1}{\det\ C} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

where

$d_{11} = ada_{11} + bda_{21} - aca_{12} - bca_{22}$
$d_{12} = aba_{11} - b^2a_{21} + a^2a_{12} + aba_{22},$
$d_{21} = cda_{11} + d^2a_{21} - c^2a_{12} - cda_{22},$
$d_{22} = -bca_{11} - bda_{21} + aca_{12} + ada_{22}$. Therefore,

$$b_{11} = \frac{1}{det\ C}d_{11}, \quad b_{12} = \frac{1}{det\ C}d_{12}, \quad b_{21} = \frac{1}{det\ C}d_{21}, \quad b_{22} = \frac{1}{det\ C}d_{22}.$$

We look for a homogeneous invariant of degree one:

$$I(\mathbf{a}) = k_1 a_{11} + k_2 a_{12} + k_3 a_{21} + k_4 a_{22}.$$

It should be $I(\mathbf{b}) = I(\mathbf{a})$, that is,

$$k_1 b_{11} + k_2 b_{12} + k_3 b_{21} + k_4 b_{22} = k_1 a_{11} + k_2 a_{12} + k_3 a_{21} + k_4 a_{22}.$$

Hence,

$$k_1 ad - k_2 ab + k_3 cd - k_4 bc = k_1 (ad - bc).$$

Thus, $k_2 = k_3 = 0$ and $k_4 = k_1$ and up to a constant multiplier $I_1(\mathbf{a}) = a_{11} + a_{22} = trA$.
Similarly we can show that each invariant of degree 2 must be of the form:

$$I(\mathbf{a}) = k_1(a_{11}^2 + a_{22}^2 + 2a_{11}a_{22}) + k_2(a_{11}a_{22} - a_{12}a_{21}) = k_1 tr^2 A^2 + k_2 \det A.$$

It yields that the homogeneous invariant of degree two is

$$I_2 = \det A = (a_{11}a_{22} - a_{12}a_{21}).$$

Any invariant of degree 3 and higher is a polynomial of $tr\,A$ and $\det A$.

Invariants of the rotation group

Consider polynomial systems on $\mathbb{C}^2$ in the form

$$\begin{aligned} \dot{x} &= -\sum_{(p,q)\in\widetilde{S}} a_{pq} x^{p+1} y^q = P(x,y), \\ \dot{y} &= \sum_{(p,q)\in\widetilde{S}} b_{qp} x^q y^{p+1} = Q(x,y), \end{aligned} \tag{2}$$

where the index set $\widetilde{S} \subset \mathbb{N}_{-1} \times \mathbb{N}_0$ is a finite set and each of its elements (p,q) satisfies $p+q \geq 0$. If ℓ is the cardinality of the set $\widetilde{S}$, we use the abbreviated notation $(a,b) = (a_{p_1,q_1}, a_{p_2,q_2}, \dots, a_{p_\ell,q_\ell}, b_{q_\ell,p_\ell}, \dots, b_{q_2,p_2}, b_{q_1,p_1})$ for the ordered vector of coefficients of system (2), let $E(a,b) = \mathbb{C}^{2\ell}$ denote the parameter space of (2), and let $\mathbb{C}[a,b]$ denote the polynomial ring in the variables a_{pq} and b_{qp}.

Consider the group of rotations

$$x' = e^{-i\varphi}x, \quad y' = e^{i\varphi}y \tag{3}$$

of the phase space $\mathbb{C}^2$ of (2). In (x', y') coordinates

$$\dot{x}' = -\sum_{(p,q)\in\widetilde{S}} a(\varphi)_{pq} x'^{p+1} y'^{q}, \quad \dot{y}' = \sum_{(p,q)\in\widetilde{S}} b(\varphi)_{qp} x'^{q} y'^{p+1},$$

where the coefficients of the transformed system are

$$a(\varphi)_{p_j q_j} = a_{p_j q_j} e^{i(p_j - q_j)\varphi}, \quad b(\varphi)_{q_j p_j} = b_{q_j p_j} e^{i(q_j - p_j)\varphi}, \tag{4}$$

for $j = 1, \dots, \ell$. For any fixed angle φ the equations in (4) determine an invertible linear mapping U_φ of the space $E(a, b)$ of parameters of (2) onto itself, which we will represent as the block diagonal $2\ell \times 2\ell$ matrix

$$U_\varphi = \begin{pmatrix} U_\varphi^{(a)} & 0 \\ 0 & U_\varphi^{(b)} \end{pmatrix},$$

where $U_\varphi^{(a)}$ and $U_\varphi^{(b)}$ are diagonal matrices that act on the coordinates a and b respectively.

Example. For the family of systems

$$\dot{x} = -a_{00}x - a_{-11}y - a_{20}x^3, \quad \dot{y} = b_{1,-1}x + b_{00}y + b_{02}y^3 \qquad (5)$$

$\widetilde{S}$ is the ordered set $\{(0,0),(-1,1),(2,0)\}$, and equation (4) gives the collection of $2\ell = 6$ equations

$$\begin{array}{lll} a(\varphi)_{00} = a_{00}e^{i(0-0)\varphi} & a(\varphi)_{-11} = a_{-11}e^{i(-1-1)\varphi} & a(\varphi)_{20} = a_{20}e^{i(2-0)\varphi} \\ b(\varphi)_{00} = b_{00}e^{i(0-0)\varphi} & b(\varphi)_{1,-1} = b_{1,-1}e^{i(1-(-1))\varphi} & b(\varphi)_{02} = b_{02}e^{i(0-2)\varphi} \end{array}$$

so that

$$U_\varphi \cdot (a,b) = \begin{pmatrix} U_\varphi^{(a)} & 0 \\ 0 & U_\varphi^{(b)} \end{pmatrix} \cdot (a,b)^T =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{-i2\varphi} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i2\varphi} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i2\varphi} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i2\varphi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_{00} \\ a_{-11} \\ a_{20} \\ b_{02} \\ b_{1,-1} \\ b_{00} \end{pmatrix} = \begin{pmatrix} a_{00} \\ a_{-11}e^{-i2\varphi} \\ a_{20}e^{i2\varphi} \\ b_{02}e^{-i2\varphi} \\ b_{1,-1}e^{i2\varphi} \\ b_{00} \end{pmatrix}.$$

Thus here

$$U_\varphi^{(a)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i2\varphi} & 0 \\ 0 & 0 & e^{i2\varphi} \end{pmatrix} \quad \text{and} \quad U_\varphi^{(b)} = \begin{pmatrix} e^{-i2\varphi} & 0 & 0 \\ 0 & e^{i2\varphi} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We write in the short form

$$(a(\varphi), b(\varphi)) = U_\varphi \cdot (a, b) = (U_\varphi^{(a)} \cdot a, U_\varphi^{(b)} \cdot b).$$

The set $U = \{U_\varphi : \varphi \in \mathbb{R}\}$ is a group, a subgroup of the group of invertible $2\ell \times 2\ell$ matrices with entries in k. In the context of U the group operation corresponds to following one rotation with another.

Definition

The group $U = \{U_\varphi : \varphi \in \mathbb{R}\}$ is called the *rotation group* of family (2). A polynomial invariant of the group U is termed an *invariant of the rotation group*, or more simply an *invariant*.

We wish to identify all polynomial invariants of this group action. The polynomials in question are elements of $\mathbb{C}[a, b]$. They identify polynomial expressions in the coefficients of elements of the family (2) that are unchanged under a rotation of coordinates. A polynomial $f \in \mathbb{C}[a, b]$ is an invariant of the group U if and only if each of its terms is an invariant, so it suffices to find the invariant monomials. Since

$$a(\varphi)_{p_j q_j} = a_{p_j q_j} e^{i(p_j - q_j)\varphi}, \quad b(\varphi)_{q_j p_j} = b_{q_j p_j} e^{i(q_j - p_j)\varphi},$$

for $\nu \in \mathbb{N}_0^{2\ell}$ the image of the corresponding monomial

$$[\nu] = a_{p_1 q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1 p_1}^{\nu_{2\ell}} \in \mathbb{C}[a, b]$$

under U_φ is the monomial

$$\begin{aligned} a(\varphi)_{p_1 q_1}^{\nu_1} \cdots a(\varphi)_{p_\ell q_\ell}^{\nu_\ell} b(\varphi)_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b(\varphi)_{q_1 p_1}^{\nu_{2\ell}} & \\ = a_{p_1 q_1}^{\nu_1} e^{i\varphi\nu_1(p_1 - q_1)} \ldots a_{p_\ell q_\ell}^{\nu_\ell} e^{i\varphi\nu_\ell(p_\ell - q_\ell)} & \\ b_{q_\ell p_\ell}^{\nu_{\ell+1}} e^{i\varphi\nu_{\ell+1}(q_\ell - p_\ell)} \ldots b_{q_1 p_1}^{\nu_{2\ell}} e^{i\varphi\nu_{2\ell}(q_1 - p_1)} & \end{aligned} \tag{6}$$

$$= e^{i\varphi[\nu_1(p_1-q_1)+\cdots+\nu_\ell(p_\ell-q_\ell)+\nu_{\ell+1}(q_\ell-p_\ell)+\cdots+\nu_{2\ell}(q_1-p_1)]}$$

$$a_{p_1 q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1 p_1}^{\nu_{2\ell}}.$$

The quantity in square brackets is $L_1(\nu) - L_2(\nu)$, where $L(\nu) = \binom{L_1(\nu)}{L_2(\nu)}$ is the linear operator on $\mathbb{N}_0^{2\ell}$ defined by

$$L(\nu) = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell}.$$

Thus, the monomial $[\nu]$ is an invariant if and only if $L_1(\nu) = L_2(\nu)$. We define the set $\mathcal{M}$ by

$$\mathcal{M} = \{\nu \in \mathbb{N}_0^{2\ell} : L(\nu) = \begin{pmatrix} k \\ k \end{pmatrix} \text{ for some } k \in \mathbb{N}_0\}. \qquad (7)$$

We have established that the monomial $[\nu]$ is invariant under the rotation group U of (2) if and only if $L_1(\nu) = L_2(\nu)$, that is, if and only if $\nu \in \mathcal{M}$.
For

$$[\nu] = a_{p_1 q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1 p_1}^{\nu_{2\ell}} \in \mathbb{C}[a, b]$$

its conjugate is defined by

$$[\hat{\nu}] = a_{p_1 q_1}^{\nu_{2\ell}} \cdots a_{p_\ell q_\ell}^{\nu_\ell + 1} b_{q_\ell p_\ell}^{\nu_\ell} \cdots b_{q_1 p_1}^{\nu_1} \in \mathbb{C}[a, b]$$

Since, for any $\nu \in \mathbb{N}_0^{2\ell}$, $L_1(\nu) - L_2(\nu) = -(L_1(\hat{\nu}) - L_2(\hat{\nu}))$, the monomial $[\nu]$ is invariant under U if and only if its conjugate $[\hat{\nu}]$ is.

Proposition

The monoid $\mathcal{M}$ consists of all ν such that

$$L_1(\nu) - L_2(\nu) = (p_1 - q_1)\nu_1 + (p_2 - q_2)\nu_2 + \cdots + (p_\ell - q_\ell)\nu_\ell + (q_\ell - p_\ell)\nu_{\ell+1} + \cdots + (q_1 - p_1)\nu_{2\ell} = 0. \tag{8}$$

Proof. Obviously every solution of (7) is also a solution of (8). Conversely, let ν be a solution of (8) and let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the ith basis vector of $\mathbb{C}^{2\ell}$. Then

$$L^1(\nu) = L^2(\nu) = k, \tag{9}$$

yielding

$$L^1(\nu) + L^2(\nu) = 2k. \tag{10}$$

Note that

$$L^1(e_i) + L^2(e_i) = L^1(e_{2\ell-i}) + L^2(e_{2\ell-i}) = p_i + q_i \geq 0 \tag{11}$$

for $i = 1, \ldots, \ell$. Taking into account the fact that $L(\nu)$ is a linear operator, we conclude from (10) and (11) that the number k on the right–hand side of (9) is non–negative. $\square$

Example. We will find all the monomials of degree at most three that are invariant under the rotation group U for the family of systems

$$\dot{x} = -a_{00}x - a_{-11}y - a_{20}x^3, \quad \dot{y} = b_{1,-1}x + b_{00}y + b_{02}y^3.$$

Since $\widetilde{S} = \{(0,0), (-1,1), (2,0)\}$, for $\nu \in \mathbb{N}_0^6$

$$\begin{aligned} L(\nu) &= \nu_1\,(0,0) + \nu_2\,(-1,1) + \nu_3\,(2,0) + \nu_4\,(0,2) + \nu_5\,(1,-1) + \nu_6\,(0,0) \\ &= (-\nu_2 + 2\nu_3 + \nu_5, \nu_2 + 2\nu_4 - \nu_5) \end{aligned}$$

so that equation (8) reads

$$-2\nu_2 + 2\nu_3 - 2\nu_4 + 2\nu_5 = 0. \qquad (12)$$

$\deg([\nu]) = 0$. The monomial 1, corresponding to $\nu = 0 \in \mathbb{N}_0^6$, is of course always an invariant.

$\deg([\nu]) = 1$. In this case $\nu = (0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0) \in \mathbb{N}_0^6$ for some j. Clearly (12) holds if and only if $\nu = e_1$ or $\nu = e_6$, yielding $a_{00}^1\, a_{-11}^0\, a_{20}^0\, b_{02}^0\, b_{1,-1}^0\, b_{00}^0 = a_{00}$ and to $a_{00}^0\, a_{-11}^0\, a_{20}^0\, b_{02}^0\, b_{1,-1}^0\, b_{00}^1 = b_{00}$ respectively.

$\deg([\nu]) = 2$. If $\nu = 2e_j$ and satisfies (12) then $j = 1$ or $j = 6$, yielding a_{00}^2 and b_{00}^2, respectively. If $\nu = e_j + e_k$ for $j < k$, then (12) holds if and only if either $(j,k) = (1,6)$ or one of j and k corresponds to a term in (12) with a plus sign and the other to a term with a minus sign, hence $(j,k) \in P := \{(2,3), (2,5), (3,4), (4,5)\}$. The former case gives $a_{00}\, b_{00}$; the latter case gives

$$\begin{aligned}
\nu &= (0,1,1,0,0,0) \quad \text{yielding} \quad a_{00}^0\, a_{-11}^1\, a_{20}^1\, b_{02}^0\, b_{1,-1}^0\, b_{00}^0 = a_{-11}\, a_{20} \\
\nu &= (0,1,0,0,1,0) \quad \text{yielding} \quad a_{00}^0\, a_{-11}^1\, a_{20}^0\, b_{02}^0\, b_{1,-1}^1\, b_{00}^0 = a_{-11}\, b_{1,-1} \\
\nu &= (0,0,1,1,0,0) \quad \text{yielding} \quad a_{00}^0\, a_{-11}^0\, a_{20}^1\, b_{02}^1\, b_{1,-1}^0\, b_{00}^0 = a_{20}\, b_{02}
\end{aligned}$$

The full set of monomial invariants of degree at most three for family

$$\dot{x} = -a_{00}x - a_{-11}y - a_{20}x^3, \quad \dot{y} = b_{1,-1}x + b_{00}y + b_{02}y^3$$

is

degree 0: 1

degree 1: $a_{00},\ b_{00}$

degree 2: $a_{00}^2,\ b_{00}^2, a_{00}\,b_{00},\ a_{-11}\,a_{20},\ a_{-11}\,b_{1,-1}, a_{20}\,b_{02},\ b_{02}\,b_{1,-1}$

degree 3: $a_{00}^3,\ b_{00}^3, a_{00}^2\,b_{00},\ a_{00}\,b_{00}^2, a_{00}\,a_{-11}\,a_{20},\ a_{00}\,a_{-11}\,b_{1,-1}, a_{00}\,a_{20}\,b_{0}$

$a_{00}\,b_{02}\,b_{1,-1},\ b_{00}\,a_{-11}\,a_{20},\ b_{00}\,a_{-11}\,b_{1,-1}, b_{00}\,a_{20}\,b_{02},\ b_{00}\,b_{02}\,b$

An algorithm for computing a generating set of invariants (A. Jarrah, R. Laubenbacher, V.R. JSC, 2003)

$$\dot{x} = -\sum_{(p,q)\in\widetilde{S}} a_{pq} x^{p+1} y^q = P(x,y),$$
$$\dot{y} = \sum_{(p,q)\in\widetilde{S}} b_{qp} x^q y^{p+1} = Q(x,y),$$

$$L(\nu) = \begin{pmatrix} L^1(\nu) \\ L^2(\nu) \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \cdots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \cdots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell}.$$

$$\mathcal{M} = \{\nu \in \mathbb{N}_0^{2\ell} : L(\nu) = \begin{pmatrix} j \\ j \end{pmatrix} \text{ for some } j \in \mathbb{N}_0\}.$$

Input: Two sequences of integers $p_1, \ldots, p_\ell$ ($p_i \geq -1$) and $q_1, \ldots, q_\ell$ ($q_i \geq 0$). (These are the coefficient labels for our system.)
Output: A finite set of generators for subalgebra of the invariant (equivalently, the Hilbert basis of $\mathcal{M}$).

1. Compute a reduced Gröbner basis G for the ideal

$$\mathcal{J} = \langle a_{p_i q_i} - y_i t_1^{p_i} t_2^{q_i}, b_{q_i p_i} - y_{\ell-i+1} t_1^{q_{\ell-i+1}} t_2^{p_{\ell-i+1}} \mid i = 1, \ldots, \ell \rangle \subset \mathbb{C}[a, b, y_1, \ldots, y_\ell, t_1, t_2]$$

   with respect to any elimination ordering for which

$$\{t_1, t_2\} > \{y_1, \ldots, y_d\} > \{a_{p_1 q_1}, \ldots, b_{q_1 p_1}\}.$$

2. $I_S = \langle G \cap \mathbb{C}[a, b] \rangle$.
3. The basis is formed by the monomials of I_S and monomials of the form $a_{ik} b_{ki}$

Time-reversible systems

$$\frac{d\mathbf{z}}{dt} = F(\mathbf{z}) \quad (\mathbf{z} \in \Omega), \tag{13}$$

$F : \Omega \mapsto T\Omega$ is a vector field and Ω is a manifold.

Definition

A time-reversible symmetry of (13) is an invertible map $R : \Omega \mapsto \Omega$, such that

$$\frac{d(R\mathbf{z})}{dt} = -F(R\mathbf{z}). \tag{14}$$

Example

$$\dot{u} = v + vf(u, v^2), \qquad \dot{v} = -u + g(u, v^2), \tag{15}$$

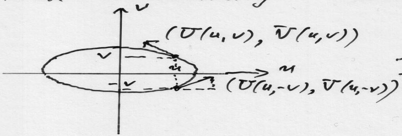
The transformation $u \to u$, $v \to -v$, $t \to -t$ leaves the system unchanged $\Rightarrow$ the u–axis is a line of symmetry for the orbits $\Rightarrow$ no trajectory in a neighborhood of $(0, 0)$ can be a spiral $\Rightarrow$ the origin is a center.
Here

$$R : u \mapsto u, \ \ v \mapsto -v. \tag{16}$$

$$\dot u = U(u,v) \qquad x = u + iv \qquad \dot x = P(x,\bar x)$$
$$\dot v = V(u,v) \qquad\qquad (P = U + iV)$$

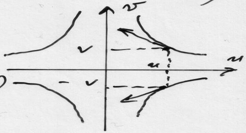
$$u \to u, \quad v \to -v$$
$$\Updownarrow$$
$$x \to \bar x, \quad \bar x \to x \qquad (A)$$

Time-reversibility



$$U(u,v) = -U(u,-v)$$
$$V(u,v) = V(u,-v)$$

Reversibility



$$U(u,v) = U(u,-v)$$
$$V(u,v) = -V(u,-v)$$

Note that,

$$P(\bar x, x) = U(u,-v) + iV(u,-v) = -U(u,v) + iV(u,v) = -\overline{P(x,\bar x)}$$

$$P(\bar x, x) = U(u,-v) + iV(u,-v) = +U(u,v) - iV(u,v) = \overline{P(x,\bar x)}$$

(A) yields $\dot{\bar x} = \overline{P(x,\bar x)}$. Therefore

$$\dot{\bar x} = -P(\bar x, x) \qquad\qquad \dot{\bar x} = P(\bar x, x)$$

Complexification

$$\dot{u} = U(u,v), \quad \dot{v} = V(u,v) \quad x = u + iv$$

$$\dot{x} = \dot{u} + i\dot{v} = U + iV = P(x,\bar{x}) \tag{17}$$

We add to (17) its complex conjugate to obtain the system

$$\dot{x} = P(x,\bar{x}), \ \dot{\bar{x}} = \overline{P(x,\bar{x})}. \tag{18}$$

The condition of time-reversibility with respect to $Ou = Im\, x$:
$P(\bar{x},x) = -\overline{P(x,\bar{x})}$.

Time-reversibility with respect to $y = \tan\varphi\, x$:

$$e^{2i\varphi}\overline{P(x,\bar{x})} = -P(e^{2i\varphi}\bar{x}, e^{-2i\varphi}x). \tag{19}$$

Consider $\bar{x}$ as a new variable y and allow the parameters of the second equation of (18) to be arbitrary. The complex system $\dot{x} = P(x,y), \quad \dot{y} = Q(x,y)$. which is is time–reversible with respect to a transformation

$$R : x \mapsto \gamma y,\ \ y \mapsto \gamma^{-1}x$$

if and only if for some γ

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y, x/\gamma)\,. \tag{20}$$

In the particular case when $\gamma = e^{2i\varphi}$, $y = \bar{x}$, and $Q = \bar{P}$ the equality (20) is equivalent to the reflection with respect a line and the reversion of time.

Systems of our interest are of the form

$$\begin{aligned} \dot{x} = & \quad x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q = P(x,y), \\ \dot{y} = & \quad -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1} = Q(x,y), \end{aligned} \qquad (21)$$

where S is the set $S = \{(p_j, q_j) \,|\, p_j + q_j \geq 0, j = 1, \ldots, \ell\} \subset (\{-1\} \cup \mathbb{N}_0) \times \mathbb{N}_0$, and $\mathbb{N}_0$ denotes the set of nonnegative integers. We will assume that the parameters $a_{p_j q_j}$, $b_{q_j p_j}$ $(j = 1, \ldots, \ell)$ are from $\mathbb{C}$ or $\mathbb{R}$. Denote by $(a, b) = (a_{p_1 q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell} \ldots, b_{q_1 p_1})$ the ordered vector of coefficients of system (21), by $E(a, b)$ the parameter space of (21) (e.g. $E(a, b)$ is $\mathbb{C}^{2\ell}$ or $\mathbb{R}^{2\ell}$), and by $k[a, b]$ the polynomial ring in the variables a_{pq}, b_{qp} over the field k.

The condition of time-reversibility

$$\gamma Q(\gamma y, x/\gamma) = -P(x,y), \quad \gamma Q(x,y) = -P(\gamma y, x/\gamma)\,.$$

yields that system (21) is time–reversible if and only if

$$b_{qp} = \gamma^{p-q} a_{pq}, \qquad a_{pq} = b_{qp}\gamma^{q-p}. \tag{22}$$

We rewrite (22) in the form

$$a_{p_k q_k} = t_k, \quad b_{q_k p_k} = \gamma^{p_k - q_k} t_k \tag{23}$$

for $k = 1, \ldots, \ell$. (23) define a surface in the affine space $\mathbb{C}^{3\ell+1} = (a_{p_1 q_1}, \ldots, a_{p_\ell q_\ell}, b_{q_\ell p_\ell}, \ldots, b_{q_1 p_1}, t_1, \ldots, t_\ell, \gamma)$. Thus, the set of all time-reversible systems is the projection of this surface onto $\mathbb{C}^{2\ell} = E(a,b)$.

Theorem (e.g. Cox D, Little J and O'Shea D 1992 *Ideals, Varieties, and Algorithms*)

Let k be an infinite field, $f_1, \ldots, f_n$ be elements of $k[t_1, \ldots, t_m]$,

$$x_1 = f_1(t_1, \ldots, t_m), \ \ldots \ x_n = f_n(t_1, \ldots, t_m),$$

and let $F : k^m \to k^n$, be the function defined by

$$F(t_1, \ldots, t_m) = (f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m)).$$

Let $J = \langle f_1 - x_1, \ldots, f_n - x_n \rangle \subset k[y, t_1, \ldots, t_m, x_1, \ldots, x_n]$, and let $J_{m+1} = J \cap k[x_1, \ldots, x_n]$. Then $\mathbf{V}(J_{m+1})$ is the smallest variety in k^n containing $F(k^m)$.

Let

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \ldots, \ell \rangle, \qquad (24)$$

Let $\mathcal{R}$ be the set of all time-reversible systems in the family (21). From the previous theorem we obtain

Theorem

$\overline{\mathcal{R}} = \mathbf{V}(\mathcal{I})$ where $\mathcal{I} = k[a, b] \cap H$, that is, the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems is the variety of the ideal $\mathcal{I}$.

Computation of $\mathcal{I} = k[a, b] \cap H$

Elimination Theorem

Fix the lexicographic term order on the ring $k[x_1, \ldots, x_n]$ with $x_1 > x_2 > \cdots > x_n$ and let G be a Groebner basis for an ideal I of $k[x_1, \ldots, x_n]$ with respect to this order. Then for every ℓ, $0 \le \ell \le n-1$, the set $G_\ell := G \cap k[x_{\ell+1}, \ldots, x_n]$ is a Groebner basis for the ideal $I_\ell = I \cap k[x_{\ell+1}, \ldots, x_n]$ (the ℓ–th elimination ideal of I).

By the theorem, to find a generating set for the ideal $\mathcal{I}$ it is sufficient to compute a Groebner basis for H with respect to a term order with $\{w, \gamma, t_k\} > \{a_{p_k q_k},\ b_{q_k p_k}\}$ and take from the output list those polynomials, which depend only on $a_{p_k q_k},\ b_{q_k p_k}$ $(k = 1, \ldots, \ell)$.

An algorithm for computing the set of all time-reversible systems

Let

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \ldots, \ell \rangle.$$

- Compute a Groebner basis G_H for H with respect to any elimination order with $\{w, \gamma, t_k\} > \{a_{p_k q_k}, b_{q_k p_k} \mid k = 1, \ldots, \ell\}$;
- the set $B = G_H \cap k[a, b]$ is a set of binomials; $\mathbf{V}(\langle B \rangle)$ is the Zariski closure of set of all time-reversible systems.

Another description of the ideal $\mathcal{I}$

Let $\mathcal{M}$ be the monoid of all solutions $\nu = (\nu_1, \nu_2, \ldots, \nu_{2l})$ with non–negative components of the equation

$$\zeta_1\nu_1+\zeta_2\nu_2+\cdots+\zeta_\ell\nu_\ell+\zeta_{\ell+1}\nu_{\ell+1}+\cdots+\zeta_{2\ell}\nu_{2\ell}=0, \ (\zeta\cdot\nu=0) \quad (25)$$

where $\zeta_j = p_j - q_j$ for $j = 1, \ldots, \ell$, $\zeta_j = q_{2\ell-j+1} - p_{2\ell-j+1}$ for $j = \ell+1, \ldots, 2\ell$, that is,

$$\zeta = (p_1 - q_1, p_2 - q_2, \ldots, p_\ell - q_\ell, q_\ell - p_\ell, \ldots, q_1 - p_1)$$

((p_j, q_j) are from the set S defining system (2)).
For $\nu = (\nu_1, \ldots, \nu_{2\ell}) \in \mathcal{M}$ we denote by $[\nu]$ the monomial

$$a_{p_1q_1}^{\nu_1} a_{p_2q_2}^{\nu_2} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} b_{q_{\ell-1}p_{\ell-1}}^{\nu_{\ell+2}} \cdots b_{q_1p_1}^{\nu_{2\ell}} \quad (26)$$

and by $\hat\nu$ the involution of the vector ν, $\hat\nu = (\nu_{2\ell}, \nu_{2\ell-1}, \ldots, \nu_1)$.
The monomials $[\nu]$ and $[\hat\nu]$ are invariants of the rotation group U_φ.
We will denote by $\mathbb{C}[\mathcal{M}]$ the monoid ring of $\mathcal{M}$ (the subalgebra generated by $\{[\nu] | \nu \in \mathcal{M}\}$).

For system (2) one can always find a function $\Psi(x, y) = xy + h.o.t.$ such that

$$\frac{\partial \Psi}{\partial x} P(x, y) + \frac{\partial \Psi}{\partial y} Q(x, y) = g_{11} \cdot (xy)^2 + g_{22} \cdot (xy)^3 + g_{33} \cdot (xy)^4 + \cdots , \tag{27}$$

where the g_{ii} are polynomials in the coefficients of (2) called *focus quantities*. System (2) is integrable if and only if $g_{ss} = 0$ for all $s = 1, 2, \ldots$.

Theorem

$g_{ss}(a, b) \in \mathbb{C}[\mathcal{M}]$ and have the form

$$g_{ss} = \sum_{\nu \in \mathcal{M}} g^{(\nu)}([\nu] - [\hat{\nu}]). \tag{28}$$

Consider the ideal

$$I_S = \langle [\nu] - [\hat{\nu}] \mid \nu \in \mathcal{M} \rangle \subset k[a, b] \quad (k \text{ is } \mathbb{C} \text{ or } \mathbb{R}).$$

We call I_S the *Sibirsky ideal* of system (2).

In the case that (2) is time-reversible, using (22) and (25) we see that for $\nu \in \mathcal{M}$

$$[\hat{\nu}] = \gamma^{\zeta\cdot\nu}[\nu] = [\nu], \tag{29}$$

where $\zeta \cdot \nu$ is the scalar product of ζ and ν, that is the left-hand side of (25). Thus, using (28), we obtain that *every time-reversible system is integrable*.

By (29) every time–reversible system $(a, b) \in E(a, b)$ belongs to $\mathbf{V}(I_S)$. The converse is false.

Theorem 1

Let $\mathcal{R} \subset E(a, b)$ be the set of all time–reversible systems in the family (2), then
(a) $\mathcal{R} \subset \mathbf{V}(I_S)$;
(b) $\mathbf{V}(I_S) \setminus \mathcal{R} = \{(a, b) \mid \exists (p, q) \in S \text{ such that } a_{pq} b_{qp} = 0 \text{ but } a_{pq} + b_{qp} \neq 0\}$.

(b) means that if in a time-reversible system (2) $a_{pq} \neq 0$ then $b_{qp} \neq 0$ as well. (b) $\Longrightarrow$ the inclusion in (a) is strict, that is $\mathcal{R} \subsetneq \mathbf{V}(I_S)$.

Theorem 2

$I_S = \mathcal{I}$ *and both ideals are prime.*

From Theorems 1 and 2 it follows

Theorem 3

The variety of the Sibirsky ideal I_S is the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems in the family (2).

Suppose we are given the system

$$x_1 = \frac{f_1(t_1,\ldots,t_m)}{g_1(t_1,\ldots,t_m)}, \ \ldots, \ x_n = \frac{f_n(t_1,\ldots,t_m)}{g_n(t_1,\ldots,t_m)}, \qquad (30)$$

where $f_j, g_j \in k[t_1,\ldots,t_m]$ for $j = 1,\ldots,n$. Let $k(t_1,\ldots,t_m)$ denote the ring of rational functions in m variable with coefficients in k (k is $\mathbb{C}$ *or* $\mathbb{R}$), and consider the ring homomorphism

$$\tilde{\psi} : k[x_1,\ldots,x_n,t_1,\ldots,t_m,w] \to k(t_1,\ldots,t_m)$$

defined by

$t_i \to t_i, \quad x_j \to f_j(t_1,\ldots,t_m)/g_j(t_1,\ldots,t_m), w \to 1/g(t_1,\ldots,t_m),$
$i = 1,\ldots,m$, $j = 1,\ldots,n$ and $g = g_1 g_2 \cdots g_n$. Let

$$\tilde{H} = \langle 1 - wg, x_1 g_1(t_1,\ldots,t_m) - f_1(t_1,\ldots,t_m), \ldots, \ x_n g_n(t_1,\ldots,t_m) - f_n(t_1,$$

$$\tilde{H} = \ker(\tilde{\psi}). \qquad (31)$$

Since $k[x_1,\ldots,x_n,t_1,\ldots,t_m,w]$ is a domain (31) yields that $\tilde{H}$ is a prime ideal.

Proof of Theorem 2.

$$H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \ldots, \ell \rangle, \quad \mathcal{I} = H \cap k[a, b].$$

Let $f \in I_S \subset \mathbb{C}[a, b]$, so that f is a finite linear combination, with coefficients in $\mathbb{C}[a, b]$, of binomials of the form $[\nu] - [\hat{\nu}]$, where $\nu \in \mathcal{M}$. $f \in \mathcal{I}$ if any such binomial is in $\mathcal{I}$. By definition of ψ

$$\begin{aligned} \psi([\nu] - [\hat{\nu}]) &= t_1^{\nu_1} \cdots t_\ell^{\nu_\ell} (\gamma^{p_\ell - q_\ell} t_\ell)^{\nu_{\ell+1}} \cdots (\gamma^{p_1 - q_1} t_1)^{\nu_{2\ell}} \\ &\quad - t_1^{\nu_{2\ell}} \cdots t_\ell^{\nu_{\ell+1}} (\gamma^{p_\ell - q_\ell} t_\ell)^{\nu_\ell} \cdots (\gamma^{p_1 - q_1} t_1)^{\nu_1} \\ &= t_1^{\nu_1} \cdots t_\ell^{\nu_\ell} t_1^{\nu_{2\ell}} \cdots t_\ell^{\nu_{\ell+1}} (\gamma^{\nu_1 \zeta_1 + \cdots + \nu_\ell \zeta_\ell} - \gamma^{\nu_{2\ell} \zeta_1 + \cdots + \nu_{\ell+1} \zeta_\ell}). \end{aligned} \tag{32}$$

Since $\nu \in \mathcal{M}$, $\zeta_1 \nu_1 + \cdots + \zeta_{2\ell} \nu_{2\ell} = \zeta \cdot \nu = 0$. But $\zeta_j = -\zeta_{2\ell - j + 1}$ for $1 \leq j \leq 2\ell$ so

$$\zeta_1 \nu_1 + \cdots + \zeta_\ell \nu_\ell = -\zeta_{\ell+1} \nu_{\ell+1} - \cdots - \zeta_{2\ell} \nu_{2\ell} = \zeta_\ell \nu_{\ell+1} + \cdots + \zeta_1 \nu_{2\ell}$$

and the exponents on γ in (32) are the same. Thus $[\nu] - [\hat{\nu}] \in \ker(\psi) = H$, hence $[\nu] - [\hat{\nu}] \in H \cap \mathbb{C}[a, b] = \mathcal{I}$, i.e. $I_S \subset \mathcal{I}$.

By (31) the ideal H defined by (24) is the kernel of the ring homomorphism

$$\psi : k[a, b, t_1, \ldots, t_\ell, \gamma, w] \longrightarrow k(\gamma, t_1, \ldots, t_\ell)$$

defined by $a_{p_k q_k} \mapsto t_k,\ b_{q_k p_k} \mapsto \gamma^{p_k - q_k} t_k,\ w \mapsto 1/(\tilde{\gamma}_1 \cdots \tilde{\gamma}_\ell)$ for $k = 1, \ldots, \ell$. We obtain a reduced Groebner basis G of $k[a, b] \cap H$ by computing a reduced Groebner basis of H using an elimination ordering with $\{a_{p_j q_j},\ b_{q_j p_j}\} < \{w, \gamma, t_j\}$ for all $j = 1, \ldots, \ell$, and then intersecting it with $k[a, b]$. Since H is binomial, any reduced Groebner basis G of H also consists of binomials. This shows that $\mathcal{I}$ is a binomial ideal.

Now suppose $f \in \mathcal{I} = H \cap \mathbb{C}[a, b] \subset \mathbb{C}[a, b]$. Since $\mathcal{I}$ has a basis consisting wholly of binomials, it is enough to restrict to the case that f is binomial, $f = a_\alpha[\alpha] + a_\beta[\beta]$. Using the definition of ψ and collecting terms

$$\psi(a_\alpha[\alpha] + a_\beta[\beta]) = a_\alpha t_1^{\alpha_1+\alpha_{2\ell}} \cdots t_\ell^{\alpha_\ell+\alpha_{\ell+1}} \gamma^{\zeta_\ell\alpha_{\ell+1}+\cdots+\zeta_1\alpha_{2\ell}} + a_\beta t_1^{\beta_1+\beta_{2\ell}} \cdots t_\ell^{\beta_\ell+\beta_{\ell+1}} \gamma^{\zeta_\ell\beta_{\ell+1}+\cdots+\zeta_1\beta_{2\ell}}.$$

Since $H = \ker(\psi)$ this is the zero polynomial, so

$$a_\beta = -a_\alpha \tag{33a}$$

$$\alpha_j + \alpha_{2\ell-j+1} = \beta_j + \beta_{2\ell-j+1} \quad \text{for} \quad j = 1, \ldots, \ell \tag{33b}$$

$$\zeta_\ell\alpha_{\ell+1} + \cdots + \zeta_1\alpha_{2\ell} = \zeta_\ell\beta_{\ell+1} + \cdots + \zeta_1\beta_{2\ell} \tag{33c}$$

For $\nu \in \mathbb{N}_0^{2\ell}$ let $R(\nu)$ denote the set of indices j for which $\nu_j \neq 0$. First suppose that $R(\alpha) \cap R(\beta) = \varnothing$. It is easy to check that condition (33b) forces $\beta_j = \alpha_{2\ell-j+1}$ for $j = 1, \ldots, 2\ell$, so that $\beta = \hat{\alpha}$. But then because $\zeta_j = -\zeta_{2\ell-j+1}$ for $1 \leq j \leq 2\ell$ condition (33c) reads

$$-\zeta_{\ell+1}\alpha_{\ell+1} - \cdots - \zeta_{2\ell}\alpha_{2\ell} = \zeta_\ell\alpha_\ell + \cdots + \zeta_1\alpha_1$$

or $\zeta_1\alpha_1 + \cdots + \zeta_{2\ell}\alpha_{2\ell} = 0$, so $\alpha \in \mathcal{M}$. Thus $f = a_\alpha([\alpha] - [\hat{\alpha}])$ and $\alpha \in \mathcal{M}$, so $f \in I_S$.

If $R(\alpha) \cap R(\beta) \neq \varnothing$, then $[\alpha]$ and $[\beta]$ contain common factors, corresponding to the common indices of some of their nonzero coefficients. Factoring out the common terms, which form a monomial $[\mu]$, we obtain $f = [\mu](a_\alpha[\alpha'] + a_\beta[\beta'])$, where $R(\alpha') \cap R(\beta') = \varnothing$. Since the ideal $\mathcal{I}$ is prime and contains no monomial we conclude that $a_\alpha[\alpha'] + a_\beta[\beta'] \in \mathcal{I}$, hence by the first case that $a_\alpha[\alpha'] + a_\beta[\beta'] \in I_S$, hence that $f \in I_S$. $\square$

Algorithm for computing $\mathcal{I}(= I_S)$

- Compute a Groebner basis G_H for $H = \langle a_{p_k q_k} - t_k, b_{q_k p_k} - \gamma^{p_k - q_k} t_k \mid k = 1, \dots, \ell \rangle$ with respect to any elimination order $\{w, \gamma, t_k\} > \{a_{p_k q_k}, b_{q_k p_k} \mid k = 1, \dots, \ell\}$;
- the set $G_H \cap k[a, b]$ is a generating set for $\mathcal{I}$ and I_S.

Theorem

Let G be a reduced Gröbner basis of $\mathcal{I}$.

1. Every element of G has the form $[\nu] - [\hat{\nu}]$, where $\nu \in \mathcal{M}$ and $[\nu]$ and $[\hat{\nu}]$ have no common factors.
2. The set

$$\mathcal{H} = \{\mu, \hat{\mu} : [\mu] - [\hat{\mu}] \in G\} \cup \{\mathbf{e}_j + \mathbf{e}_{2\ell-j+1} : j = 1, \dots, \ell \text{ and } \pm([\mathbf{e}_j] - [\mathbf{e}_{2\ell-j+1}]) \notin G\},$$

where $\mathbf{e}_j = (0, \dots, 0, \overset{j}{1}, 0, \dots, 0)$, is a Hilbert basis of $\mathcal{M}$.

As an example consider the system

$$\begin{aligned} \dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\ \dot{y} &= -y + b_{10}xy + b_{01}y^2 + b_{2,-1}x^2. \end{aligned} \tag{34}$$

Computing a Groebner basis of the ideal

$$\mathcal{J} = \langle 1-w\gamma^4, a_{10}-t_1, b_{01}-\gamma t_1, a_{01}-t_2, \ \gamma b_{10}-t_2, a_{-12}-t_3, \gamma^3 b_{2,-1}-t_3 \rangle$$

with respect to the lexicographic order with $w > \gamma > t_1 > t_2 > t_3 > a_{10} > a_{01} > a_{-12} > b_{10} > b_{01} > b_{2,-1}$ we obtain a list of polynomials.

In[10]:=
```
GroebnerBasis[{a10 - t1, b01 - γ t1, a01 - t2, γ b10 - t2, a12 - t3, γ^3 b21 - t3,
  1 - w γ^4}, {w, γ, t1, t2, t3, b10, b01, a10, a01, a12, b21}]
```

Out[10]=

$$\{-a10^3\, a12 + b01^3\, b21,\ a10^2\, a12\, b10 - a01\, b01^2\, b21,\ -a01\, a10 + b01\, b10,$$
$$a10\, a12\, b10^2 - a01^2\, b01\, b21,\ a12\, b10^3 - a01^3\, b21,\ -a12 + t3,\ -a01 + t2,\ -a10 + t1,$$
$$-a12\, b10^2 + a01^2\, b21\, \gamma,\ -b01 + a10\, \gamma,\ -a10\, a12\, b10 + a01\, b01\, b21\, \gamma,\ -a10^2\, a12 + b01^2\, b21\, \gamma,$$
$$-a01 + b10\, \gamma,\ -a12\, b10 + a01\, b21\, \gamma^2,\ -a10\, a12 + b01\, b21\, \gamma^2,\ -a12 + b21\, \gamma^3,\ a12^2\, w - b21^2\, \gamma^2,$$
$$-b10\, b21 + a01\, a12\, w,\ -b10^4 + a01^4\, w,\ -a10\, b21 + a12\, b01\, w,\ -a10\, b10^3 + a01^3\, b01\, w,$$
$$-a10^2\, b10^2 + a01^2\, b01^2\, w,\ -a10^3\, b10 + a01\, b01^3\, w,\ -a10^4 + b01^4\, w,\ -b21 + a12\, w\, \gamma,\ -b10^3 + a01^3\, w\, \gamma,$$
$$-a10\, b10^2 + a01^2\, b01\, w\, \gamma,\ -a10^2\, b10 + a01\, b01^2\, w\, \gamma,\ -a10^3 + b01^3\, w\, \gamma,\ -b10^2 + a01^2\, w\, \gamma^2,$$
$$-a10\, b10 + a01\, b01\, w\, \gamma^2,\ -a10^2 + b01^2\, w\, \gamma^2,\ -b10 + a01\, w\, \gamma^3,\ -a10 + b01\, w\, \gamma^3,\ -1 + w\, \gamma^4\}$$

According to step 2 of the algorithm we pick up the polynomials that do not depend on w, γ, t_1, t_2, t_3:
$f_1 = a_{01}^3 b_{2,-1} - a_{-12} b_{10}^3$, $f_2 = a_{10} a_{01} - b_{01} b_{10}$,
$f_3 = a_{10}^3 a_{-12} - b_{2,-1} b_{01}^3$, $f_4 = a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}$,
$f_5 = a_{10}^2 a_{-12} b_{10} - a_{01} b_{2,-1} b_{01}^2$. Thus, for system (34)

$$I_S = \mathcal{I} = \langle f_1, \ldots, f_5 \rangle.$$

- $\mathbf{V}(\langle f_1, \ldots, f_5 \rangle)$ is the Zariski closure of the set of all time-reversible systems inside of (34)
- The monomials of f_i together with $a_{10} b_{01}$, $a_{01} b_{10}$, $a_{-12} b_{2,-1}$ generate the subalgebra $\mathbb{C}[\mathcal{M}]$ for invariants of U_φ and the exponents of the monomials form the Hilbert basis of the monoid $\mathcal{M}$.
- Focus quantities g_{ii} of (34) belong to $\mathbb{C}[\mathcal{M}]$.

We now show a further interconnection of time-reversibility and invariants of a group of transformations of the phase space of

$$\begin{aligned} \dot{x} &= -\sum_{(p,q)\in\widetilde{S}} a_{pq}x^{p+1}y^{q} = P(x,y), \\ \dot{y} &= \sum_{(p,q)\in\widetilde{S}} b_{qp}x^{q}y^{p+1} = Q(x,y), \end{aligned} \tag{35}$$

Consider the transformations of the phase space of (35)

$$x' = \eta x, \quad y' = \eta^{-1}y \ \ (x, y, \eta \in \mathbb{C}, \ \eta \neq 0). \tag{36}$$

In (x', y') coordinates (35) has the form

$$\dot{x}' = \sum_{(p,q)\in S} a(\eta)_{(p,q)}x'^{p+1}y'^{q}, \ \dot{y}' = \sum_{(p,q)\in S} b(\eta)_{(q,p)}x'^{q}y'^{p+1}$$

and the coefficients of the transformed system are

$$a(\eta)_{p_k q_k} = a_{p_k q_k}\eta^{q_k - p_k}, \quad b(\eta)_{q_k p_k} = b_{q_k p_k}\eta^{p_k - q_k}, \tag{37}$$

where $k = 1, \dots, \ell$. Let U_η denote the transformation (37). We write (37) as $(a(\eta), b(\eta)) = U_\eta(a, b)$.

The action of U_η on the coefficients a_{ij}, b_{ji} of the system of differential equations (35) yields the following transformation of the monomial $[\nu]$ defined by (26):

$$U_\eta[\nu] = a(\eta)^{\nu_1}_{p_1 q_1} \cdots a(\eta)^{\nu_\ell}_{p_\ell q_\ell} \; b(\eta)^{\nu_{\ell+1}}_{q_\ell p_\ell} \cdots b(\eta)^{\nu_{2\ell}}_{q_1 p_1} = \tag{38}$$
$$\eta^{\zeta\cdot\nu} a^{\nu_1}_{p_1 q_1} \cdots a^{\nu_\ell}_{p_\ell q_\ell} \; b^{\nu_{\ell+1}}_{q_\ell p_\ell} \cdots b^{\nu_{2\ell}}_{q_1 p_1} = \eta^{\zeta\cdot\nu}[\nu].$$

Thus we see that *the monomial $[\nu]$ is invariant under the action of U_η if and only if $\zeta \cdot \nu = 0$, i.e., if and only if $\nu \in \mathcal{M}$.*

Denote by $\widehat{(a, b)}$ the involution of (a, b),

$$\widehat{(a, b)} = (b_{q_1 p_1}, \ldots, b_{q_l p_l}, a_{p_l q_l}, \ldots, a_{p_1 q_1}). \tag{39}$$

The orbit $\mathcal{O}$ of the group U_η is invariant under the involution (39) if for any $(a, b) \in \mathcal{O}$ the system $\widehat{(b, a)}$ also belongs to $\mathcal{O}$.

Theorem

(a) The set of the orbits of U_η is divided into two not intersecting subsets: one consists of all time-reversible systems and only time-reversible systems, and there are no time-reversible systems in the other subset.
(b) The variety $\mathbf{V}(I_S)$ is the Zariski closure of all orbits of the group U_η invariant under the involution (39).

Conclusions

- The theory of invariants of ODEs is almost untouched field for applications of methods and algorithms of computational algebra
- Two interesting problems for studying:
  - generalization of the presented methods to higher dimensional systems of ODEs
  - studying invariants of another groups of transformations of the phase space