Algorithms for algebraic analysis

by

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Abstract

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One of the major goals in the field of symbolic computation of differential equations is to develop algorithms for exact or closed-form solutions. This thesis studies symbolic computation of maximally overdetermined systems of linear partial differential equations by using constructions in the corresponding ring of differential operators with polynomial coefficients, which is called the Weyl algebra D. We develop algorithms to find polynomial solutions, rational function solutions, and more generally holonomic solutions. By holonomic solutions, we mean the following: sometimes the best way to specify a function F is as the solution of a system of differential equations – this is for instance how many special functions are classically described. Our algorithm takes as input the differential equations describing F as well as the system S that we wish to solve, and returns as output any solutions to S existing within the D-module generated by F. We also study aspects of the opposite problem, namely given a function F, how can differential equations describing F be produced? We introduce the Weyl closure of an ideal I of the Weyl algebra, which is the set of all differential operators annihilating the common holomorphic solutions of I at a generic point. We give an algorithm to compute Weyl closure, which has applications to symbolic integration, and which we also use to make a detailed study of ideals in the first Weyl algebra.

Professor Bernd Sturmfels
Dissertation Committee Chair

Contents

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Introduction

A system of linear partial differential equations can be viewed as a module over a ring of differential operators, and *algebraic analysis* is the study of a system through the algebraic structure of its corresponding module. This field was pioneered in the 1960's and 1970's by Bernstein, Kashiwara, Malgrange, Sato, and others, who haved coined the term *D-modules* to refer to the general study of modules over rings of differential operators on algebraic varieties or analytic spaces. Since its inception, the theory of D-modules has proved useful to a wide range of modern mathematics, including representation theory, mathematical physics, singularity theory, and of course differential equations.

In recent years, there has been an active development in the computational side of D-modules, and these efforts have similarly led to interesting and diverse applications. For instance, in the 1980's and 1990's, Zeilberger and his collaborators applied ideas of D-modules to create an algorithmic machine for proving and generating combinatorial identities. An excellent summary of this work can be found in their book [38]. Another focus of algorithmic attention, which will be the point of view of this thesis, has been the Weyl algebra

$$D = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle,$$

which is the ring of differential operators on affine space K^n, where K is a field of characteristic 0. In terms of generators and relations, it is a free associative algebra modulo the relations

$$x_i x_j - x_j x_i = 0 \qquad \partial_i \partial_j - \partial_j \partial_i = 0 \qquad \partial_i x_j - x_j \partial_i = \delta_{ij}.$$

The algorithmic foundations in the Weyl algebra were laid by Galligo [21], Takayama [45], and others in the mid-1980's, who established a working theory of Gröbner bases in this slightly noncommutative setting. One of Takayama's motivations was to apply Gröbner bases to improving and extending Zeilberger's theory [47]. In recent years, Chyzak has continued these efforts and has considerably advanced the study of special functions and combinatorial identities by Gröbner bases techniques [13].

In general, Gröbner bases have become a highly useful tool in computational algebra. In the field of computational algebraic geometry, Gröbner bases have made many constructions of commutative algebra possible, such as free resolutions, Hilbert functions, and homological functors. In the field of computational algebraic D-modules, similar sorts of constructions for modules over the Weyl algebra have only recently been made effective by Oaku and Takayama. We mention here three of their key results: they are algorithms for computing *Gröbner bases adapted to the V-filtration* [32] (which we expain at the end of the preface), *b-functions* [31] (which we use and explain in Chapter 3), and *derived restrictions* [33] (which we use in Chapter 2 and Chapter 4 and explain in the Appendix). These three algorithms have led to the ability to compute a collection of other algebraic operations such as localization [36], tensor product [33], and integration [34]. Moreover, they have also led to interesting applications in computational algebraic geometry, such as algorithms for computing local cohomology and deRham cohomology in a wide class of situations (see the work of Walther [51], [52], [53] and Oaku and Takayama [33], [34]).

In this thesis, we will apply the Gröbner basis approach from the point of view of algebraic analysis, that is to say, we wish to determine properties about a given system of linear partial differential equations and in particular its solutions by using constructions in the Weyl algebra. This point of view has been developed recently by Saito, Sturmfels, and Takayama in their book [40], which is also an excellent starting place to learn about algorithms for D-modules based on Gröbner bases. For instance, they give Gröbner basis methods to construct power series solutions of a holonomic system of linear partial differential equations at a regular singular point, and they use these techniques to make a systematic study of Gel'fand-Kapranov-Zelevinsky systems of hypergeometric differential equations. An important notion introduced in their book is a *Gröbner deformation*, which generalizes the notion of Gröbner bases adapted to the V-filtration and which we explain at the end of the preface.

The algorithmic development of D-modules has also been accompanied by implementation in computer algebra systems. The work of Chyzak is implemented in his computer package `Mgfun` for `Maple` and is available at `http://www-rocq.inria.fr/algo/chyzak`. Takayama was the first to implement Gröbner bases in the Weyl algebra and has developed a specialized system called `kan/sm1`, which has one-line commands for many of the algorithms and applications described here and which is available at `http://www.kobe-u.ac.jp/KAN`. Similarly, together with Anton Leykin and Mike Stillman, we have implemented a package for D-modules for the computer algebra system `Macaulay 2` [23]. One of the nice features of Macaulay 2 is that it has a top level programming environment which makes the system flexible for applications. Our package will shortly be included in the Macaulay 2 distribution and is currently available at `http://www.math.berkeley.edu/~htsai/Dmodules.html`. We will include output from Macaulay 2 sessions throughout this thesis wherever appropriate. A typical session will have the form

```
i1 : (statement 1;
      statement 2;
      statement 3)

o1 = mathematical output
```

It means that Macaulay 2 has executed statements 1, 2, and 3, and has returned the output of statement 3 (the ending semi-colons causes the output of statements 1 and 2 to be suppressed).

Let us now give a brief summary of the contents of this thesis. In Chapter 1, we study linear ordinary differential equations, which correspond to ideals of operators in the first Weyl algebra. We show in particular how the *characteristic ideal* contains information about the solution spaces of a differential equation. In Chapter 2, we study the *Weyl closure* of a system of linear partial differential equations. Roughly speaking, this is the ideal of differential operators which annihilate the common solutions of the original system, much in the same way that the radical operation of commutative algebra consists of the the ideal of functions which vanish on the common zeros of a system of polynomial equations. The Weyl closure turns out to be useful in the combinatorial applications of Chyzak, and we give an algorithm to compute it. In Chapter 3, we give algorithms to find polynomial and rational solutions to *finite rank* systems of linear partial differential equations. Finally in Chapter 4, we give an algorithm to compute homomorphisms between *holonomic* D-modules M and N. These homomorphisms can also be viewed as "solutions" to differential equations, namely the image of a homomorphism is a "function" in the module N which solves the system of differential equations corresponding to M.

Preliminaries and Conventions. This thesis should be understandable to someone who has read the recent book *Gröbner Deformations of Hypergeometric Differential Equations* by Saito, Sturmfels, and Takayama [40]. For the reader's convenience we recall a handful of notions introduced there which will be fundamental to many of our algorithms. A real vector $(u,v)=(u_1,\ldots,u_n,v_1,\ldots,v_n)\in\mathbb{R}^{2n}$ with $u_i+v_i\geq 0$ for all $i=1,\ldots,n$ is called a *weight vector* and defines a filtration $D=\cup_{i\in\mathbb{Z}}F_i^{(u,v)}$ of the Weyl algebra by the linear subspaces

$$F_i^{(u,v)}=\mathrm{Span}_K\left\{\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta : u\cdot\alpha+v\cdot\beta\leq i\right\}.$$

For specific values of (u,v), these filtrations specialize to a number of well-studied filtrations. For $(u,v)=(-e_1-\cdots-e_d,e_1+\cdots+e_d)$, the filtration $F^{(u,v)}$ is also referred to as the V-filtration with respect to the subspace $Y=\{x_1=\cdots=x_d=0\}$. Its role in computing restriction to Y is discussed in the appendix. For $(u,v)=(0,e)$ where $e=e_1+\cdots+e_n$, the filtration $F^{(u,v)}$ is more commonly known as the order filtration, and for $(u,v)=(e,e)$, the filtration $F^{(u,v)}$ is more commonly known as the Bernstein filtration.

When $u+v>0$, meaning that each coordinate is positive, then the associated graded ring $\mathrm{gr}D$ is the polynomial ring $K[\boldsymbol{x},\boldsymbol{\xi}]$ while if $u+v=0$, then the associated graded ring $\mathrm{gr}D$ is again D. For an operator $L=\sum_{\alpha,\beta}c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta\in D$, we thus define the *initial form with respect to* (u,v) to be the subsum,

$$\mathrm{in}_{(u,v)}(L)=\begin{cases}\sum_{\{\alpha,\beta\ :\ c_{\alpha\beta}\neq 0,u\cdot\alpha+v\cdot\beta\ maximal\}}c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\xi}^\beta\in K[\boldsymbol{x},\xi] & \text{if } u+v>0\\ \sum_{\{\alpha,\beta:c_{\alpha\beta}\neq 0,u\cdot\alpha+v\cdot\beta\ maximal\}}c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta\in D & \text{if } u+v=0\end{cases}$$

Similarly, for a left ideal $I\subset D$, we define the *initial ideal with respect to* (u,v) to be the left ideal of $\mathrm{gr}D$ given by

$$\mathrm{in}_{(u,v)}(I)=\mathrm{Span}_K\{\mathrm{in}_{(u,v)}(L) : L\in I\}.$$

When $u+v=0$ so that $(u,v)=(-w,w)$ for some vector w, then $\mathrm{in}_{(-w,w)}(I)$ is called the *Gröbner deformation of* I *with respect to* w. When $(u,v)=(0,e)$ so that the corresponding filtration is the order filtration, then $\mathrm{in}_{(0,e)}(I)$ is called the *characteristic ideal* and its zero locus is called the *characteristic variety* of the quotient module D/I.

Finally, a finite subset G is called a *Gröbner basis with respect to* (u,v) if I is generated by G and $\mathrm{in}_{(u,v)}(I)$ is generated by $\mathrm{in}_{(u,v)}(g)$ for $g\in G$.

Let us also explain some conventions we will adopt throughout this thesis. K will denote a computable subfield of the complex field $\mathbb{C}$, unless otherwise stated. D or D_n will denote the n-th Weyl algebra except in Chapter 1, where D will denote the first Weyl algebra. We use θ_i to denote $x_i\partial_i$. We use $\boldsymbol{x}$ to stand for $x_1, \ldots, x_n$, and similarly for $\boldsymbol{\partial}$, $\boldsymbol{\xi}$, $\boldsymbol{\theta}$, and other bold-face symbols. A left submodule of D^r generated by $\{g_1, \ldots, g_s\}$ will be denoted $D \cdot \{g_1, \ldots, g_s\}$ and a right submodule will be denoted $\{g_1, \ldots, g_s\} \cdot D$.

Chapter 1

Ideals of the first Weyl algebra

In this chapter, we study left ideals of the first Weyl algebra $K\langle x, \partial\rangle$, which is the free associative algebra modulo the relation $\partial x - x\partial = 1$. We will denote this algebra by D or D_1 in this chapter. In later chapters, D will stand more generally for the n-th Weyl algebra. The material in this chapter is based upon the paper [48]. Our point of view will be to compare ideals in D to ideals in $R = K(x)\langle\partial\rangle$, and we start by studying contractions $J \cap D$ for ideals $J \subset R$. In particular, since R is a principal ideal domain, $J = R \cdot L$ for some $L \in R$ (we may also assume $L \in D$ by replacing L with a suitable $f(x)L$). On the other hand, the contraction $R \cdot L \cap D$ need not be principal and we would like to understand these situations. For instance, if $L = x\partial - 1$ then $R \cdot L \cap D = D\{x\partial - 1, \partial^2\}$ and is no longer principal.

Definition 1.0.1. *The Weyl closure of a linear differential operator $L \in D$ is*

$$\mathrm{Cl}(L) := R \cdot L \cap D.$$

We will be particularly interested in understanding the characteristic ideal of the Weyl closure $\mathrm{Cl}(L)$, which is the initial ideal of $\mathrm{Cl}(L)$ under the order filtration. The zero locus of the characteristic ideal is the characteristic variety, which is an important invariant of the quotient module $D/\mathrm{Cl}(L)$. One of our aims is to show how the characteristic ideal provides finer information which may also be of interest. An analytic interpretation of the Weyl closure is the following. Suppose the operator L has order n. If $K = \mathbb{C}$ and $\lambda \in \mathbb{C}$ is a nonsingular point of L, then the holomorphic solutions of L in a neighborhood of λ form a $\mathbb{C}$-vector space $V_\lambda(L)$ of dimension n. In this case, the Weyl closure of L is equal to the ideal of operators in D which annihilate $V_\lambda(L)$. From an algebraic perspective, the Weyl closure of L also arises naturally when considering the support of D/DL. In particular, $\mathrm{Cl}(L)/DL$ is the submodule of D/DL consisting of elements which are supported on a finite subset of Spec $K[x]$.

Let us give a brief outline of the contents of this chapter. In Sections 1.1 and 1.2, we give an algorithm to compute the Weyl closure $\mathrm{Cl}(L)$, and we describe the characteristic ideal of any ideal I with the property that $D \cdot L \subset I \subset \mathrm{Cl}(L)$. The Weyl closure algorithm is an application of the restriction algorithm due to Oaku and Takayama [33]. In Section 1.3, we give a combinatorial description of the characteristic ideal of $\mathrm{Cl}(L)$ in terms of certain solution spaces of L. In Section 1.4, we use the Weyl closure to give an algorithm for constructing a Jordan-Hölder series for a holonomic D-module and a formula for its length. As a corollary we obtain a criteria for when an operator L generates a maximal ideal in D. In Section 1.5, we give an alternative proof of Strömbeck's inequality [44] based on the Weyl closure. This inequality describes all possible characteristic ideals of left ideals in D. In Section 1.6, we use the Weyl closure to describe the space of isomorphism classes of left ideals of D, a result first obtained by Cannings and Holland [9], and we also obtain an algorithm to determine the isomorphism class of a left ideal from its generators. Let us give now an example before beginning with the actual technical details.

Example 1.0.2. Consider the operator

$$L = x^2(x-1)(x-3)\partial^2 - (6x^3 - 20x^2 + 12x)\partial + (12x^2 - 32x + 12),$$

whose classical solution space $V = \mathrm{Span}_{\mathbb{C}}\{x^4, x(x-1)^2\}$ consists entirely of polynomials. Then $\partial^5 \in \mathrm{Cl}(L)$ but $\partial^5 \notin D \cdot L$. In fact, our algorithm finds

$$\mathrm{Cl}(L) = D \cdot \{L, \partial^5\} \subset D.$$

Now let us consider the initial ideal $\mathrm{in}_{(0,1)}(\mathrm{Cl}(L))$ of the closure of L with respect to the order filtration (see Definition 1.2.6 for the precise definition of initial ideal). We find that

$$\mathrm{in}_{(0,1)}(\mathrm{Cl}(L)) = \langle x^2(x-1)(x-3)\xi^2, x\xi^3, \xi^5 \rangle \subset \mathbb{C}[x, \xi].$$

Finally, we observe in Corollary 1.3.3 a combinatorial relationship between the above generators of the initial ideal and the following property of the solution space V. At $x = 0$, the solutions have multiplicity 1 and 4, at $x = 1$ and $x = 3$, the solutions have multiplicity 0 and 2, and at all remaining points, the solutions have multiplicity 0 and 1. To see that there is indeed a solution with multiplicity 2 at $x = 3$, consider the linear combination $4x^4 + 27x(x-1)^2 = 4x(x+3)(x-3)^2$. This relationship will be made precise in Section 1.3 using the notion of cotype. For general L, we shall see that the correct relationship is between the initial ideal $\mathrm{in}_{(0,1)}(\mathrm{Cl}(L))$ and "solutions" of L in various spaces $\mathbb{C}[x]/\langle (x-\lambda)^i \rangle$.

1.1 Local closure

In this section, we define the local closure, give an algorithm to compute it, and describe its initial ideal with respect to the order filtration refined by the V-filtration. These results will then be applied in Section 1.2 to the study of $\mathrm{Cl}(L)$.

Definition 1.1.1. *The local closure* $\mathrm{Cl}_\lambda(L)$ *of* $L \in D$ *at* $x = \lambda$ *is the ideal*

$$\mathrm{Cl}_\lambda(L) = K[x, (x-\lambda)^{-1}]\langle \partial \rangle L \cap D$$

The local closure arises naturally when considering the following torsion of D/DL:

$$H^0_{x-\lambda}\left(\frac{D}{DL}\right) = \{T \in D/DL : (x-\lambda)^i T = 0 \text{ for } i \gg 0\} = \frac{\mathrm{Cl}_\lambda(L)}{DL} \subset \frac{D}{DL}.$$

To compute the local closure, it thus suffices to compute generators of $H^0_{x-\lambda}(D/DL)$ and lift them to D. The lifted generators together with the element L are then a set of generators for $\mathrm{Cl}_\lambda(L)$. The reason why this reformulation is useful is Kashiwara's equivalence, which implies that the torsion module $H^0_{x-\lambda}(D/DL)$ is generated as a D-module by the subspace $\{T \in D/DL : (x-\lambda)T = 0\}$. We will give a precise statement of Kashiwara's equivalence in the proof of correctness of the algorithm.

Algorithm 1.1.2. (Local closure at $x = \lambda$)
INPUT: an operator $L = p_n(x)\partial^n + \cdots + p_0(x) \in D$
OUTPUT: a set of generators of $\mathrm{Cl}_\lambda(L)$.

1. Rewrite L as
$$L = \sum_{i=r}^{s} \zeta_i q_i(\theta_\lambda) \qquad \zeta_i = \begin{cases} \partial^{-i} & \text{if } i \le 0 \\ (x-\lambda)^i & \text{if } i > 0 \end{cases}$$
where $\theta_\lambda = (x-\lambda)\partial$ and $q_r(\theta_\lambda) \neq 0$.

2. Set m equal to the maximum integer root of the lowest term $q_r(\theta_\lambda)$ if it is greater than 0. Otherwise, set m equal to 0.

3. If $m + r < 0$, set $B = 0$. Otherwise, compute a basis B for the kernel of the $(m+1) \times (m+r+1)$ matrix $[R_\lambda(L)_{ij}]_{0 \le i \le m, 0 \le j \le m+r}$, whose entry in row i and column j is the following element of the ground field K:
$$R_\lambda(L)_{ij} = \begin{cases} q_{j-i}(i) & \text{if } i \ge j \\ j(j-1)\cdots(i+1)q_{j-i}(i) & \text{if } i < j. \end{cases} \tag{1.1}$$
For each $\vec{v} = [v_0, v_1, \ldots, v_{m+r}]^t \in B$, set $p_v = \sum_{i=0}^{m+r} v_i \partial^i$.

4. Return $\{L, (x-\lambda)^{-1} p_v L : \vec{v} \in B\}$.

Proof. (Correctness of Algorithm 1.1.2) The algorithm is an application of the restriction algorithm due to Oaku and Takayama for the special case of D/DL restricted to the point $x = \lambda$. The details which we present are implicit in their paper [33].

Let us discuss the computation of $H^0_{x-\lambda}(D/DL)$. In our situation, Kashiwara's equivalence [15, Theorem 17.2.4] states that
$$H^0_{x-\lambda}\left(\frac{D}{DL}\right) = \bigoplus_{i=0}^{\infty} \partial^i \ker[x-\lambda]$$
where
$$\ker[x-\lambda] = \{T \in D/DL : (x-\lambda)T = 0\}.$$
In particular, $H^0_{x-\lambda}(D/DL)$ is generated as a D-module by $\ker[x-\lambda]$, which we prefer to think of as the cohomology in degree -1 of the complex
$$0 \to \frac{D}{DL} \xrightarrow{(x-\lambda)\cdot} \frac{D}{DL} \to 0 \qquad T \mapsto (x-\lambda)T.$$

This complex is the restriction of D/DL to the point $x = \lambda$ and is equivalent to the complex

$$0 \to \frac{D}{(x-\lambda)D} \xrightarrow{\cdot L} \frac{D}{(x-\lambda)D} \to 0 \qquad T \mapsto TL \tag{1.2}$$

since both complexes are quasi-isomorphic to the total complex of

$$\begin{tikzcd} D \arrow[r, "(x-\lambda)\cdot"] & D \\ D \arrow[u, "\cdot L"] \arrow[r, "(x-\lambda)\cdot"] & D \arrow[u, "\cdot L"'] \end{tikzcd}$$

where the horizontal maps are left multiplication by $x - \lambda$ and the vertical maps are right multiplication by L.

So it suffices to analyze the complex (1.2). The module $D/(x-\lambda)D$ has the basis $\{\partial^j\}_{j=0}^{\infty}$ and can thus be viewed (as vector space) as the polynomial ring $C[\partial]$. With respect to this basis, right multiplication by $(x - \lambda)$ becomes differentiation, i.e.,

$$\partial^j(x-\lambda) = (x-\lambda)\partial^j + j\partial^{j-1} = j\partial^{j-1} \in \frac{D}{(x-\lambda)D}.$$

It follows that right multiplication of ∂^j by $\zeta_k q_k(\theta_\lambda)$ is

$$\partial^j \zeta_k q_k(\theta_\lambda) = \begin{cases} q_k(j-k)\partial^{j-k} & \text{if } k \leq 0 \\ [j]_k q_k(j-k)\partial^{j-k} & \text{if } k > 0 \end{cases}$$

where $[j]_k = j(j-1)\cdots(j-(k-1))$. In particular, if we identify the element $\sum_i a_i\partial^i \in D/(x-\lambda)D$ with the column vector $\vec{a} = [a_0, a_1, \ldots]^t$, then the element $(\sum_i a_i\partial^i)L \in D/(x-\lambda)D$ is identified with $R_\lambda(L)\vec{a}$, where $[R_\lambda(L)_{ij}]_{i,j\in\mathbb{N}}$ is the infinite matrix with entries given by (1.1). Written out, the matrix $R_\lambda(L)$ looks like,

$$R_\lambda(L) = \begin{bmatrix} q_0(0) & [1]_1 q_1(0) & [2]_2 q_2(0) & [3]_3 q_3(0) & [4]_4 q_4(0) & \cdots \\ q_{-1}(1) & q_0(1) & [2]_1 q_1(1) & [3]_2 q_2(1) & [4]_3 q_3(1) & \cdots \\ q_{-2}(2) & q_{-1}(2) & q_0(2) & [3]_1 q_1(2) & [4]_2 q_2(2) & \cdots \\ q_{-3}(3) & q_{-2}(3) & q_{-1}(3) & q_0(3) & [4]_1 q_1(3) & \cdots \\ q_{-4}(4) & q_{-3}(4) & q_{-2}(4) & q_{-1}(4) & q_0(4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Observe that $R_\lambda(L)$ is identically 0 below the diagonal with entries $q_r(i)$. Consequently, if m is the maximum integer root of $q_r(\theta)$, then the kernel of $R_\lambda(L)$ comes from the kernel of the $(m+1) \times (m+r+1)$ upper left submatrix of $R_\lambda(L)$.

It follows that the elements $\{p_v : \vec{v} \in B\}$ of Step 4 form a basis for the subspace $\ker\left(D/(x-\lambda)D \xrightarrow{\cdot L} D/(x-\lambda)D\right)$. From the equivalence of complexes observed earlier in the proof, the elements $\{(x-\lambda)^{-1}p_v L : \vec{v} \in B\}$ form a basis for $\ker[x-\lambda]$. This concludes the proof of correctness. $\square$

Example 1.1.3. Let us compute the local closure of L in Example 1.0.2 at the point $x = 0$. For step 2, we rewrite L as

$$L = (3\theta^2 - 15\theta + 12) + x(-4\theta^2 + 24\theta - 32) + x^2(\theta^2 - 7\theta + 12).$$

For step 3, the maximum integral root of $3\theta^2 - 15\theta + 12$ is $\theta = 4$. For step 4, we form the matrix

$$R_0(L) = \begin{bmatrix} 12 & -32 & 24 & 0 & 0 \\ 0 & 0 & -24 & 36 & 0 \\ 0 & 0 & -6 & 0 & 24 \\ 0 & 0 & 0 & -6 & 16 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

whose kernel has basis $B = \{[8,3,0,0,0]^t, [0,9,12,8,3]^t\}$. Then $p_{[8,3,0,0,0]^t} = 3\partial + 8$ and $p_{[0,9,12,8,3]^t} = 3\partial^4 + 8\partial^3 + 12\partial^2 + 9\partial$. For step 5, generators of the local closure $\mathrm{Cl}_0(L)$ are thus,

$$\begin{array}{lcl} L & = & (x^4 - 4x^3 + 3x^2)\partial^2 - (6x^3 - 20x^2 + 12x)\partial + (12x^2 - 32x + 12) \\ \frac{1}{x}p_{[8,3,0,0,0]^t}L & = & (3x^3 - 12x^2 + 9x)\partial^3 + (8x^3 - 38x^2 + 48x - 18)\partial^2 \\ & & -(48x^2 - 142x + 72)\partial + (96x - 184) \\ \frac{1}{x}p_{[0,9,12,8,3]^t}L & = & (3x^3 - 12x^2 + 9x)\partial^6 + (8x^3 - 2x^2 - 60x + 36)\partial^5 \\ & & +(12x^3 - 56x)\partial^4 + (9x^3 - 12x^2 - 69x + 56)\partial^3 \\ & & -(18x^2 + 72x - 138)\partial^2 - (54x - 216)\partial + 216. \end{array}$$ □

Let us now describe the initial ideal of $\mathrm{Cl}_\lambda(L)$ coming from the order filtration refined by the V-filtration at the point $x = \lambda$. Equivalently, we describe a standard basis for $\mathrm{Cl}_\lambda(L)$ in the sense of Briançon and Maisonobe [8] at $x = \lambda$.

Definition 1.1.4. *For $T = p_n(x)\partial^n + \cdots + p_0(x)$ and $I \subset D$ a left ideal, let*

$$\begin{array}{lcl} \mathrm{in}_\lambda(T) & := & (x-\lambda)^{\mathrm{ord}_\lambda(p_n)}\xi^n \ \in \ \mathbb{C}[x,\xi] \\ \mathrm{in}_\lambda(I) & := & \mathrm{Span}_K\{\mathrm{in}_\lambda(T) | T \in I\} \ \subset \ \mathbb{C}[x,\xi] \end{array}$$

where $\mathrm{ord}_\lambda(f)$ is the order of vanishing of a polynomial f at the point $x = \lambda$.

In the following theorem, we describe the initial ideal of an ideal I having the property that $DL \subset I \subset \mathrm{Cl}_\lambda(L)$.

Theorem 1.1.5. *Let $V \subset \ker\left(D/(x-\lambda)D \xrightarrow{\cdot L} D/(x-\lambda)D\right)$ be a finite dimensional vector subspace, let $\{f_0(\partial), \ldots, f_s(\partial)\}$ be a basis of V with the property that $\deg(f_i) < \deg(f_{i+1})$ for all i, and let $I(V) \subset D$ be the left ideal $D \cdot \{L, (x-\lambda)^{-1}vL : v \in V\}$. Then*

$$\mathrm{in}_\lambda(I(V)) = \langle \mathrm{in}_\lambda(L), (x-\lambda)^{-(i+1)}\xi^{\deg(f_i)-i}\,\mathrm{in}_\lambda(L) : 0 \le i \le s\rangle.$$

Recall that $\mathrm{Cl}_\lambda(L) = I(V)$ if $V = \ker\left(D/(x-\lambda)D \xrightarrow{\cdot L} D/(x-\lambda)D\right)$.

Proof. To simplify notation, we assume that $\lambda = 0$ and we write "in" for "in_λ." We wish to construct a Gröbner basis for $I(V) = D \cdot \{L, x^{-1}f_0(\partial)L, \ldots, x^{-1}f_s(\partial)L\}$. Since all of the generators are left multiples of L in $K[x,x^{-1}]\langle\partial\rangle$, let us consider the D-submodule

$$M = D \cdot \{1, x^{-1}f_0(\partial), \ldots, x^{-1}f_s(\partial)\} \subset K[x,x^{-1}]\langle\partial\rangle.$$

We can extend Definition 1.1.4 to $K[x,x^{-1}]\langle\partial\rangle$ as follows. For an element $T = p_n(x)\partial^n + \cdots + p_0(x) \in K[x,x^{-1}]\langle\partial\rangle$, let

$$\begin{aligned} \mathrm{in}(T) &:= x^{\mathrm{ord}_0(p_n)}\xi^n \in K[x,x^{-1},\xi] \\ \mathrm{in}(M) &:= \mathrm{Span}_K\{\mathrm{in}(T) | T \in M\} \subset K[x,x^{-1},\xi] \end{aligned}$$

where $\mathrm{ord}_0(f)$ is the order of f at the point $x=0$. Observe that $\mathrm{in}(M)$ is not an ideal of $K[x,x^{-1},\xi]$ but a $K[x,\xi]$-submodule. Let us now define a Gröbner basis of M to be a set of elements $\{T_1,\ldots,T_m\} \subset M$ such that $\mathrm{in}(M)$ is generated by $\{\mathrm{in}(T_1),\ldots,\mathrm{in}(T_m)\}$ as a $K[x,\xi]$-module. It then follows that the set $\{T_1L,\ldots,T_mL\} \subset I(V)$ is a Gröbner basis for $I(V)$ because $I(V) = ML$ and $\mathrm{in}(I(V)) = \mathrm{in}(M)\,\mathrm{in}(L)$. So to construct a Gröbner basis for $I(V)$, it suffices to construct a Gröbner basis for M and multiply it on the right by L.

In the remainder of the proof, we describe a Gröbner basis of M. An arbitrary element $T \in M$, after being reduced against $\{1\} \in M$, can be written as

$$T = p_0(\partial)x^{-1}f_0(\partial) + \cdots + p_s(\partial)x^{-1}f_s(\partial)$$

where $p_i(\partial) \in K[\partial]$. Then T has a left normally ordered form,

$$T = \sum_{i\geq 1, j\geq 0} t_{ij}x^{-i}\partial^j.$$

CASE 1: Let us first assume that $f_i(\partial) = \partial^{r_i}$ with $r_i < r_{i+1}$ for all i. Then M is torus invariant, or equivalently, homogeneous with respect to the weight vector $(-1,1)$. In this case, an arbitrary element $T \in M$ can be decomposed into its homogeneous components $T_i \in M$, and hence to compute the initial ideal, it suffices to consider only the initial forms of homogenous elements.

So let $T \in M$ be a homogeneous element of weight $d+1$, and define n to be such that $r_n \leq d < r_{n+1}$. Then we can write

$$T = \sum_{i=0}^{n}(-1)^j a_i\partial^{d-r_i}x^{-1}\partial^{r_i} = \sum_{i=0}^{n}\sum_{j=0}^{d-r_i} a_i[d-r_i]_j x^{-j-1}\partial^{d-j} = \sum_{j\geq 0}\sum_{i=0}^{n} a_i[d-r_i]_j x^{-j-1}\partial^{d-j}$$

where

$$[d-r_i]_j = \left\{ \begin{array}{cc} 1 & \text{if } j=0 \\ (d-r_i)(d-r_i-1)\ldots(d-r_i-j+1) & \text{if } j>0 \end{array} \right\}.$$

If we set $b_j = \sum_{i=0}^n a_i[d-r_i]_j$, then $\mathrm{in}(T) = x^{-k-1}\xi^{d-k}$ where k is the least integer such that $b_k \neq 0$. Now we claim that there exists $\{a_i\}_{i=0}^n$ such that $\mathrm{in}(T) = x^{-n-1}\xi^{d-n}$ while there does not exist $\{a_i\}_{i=0}^n$ such that $\mathrm{in}(T) = x^{-n-2}\xi^{d-n-1}$. To see this, we form the $(n+1)\times(n+1)$ matrix,

$$A = \begin{bmatrix} [d-r_0]_0 & \cdots & [d-r_0]_n \\ \vdots & & \vdots \\ [d-r_n]_0 & \cdots & [d-r_n]_n \end{bmatrix}.$$

Then $[b_0, \ldots, b_n] = [a_0, \ldots, a_n]A$ so that our claim is equivalent to showing that A is nonsingular. To see that A is nonsingular, consider a vector $\vec{v} = [v_0, \ldots, v_n]^t$. Then $A\vec{v} = [p_v(d-r_0), \ldots, p_v(d-r_n)]^t$, where $p_v(x) = \sum_{i=0}^n v_i[x]_i \in K[x]$. If $\vec{v} \neq 0$, then $p_v(x) \neq 0$ and $\deg(p_v) \leq n$. Therefore, p_v has at most n roots and $A\vec{v} \neq 0$. It now follows immediately that

$$\mathrm{in}(M) = \langle 1, x^{-1}\xi^{r_0}, x^{-2}\xi^{r_1-1}, \ldots, x^{-s-1}\xi^{r_s-s}\rangle.$$

Case 2: Let us now return to the general case where $f_i(\partial)$ are arbitrary monic polynomials of increasing degree r_i.

(i) First, let us demonstrate the inclusion

$$\langle 1, x^{-n-1}\xi^{r_n-n} : 0 \leq n \leq s\rangle \subset \mathrm{in}(M).$$

To do this, we shall inductively construct $T^{(n)} \in M$ with the properties that $\mathrm{in}(T^{(n)}) = x^{-n-1}\xi^{r_n-n}$ and that $T^{(n)}$ has weight r_n+1 with respect to the weight vector $(-1,1)$. The case $n=0$ is given by $T^{(0)} = x^{-1}f_0(\partial)$. For the case of general n, we consider the class of elements $T \in M$ which can be written,

$$T = p_0(\partial)x^{-1}f_0(\partial) + \cdots + p_n(\partial)x^{-1}f_n(\partial)$$

where p_i is either 0 or of degree $(r_n - r_i)$.

Let us denote by T_d the homogeneous component of T of weight d with respect to the weight vector $(-1,1)$. Then by Case 1, we can find a set $\{p_i = a_i\partial^{r_n-r_i}\}$ such that $\mathrm{in}(T_{r_n+1}) = x^{-n-1}\xi^{r_n-n}$. On the other hand, for $k < r_n$,

$$T_{k+1} = \sum_{i+j=k} p_0(\partial)_i x^{-1} f_0(\partial)_j + \cdots + \sum_{i+j=k} p_n(\partial)_i x^{-1} f_n(\partial)_j.$$

Expanding into normal form, we may write

$$T_{k+1} = \sum_{h=0}^{k} c_h x^{-h-1}\partial^{k-h}.$$

Now suppose the term $c_h x^{-h-1}\partial^{k-h}$ of highest order in the above sum has order $k-h \geq r_n - n$. Then $h \leq n + k - r_n < n$ since $k < r_n$.This also implies that $k - h \geq r_n - n \geq r_h - h$ since the r_i are strictly increasing. We conclude that $k \geq r_h$.

By induction, we have constructed an element $T^{(h)} \in M$ which has initial form $x^{-h-1}\xi^{r_h-h}$ and weight r_h+1. Then we may replace T by $T - a_h\partial^{k-r_h}T^{(h)}$ for suitable $a_h \in K$, for which T_{k+1} has order strictly less than $k-h$. Continuing to replace if necessary, we arrive at T such that T_{k+1} has order less than $r_n - n$. Furthermore, since $\partial^{k-r_h}T^{(h)}$ always has weight $k+1$, the new T differs from our original T only in weights less than or equal to $k+1$. Thus, doing this for each $k < r_n$ in decreasing order, we eventually obtain $T^{(n)}$ of weight $r_n + 1$ such that $\mathrm{in}(T) = x^{-n-1}\xi^{r_n - n}$. This completes the induction.

(ii) Now let us demonstrate the opposite inclusion

$$\mathrm{in}(M) \subset \langle 1, x^{-n-1}\xi^{r_n-n} : 0 \leq n \leq s\rangle.$$

The argument is essentially the same. After reduction against $\{1\} \in M$, we may write an arbitrary $T \in M$ as

$$T = p_0(\partial)x^{-1}f_0(\partial) + \cdots + p_s(\partial)x^{-1}f_s(\partial).$$

Let us decompose T into "homogeneous syzygy components" $T(d+1)$, which are defined as

$$T(d+1) := \sum_{\{i:\deg(p_i)+r_i=d\}} p_i(\partial)x^{-1}f_i(\partial).$$

Let $n(d)$ be defined to satisfy $r_{n(d)} \leq d < r_{n(d)+1}$ (here we set $r_{s+1} = \infty$). We claim that any nonzero monomial in the left normally ordered form of $T(d+1)$ either has order less than $d - n(d)$ or is the initial monomial of some $\partial^i T^{(h)}$, with $T^{(h)}$ defined as in (i).

To see this claim, we note that $T(d+1)$ has weight $d+1$ with respect to the weight vector $(-1,1)$. Therefore, any nonzero monomial in the left normally ordered form of $T(d+1)$ is of some weight $k+1 \leq d+1$ and can be written as $cx^{-h-1}\partial^{k-h}$ with $h \leq k \leq d$. If the order $k - h \geq d - n(d)$, then $h \leq n(d) + k - d \leq n(d)$. This implies that $k - h \geq d - n(d) \geq r_{n(d)} - n(d) \geq r_h - h$, and it follows that $cx^{-h-1}\partial^{k-h}$ is the initial monomial of $c\partial^{k-r_h}T^{(h)}$.

At this point, we set d' to be the maximum integer such that $T(d'+1) \neq 0$. For every $d < d'$, we have $d - n(d) \leq d' - n(d')$. Then by the above, every nonzero monomial of the left normally ordered expression for T is either of order less than $d' - n(d')$ or is equal to the initial monomial of some $\partial^i T^{(h)}$. Thus, as long as there is a nonzero monomial with order $\geq d' - n(d')$ occuring in the left normally ordered expression for T, then $\mathrm{in}(T) \in \langle 1, x^{-n-1}\xi^{r_n-n} : 0 \leq n \leq s\rangle$.

To show the existence of this monomial, we observe that the only terms of weight $d'+1$ in T come from $T(d'+1)$. In other words, the homogeneous component of weight $d'+1$,

$$\begin{aligned} T_{d'+1} = T(d'+1)_{d'+1} &= \left(\textstyle\sum_{\{i:\deg(p_i)+r_i=d'\}} p_i(\partial)x^{-1}f_i(\partial)\right)_{d'+1} \\ &= \textstyle\sum_{\{i:\deg(p_i)+r_i=d'\}} c_i\partial^{\deg(p_i)}x^{-1}\partial^{r_i}. \end{aligned}$$

By Case 1, we conclude that $\mathrm{in}(T_{d'+1}) = x^{-m-1}\xi^{d'-m}$ for some $0 \leq m \leq n(d')$, so that the monomial $c_m x^{-m-1}\partial^{d'-m}$ occurs in the left normally ordered form of T with order $d' - m \geq d' - n(d')$. $\square$

Example 1.1.6. For the ideal $\mathrm{Cl}_0(L)$ of Example 1.1.3, we already computed a suitable basis $\{3\partial + 8, 3\partial^4 + 8\partial^3 + 12\partial^2 + 9\partial\}$ for $\ker(D/xD \xrightarrow{L\cdot} D/xD)$. According to Theorem 1.1.5, the initial ideal is

$$\mathrm{in}_0(\mathrm{Cl}_0(L)) = \langle \mathrm{in}_0(L), x^{-1}\xi\,\mathrm{in}_0(L), x^{-2}\xi^3\,\mathrm{in}_0(L)\rangle = \langle x^2\xi^2, x\xi^3, \xi^6\rangle.$$

1.2 Global closure

In this section, we give an algorithm to compute the Weyl closure $\mathrm{Cl}(L)$ and we describe its initial ideal under the order filtration.

Theorem 1.2.1. *Let $L = p_n(x)\partial^n + \cdots + p_0(x)$, let $K = \mathbb{C}$ and let $\{\lambda_1, \ldots, \lambda_k\}$ be the distinct roots of $p_n(x)$. Then $\mathrm{Cl}(L) = \mathrm{Cl}_{\lambda_1}(L) + \cdots + \mathrm{Cl}_{\lambda_k}(L)$.*

The key to proving the theorem is the following observation.

Lemma 1.2.2. *Let $L = p_n(x)\partial^n + \cdots + p_0(x)$, and let $p(x) = \gcd(p_n(x), p_n'(x))$ be the squarefree part of p_n. Then $\mathrm{Cl}(L) = K[x, p^{-1}]\langle\partial\rangle L \cap D$.*

Proof. (Lemma 1.2.2) Suppose that $T \in \mathrm{Cl}(L)$. Then $T = SL$ for some $S \in K(x)\langle\partial\rangle$. By collecting denominators, S can be written as

$$S = \frac{1}{h(x)}(g_m(x)\partial^m + \cdots + g_0(x)).$$

The hypothesis $T \in \mathrm{Cl}(L)$ can now be written

$$T = \frac{1}{h(x)}(g_m(x)\partial^m + \cdots + g_0(x))(p_n(x)\partial^n + \cdots + p_0(x)) \in D.$$

Expanding out the right hand side, we find that $h(x)$ divides

$$\begin{array}{l} g_m(x)p_n(x) \\ g_m(x)(p_{n-1}(x) + mp_n'(x)) + g_{m-1}(x)p_n(x) \\ \quad\vdots \\ g_m(x)(p_{n-m}(x) + mp_{n-m+1}'(x) + \cdots) + \cdots + g_0(x)p_n(x). \end{array}$$

If we factor $h(x) = a(x)b(x)$ such that $\gcd(a(x), p_n(x)) = 1$ and the squarefree part$b(x)$ divides $p(x)$, then $a(x)$ divides $g_m(x)$ and by descending induction divides $g_i(x)$ for all i. Therefore, S can also be written as $b(x)^{-1}S'$ with $S' \in D$. As a consequence, $T = b(x)^{-1}S'L \in K[x, p^{-1}]L \cap D$ as required. ☐

Proof. (Theorem 1.2.1) From Lemma 1.2.2, it follows that $\mathrm{Cl}(L)/DL = H_p^0(D/DL)$. Thus in order to compute $\mathrm{Cl}(L)$, it suffices to compute the torsion of D/DL with respect to $p(x)$. Again by Kashiwara's equivalence,

$$H_p^0(D/DL) = D(\ker[p])$$

$$\ker[p] = \{T \in D/DL : p(x)T = 0\}.$$

As before, $\ker[p]$ is the cohomology in degree 0 of the complex

$$0 \to \frac{D}{DL} \xrightarrow{p(x)\cdot} \frac{D}{DL} \to 0 \qquad T \mapsto p(x)T$$

which is equivalent to the complex

$$0 \to \frac{D}{p(x)D} \xrightarrow{\cdot L} \frac{D}{p(x)D} \to 0 \qquad T \mapsto TL.$$

Since K is algebraically closed, we have the factorization $p(x) = (x-\lambda_1)\cdots(x-\lambda_k)$, with distinct $\lambda_i \in K$ (recall that $p(x)$ is squarefree). Then the projection maps $\{D/p(x)D \to D/(x-\lambda_i)D\}_{i=1}^k$ together yield an isomorphism of right D-modules,

$$\frac{D}{p(x)D} \xrightarrow{\cong} \bigoplus_{i=1}^{k} \frac{D}{(x-\lambda_i)D}$$

with inverse given by

$$e_j \mapsto \frac{\prod_{i\neq j}(x-\lambda_i)}{\prod_{i\neq j}(\lambda_j-\lambda_i)}.$$

As an isomorphism of right D-modules, these maps are compatible with right multiplication by L, so that we have an isomorphism of complexes,

$$\begin{tikzcd} 0 \arrow[r] & \frac{D}{p(x)D} \arrow[r, "\cdot L"] \arrow[d, "\simeq"'] & \frac{D}{p(x)D} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \bigoplus_{i=1}^{k} \frac{D}{(x-\lambda_i)D} \arrow[r, "\cdot L"] & \bigoplus_{i=1}^{k} \frac{D}{(x-\lambda_i)D} \arrow[r] & 0. \end{tikzcd}$$

The closure $\mathrm{Cl}(L)$ is generated by L and the vector space of the left hand side. By Algorithm 1.1.2, the local closure $\mathrm{Cl}_{\lambda_i}(L)$ is generated by L and the i-th vector space component of the right hand side. This proves the equality of the theorem. $\square$

The theorem says that if we know the factorization of $p_n(x)$ over $K = \mathbb{C}$, then the Weyl closure of L can be computed by computing local closures at the singular points.

Algorithm 1.2.3. (Weyl closure of L assuming knowledge of singular points)
INPUT: $L = p_n(x)\partial^n + \cdots + p_0(x) \in D$, and the distinct roots $\{\lambda_1, \ldots, \lambda_t\}$ of $p_n(x)$.
OUTPUT: a set of generators of $\mathrm{Cl}(L)$.

1. For each i, let S_i be generators of $\mathrm{Cl}_{\lambda_i}(L)$ as obtained by Algorithm 1.1.2.
2. Return $\cup_{i=1}^t S_i$.

In practice, however, we may not know the factorization of $p_n(x)$ over $K = \mathbb{C}$. We shall give an algorithm when $L \in \mathbb{Q}\langle x, \partial\rangle$, where $\mathbb{Q}$ is the field of rational numbers. We assume the ability to factor a polynomial $f(x) \in \mathbb{Q}[x]$ into irreducibles over $\mathbb{Q}$ and the ability to bound the integer roots of a polynomial $g(x) \in \mathbb{Q}(\alpha)[x]$, where $\mathbb{Q}(\alpha)$ is an algebraic extension of $\mathbb{Q}$. These requirements are within the capabilities of a standard computer algebra system.

Algorithm 1.2.4. (Weyl closure of $L \in \mathbb{Q}\langle x, \partial\rangle$ without knowledge of singular points)

INPUT: $L = p_n(x)\partial^n + \cdots + p_0(x) \in \mathbb{Q}\langle x, \partial\rangle$
$p(x) = \gcd(p_n, p_n') = \prod_{k=1}^{t} f_k(x)$, irreducible factorization over $\mathbb{Q}[x]$.

OUTPUT: a set of generators of $\mathrm{Cl}(L)$.

1. For each $1 \le k \le t$, let $\theta_\alpha = (x-\alpha)\partial$ and rewrite L as

$$L = \sum_{i=r_k}^{s_k} \zeta_i q_i(\theta_\alpha) \in \frac{\mathbb{Q}[\alpha]}{f_k(\alpha)}\langle x, \partial\rangle \qquad \zeta_i = \begin{cases} \partial^{-i} & \text{if } i \le 0 \\ (x-\alpha)^i & \text{if } i > 0 \end{cases}$$

such that $q_{r_k}(\theta) \neq 0$. This new expression for L can be obtained by expanding

$$L = p_n((x-\alpha)+\alpha)\partial^n + \cdots + p_0((x-\alpha)+\alpha) \in \mathbb{Q}\langle x, \partial\rangle \subset \frac{\mathbb{Q}[\alpha]}{f_k(\alpha)}\langle x, \partial\rangle.$$

2. For each $1 \le k \le t$, let m_k be the maximum integer root of $q_{r_k}(\theta_\alpha)$ if it is greater than 0. Otherwise, let m_k be equal to 0. Finally, set $m = \max_k\{m_k + r_k\}$.

3. Let $W \subset D/p(x)D$ be the linear subspace with basis given by $\{x^i\partial^j\}$ for $0 \le i < \deg(p)$ and $0 \le j \le m$. Using linear algebra, compute a basis B of the kernel of the map

$$W \xrightarrow{\cdot L} \frac{D}{p(x)D}.$$

4. Return $\{L, p(x)^{-1}vL : v \in B\}$.

Proof. (Correctness of Algorithm 1.2.4) We know from the proof of Theorem 1.2.1 that generators of $\mathrm{Cl}(L)$ can be taken to be L and $\{p(x)^{-1}uL : u \in U\}$, where

$$U = \ker\left(\frac{D}{p(x)D} \xrightarrow{\cdot L} \frac{D}{p(x)D}\right).$$

We also know from the isomorphism of complexes that $U = \bigoplus_{\{\lambda : p(\lambda)=0\}} \phi_\lambda(U_\lambda)$, where

$$U_\lambda = \ker\left(\frac{D}{(x-\lambda)D} \xrightarrow{\cdot L} \frac{D}{(x-\lambda)D}\right)$$

$$\phi_\lambda : \frac{D}{(x-\lambda)D} \to \frac{D}{p(x)D} \qquad 1 \mapsto \prod_{\{\mu\neq\lambda : p(\mu)=0\}} \frac{x-\mu}{\lambda-\mu}.$$

Now let us suppose λ is a root of the irreducible factor $f_i(x)$ of $p(x)$. Then by the proof of Algorithm 1.1.2, U_λ is contained in the linear subspace $W_\lambda \subset D/(x-\lambda)D$ spanned by $\{\partial^j\}_{j=0}^{m_k+r_k}$, where m_k is the integer computed in Step 3. Under the map ϕ_λ, $\{\partial^j\}_{j=0}^{m_k+r_k}$ is sent to $\{\prod_{\{\mu\neq\lambda:p(\mu)=0\}}(x-\mu/\lambda-\mu)\partial^j\}_{j=0}^{m_k+r_k} \subset \frac{D}{p(x)D}$. All of these elements are contained in the linear subspace $W \subset D/p(x)D$ spanned by $\{x^i\partial^j\}$ for $0 \le i < \deg(p)$ and $0 \le j \le m$. Therefore, U is contained in the subspace W, which proves the correctness of the algorithm. $\square$

Example 1.2.5. Let us illustrate the algorithm on a very simple example, $L = (x^3+2)\partial - 3x^2$, whose solution space is spanned by the polynomial x^3+2. For the preprocessing, x^3+2 is already irreducible in $\mathbb{Q}[x]$, hence for step 1, we write

$$L = ([(x-\alpha)+\alpha]^3+2)\partial - 3((x-\alpha)+\alpha)^2 \in \frac{\mathbb{Q}[\alpha]}{(\alpha^3+2)}\langle x, \partial\rangle$$

$$= (3\alpha^2\theta_\alpha - 3\alpha^2) + (x-\alpha)(3\alpha\theta_\alpha - 6\alpha) + (x-\alpha)^2(\theta_\alpha - 3)$$

For step 2, the maximum integer root of $(3\alpha^2\theta_\alpha - 3\alpha^2)$ is $\theta_\alpha = 1$, hence for step 3, we wish to compute the kernel of

$$\mathrm{Span}_K\{1, x, x^2, \partial, x\partial, x^2\partial\} \xrightarrow{L\cdot} \frac{D}{(x^3+2)D}$$

The image in $D/(x^3+2)D$ is easily computed to be,

$$\begin{array}{lcllcl} L\cdot 1 &=& -3x^2 & L\cdot\partial &=& -6x \\ L\cdot x &=& 6 & L\cdot x\partial &=& -6x^2 \\ L\cdot x^2 &=& 6x & L\cdot x^2\partial &=& 12 \end{array}$$

so that the kernel is spanned by $\{\partial + x^2, x\partial - 2, x^2\partial - 2x\}$. The output of step 4 is thus,

$$\begin{array}{lcl} L &=& (x^3+2)\partial - 3x^2 \\ \frac{1}{x^3+2}(\partial + x^2)L &=& \partial^2 + x^2\partial - 3x \\ \frac{1}{x^3+2}(x\partial - 2)L &=& x\partial^2 - 2\partial \\ \frac{1}{x^3+2}(x^2\partial - 2x)L &=& x^2\partial^2 - 2x\partial. \end{array}$$

Let us now describe the initial ideal of $\mathrm{Cl}(L)$ with respect to the order filtration.

Definition 1.2.6. *For $T = p_n(x)\partial^n + \cdots + p_0(x)$ and $I \subset D$ a left ideal, let*

$$\begin{array}{lcl} \mathrm{in}_{(0,1)}(T) &:=& p_n(x)\xi^n \ \in\ K[x,\xi] \\ \mathrm{in}_{(0,1)}(I) &:=& \mathrm{Span}_K\{\mathrm{in}_{(0,1)}(T) | T \in I\} \ \subset\ K[x,\xi] \end{array}$$

Theorem 1.2.7. *Let $L = p_n(x)\partial^n + \cdots + p_0(x)$, let $K = \mathbb{C}$, and let $\{\lambda_1, \ldots, \lambda_t\}$ be the distinct roots of $p_n(x)$. For each $1 \le k \le t$, let $V_k \subset \ker\left(D/(x-\lambda_k)D \xrightarrow{\cdot L} D/(x-\lambda_k)D\right)$ be a linear subspace, and let $\{f_{k0}(\partial), \ldots, f_{ks_k}(\partial)\}$ be a basis of V_k with the property that $deg(f_{ki}) < deg(f_{k,i+1})$ for all i. Finally, let*

$$I := I(V_1) + \cdots + I(V_t) = D\cdot\{L, (x-\lambda_k)^{-1}vL : v \in V_k, 1 \le k \le t\}$$

with $I(V_k)$ defined as in Theorem 1.1.5. Then

1. $\mathrm{in}_{\lambda_k}(I) = \mathrm{in}_{\lambda_k}(I(V_k))$ *(described in Theorem 1.1.5).*
2. $\mathrm{in}_{(0,1)}(I) = \langle(\prod_{k=1}^t (x-\lambda_k)^{j_k})\xi^m : (x-\lambda_k)^{j_k}\xi^m \in \mathrm{in}_{\lambda_k}(I)\rangle.$

Recall that $I = \mathrm{Cl}(L)$ if $V_k = \ker\left(D/(x-\lambda_k)D \xrightarrow{\cdot L} D/(x-\lambda_k)D\right)$ for all k.

Proof. We have the decomposition $I/DL = \oplus_{k=1}^t I(V_k)/DL = \oplus_{k=1}^t H^0_{x-\lambda_k}(I/DL)$. An element $T \in I$ thus has the property that $\prod_{i=1}^t (x-\lambda_i)^{e_i} T \in DL$ for some set of non-negative integers $\{e_i\}$. The element $T' = \prod_{\{i \neq k\}} (x-\lambda_i)^{e_i} T$ then has the property that $(x-\lambda_k)^{e_k} T' \in DL$ so that $T' \in I(V_k)$. Finally, $\mathrm{in}_{\lambda_k}(T) = \mathrm{in}_{\lambda_k}(T') \in \mathrm{in}_{\lambda_k}(I(V_k))$, which proves (1).

Now let $T \in I$ with $\mathrm{in}_{(0,1)}(T) = f(x)\xi^m$. If $f(x) = \prod_{k=1}^t (x-\lambda_k)^{j_k} g(x)$ such that $g(\lambda_k) \neq 0$ for all k, then $(x-\lambda_k)^{j_k}\xi^m = \mathrm{in}_{\lambda_k}(T) \in \mathrm{in}_{\lambda_k}(I)$. This proves the inclusion "$\subset$" of (2). To prove the opposite inclusion "$\supset$" of (2), let $\{j_k\}_{k=1}^t$ be a set of non-negative integers such that $(x-\lambda_k)^{j_k}\xi^m \in \mathrm{in}_{\lambda_k}(I)$ for all k. Then there exists $T_k \in I$ such that $\mathrm{in}_{\lambda_k}(T_k) = (x-\lambda_k)^{j_k}\xi^m$. This implies that $\mathrm{in}_{(0,1)}(T_k) = (x-\lambda_k)^{j_k} g_k(x)\xi^m$ for some $g_k(x)$ with $g_k(\lambda_k) \neq 0$. Now consider the element $S = h_0(x)\partial^{m-n}L + \sum_{k=1}^t h_k(x)T_k$, and let $H_S(x) = h_0(x)p_n(x) + \sum_{k=1}^t h_k(x)(x-\lambda_k)^{j_k} g_k(x)$. Then $\mathrm{in}_{(0,1)}(S) = H_S(x)\xi^m$ as long as $H_S(x) \neq 0$. In particular, we may choose S so that $H_S(x) = \gcd\{p_n(x), (x-\lambda_k)^{j_k} g_k(x)\}_{k=1}^t$, which divides $\prod_{k=1}^t (x-\lambda_k)^{j_k}$. □

Corollary 1.2.8. *Let $L = p_n(x)\partial^n + \cdots + p_0(x) \in D$ and let $\{\lambda_1, \ldots, \lambda_t\}$ be the distinct roots of $p_n(x)$. Then the set of ideals which contain L and which contain no operator of lower order can be parameterized by the space*

$$\prod_{k=1}^t \mathbf{Gr}\left[\ker\left(\frac{D}{(x-\lambda_k)D} \xrightarrow{\cdot L} \frac{D}{(x-\lambda_k)D}\right)\right]$$

where $\mathbf{Gr}[V]$ denotes the Grassmannian of all vector subspaces of V. Furthermore, the possible initial ideals are described by Theorem 1.2.7.

Proof. Given I satisfying the above hypothesis, we claim that $DL \subset I \subset \mathrm{Cl}(L)$. To see this, suppose $T \notin \mathrm{Cl}(L)$. Then L does not divide T in $R = K(\boldsymbol{x})\langle\boldsymbol{\partial}\rangle$. In particular, the greatest common right divisor G of L and T in R has order less than n. The left ideal generated by L and T in D will contain a multiple $f(x)G$, and since L is an element of minimal order in I, we must have that $T \notin I$. The corollary now follows from Theorem 1.2.7, since any ideal between DL and $\mathrm{Cl}(L)$ can be associated uniquely with vector subspaces $V_1, \ldots, V_t$. □

1.3 Cotype of a linear differential operator

In this section, we relate $\mathrm{Cl}(L)$ and $\mathrm{Cl}_\lambda(L)$ to more familiar solution spaces of L. In particular, we establish an equivalence between $\mathrm{in}_\lambda(\mathrm{Cl}(L))$ and the dimensions of "solution spaces" of L in $K[x]/\langle(x-\lambda)^i\rangle$ for $i \in \mathbb{N}$. We then compare $\mathrm{Cl}_\lambda(L)$ with power series solutions of L in $K[[x-\lambda]]$, and at the end of the section, we give some simple applications coming from this point of view.

It will be convenient to replace L with its balanced form, which we define as follows.

Definition 1.3.1. *The balanced form $B_\lambda(L)$ of L at $x = \lambda$ is the unique operator*

$$B_\lambda(L) := (x-\lambda)^k L = q_0(\theta_\lambda) + (x-\lambda)q_1(\theta_\lambda) + \cdots + (x-\lambda)^r q_r(\theta_\lambda)$$

with $k \in \mathbf{Z}$ such that $q_0 \neq 0$. L is balanced at $x = \lambda$ if $L = B_\lambda(L)$. L is balanced if it is balanced at each of its singular points.

Note we can make L balanced at a point λ by muliplying L by $(x-\lambda)^i$ for a suitable $i \in \mathbb{Z}$. Moreover, if L was already balanced at some other $x = \lambda'$, then $(x-\lambda)^i L$ remains balanced at λ'. Since there are only finitely many singular points, we may thus make L balanced by multiplication by a suitable rational function $p(x)/q(x)$. The advantage is that L then acts on the ideals $\langle (x-\lambda)^i \rangle$, and we get induced maps

$$L_n : \frac{K[x]}{\langle (x-\lambda)^{n+1} \rangle} \to \frac{K[x]}{\langle (x-\lambda)^{n+1} \rangle} \qquad [\sum a_i (x-\lambda)^i] \mapsto [L \bullet \sum a_i (x-\lambda)^i]$$

whose kernels represent the solutions of L in $K[x]/\langle (x-\lambda)^{n+1} \rangle$. If we let $E = K\langle x, \theta_\lambda \rangle$, then these kernels can also be described as,

$$W_n := \ker(L_n) \cong \mathrm{Hom}_E \left(\frac{E}{EL}, \frac{K[x]}{\langle (x-\lambda)^{n+1} \rangle} \right).$$

By applying $\mathrm{Hom}_E(E/EL, -)$ to the inverse system (and its inverse limit)

$$0 \leftarrow \frac{K[x]}{\langle x-\lambda \rangle} \leftarrow \cdots \leftarrow \frac{K[x]}{\langle (x-\lambda)^n \rangle} \leftarrow \frac{K[x]}{\langle (x-\lambda)^{n+1} \rangle} \leftarrow \cdots \qquad (\leftarrow \varprojlim = K[[x-\lambda]])$$

we obtain the inverse system of solution spaces

$$0 \leftarrow W_0 \leftarrow \cdots \leftarrow W_{n-1} \leftarrow W_n \leftarrow \cdots \qquad (\leftarrow W_\infty = \mathrm{Hom}_E \left(\frac{E}{EL}, K[[x-\lambda]] \right)).$$

We now use the solutions spaces $\{W_j\}$ to make the following definition. For this definition, let us also set $W_{-1} = 0$.

Definition 1.3.2. *The cotype of L at $x = \lambda$ is the ordered sequence,*

$$\mathrm{cotype}_\lambda(L) = \{ j \in \mathbb{N} | \dim(W_j) \neq \dim(W_{j-1}) \}.$$

We can now formulate the equivalence between cotype and $\mathrm{in}_\lambda(\mathrm{Cl}(L))$.

Corollary 1.3.3. *Given $L \in D$ and a singular point λ, the following are equivalent.*

1. $\mathrm{cotype}_\lambda(L) = \{ j_0, \ldots, j_{s_\lambda} \}$.
2. $\mathrm{in}_\lambda(\mathrm{Cl}(L)) = \langle \mathrm{in}_\lambda(B_\lambda(L)), (x-\lambda)^{-(i+1)} \xi^{j_i - i} \mathrm{in}_\lambda(B_\lambda(L)) : 0 \leq i \leq s_\lambda \rangle$.

Proof. We may as well assume that L is balanced at λ. Let us understand the linear algebra that is involved in computing the cotype. If $\sum a_i (x-\lambda)^i \in K[[x-\lambda]]$ is denoted by the vector $\vec{a} = [a_0, a_1, a_2, \ldots]^t$, then $L \bullet \sum a_i (x-\lambda)^i = Q_\lambda(L) \vec{a}$ where

$$Q_\lambda(L)\vec{a} = \begin{bmatrix} q_0(0) & & & & & & \\ q_1(0) & q_0(1) & & & & & \\ q_2(0) & q_1(1) & q_0(2) & & & & \\ q_3(0) & q_2(1) & q_1(2) & q_0(3) & & & \\ q_4(0) & q_3(1) & q_2(2) & q_1(3) & q_0(4) & & \\ q_5(0) & q_4(1) & q_3(2) & q_2(3) & q_1(4) & q_0(5) & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{bmatrix}.$$

The matrix $Q_\lambda(L)$ is derived from the computation $q(\theta_\lambda) \bullet x^i = q(i)x^i$. We observe that $W_j = \ker(Q_j)$, where Q_j denotes the left upper $(j+1) \times (j+1)$ submatrix of $Q_\lambda(L)$. The corollary now follows immediately from Algorithm 1.1.2, Theorem 1.1.5, and the fact that for balanced L, $Q_\lambda(L) = (SR_\lambda(L)S^{-1})^t$, where S is the diagonal matrix with entries $M_{ii} = i!$ and R_λ is as in Algorithm 1.1.2. $\square$

Corollary 1.3.4. (Closed operators) *Let $L = p_n(x)\partial^n + \cdots + p_0(x) \in D$. Then L generates a closed ideal if and only if*

1. $\gcd(p_n(x), \ldots, p_0(x)) = 1$
2. *For each singular point λ of L, type$_\lambda(L) = \emptyset$ or $\{0, \ldots, n_\lambda\}$ for some $n_\lambda \in \mathbb{N}$*

Example 1.3.5. The Gauss hypergeometric equation corresponds to the family of operators

$$L_{a,b,c} = x(1-x)\partial^2 + (c - x(a+b+1))\partial - ab$$

with singular points 0 and 1. By analyzing the matrices $Q_0(L_{a,b,c})$ and $Q_1(L_{a,b,c})$, we obtain that $L_{a,b,c}$ generates a closed ideal except when

$$(\text{either } c \in \mathbb{Z}_{\leq 0} \text{ or } c \in \mathbb{Z}_{\geq a+b}) \text{ and } (\text{either } a \in \mathbf{Z}_{\leq 0} \text{ or } b \in \mathbf{Z}_{\leq 0}).$$

Let us now discuss some elementary properties of the cotype, which are easily seen using the matrix representation $Q(L)$.

Lemma 1.3.6. *The ranks of Q_j either stay constant or increase by one as j increases by one. Therefore, the dimensions of $\{W_j\}$ either stay constant or increase by one:*

1. *If* $\operatorname{rank}(Q_j) = \operatorname{rank}(Q_{j-1}) + 1$, *then* $\dim(W_j) = \dim(W_{j-1})$.
2. *If* $\operatorname{rank}(Q_j) = \operatorname{rank}(Q_{j-1})$, *then* $\dim(W_j) = \dim(W_{j-1}) + 1$.

Lemma 1.3.7. *If $\dim(W_j) \neq \dim(W_{j-1})$, then $q_0(j) = 0$. Therefore, letting d denote the number of distinct non-negative integral roots of $q_0(\theta_\lambda)$ and m denote the maximum non-negative integral root,*

$$\dim(W_j) \leq d \qquad \lim_{j\to\infty} \dim(W_j) = \dim(W_m).$$

Corollary 1.3.8. *The cotype is a subset of the non-negative integral exponents of L at $x = \lambda$. In particular, it is a finite ordered set which we shall usually write as $\{j_0, \ldots, j_s\}$. By definition, the exponents of L are the roots of the indicial polynomial $q_0(\theta)$.*

To continue our comparison of $\mathrm{Cl}(L)$ with solution spaces, let us remark that there is another distinguished subset of the non-negative integral exponents coming from the power series solutions $W_\infty \subset K[[x-\lambda]]$. This subset of exponents can be described by the following definition.

Definition 1.3.9. *The type of L at $x = \lambda$ is the finite ordered sequence, usually written $\{i_0, \ldots, i_r\}$,*

$$\text{type}_\lambda(L) = \{i \in \mathbb{N} | (x-\lambda)^i u_i \in W_\infty \textit{ for some unit } u_i \in K[[x-\lambda]]\}.$$

The type and cotype are related via the matrix Q_m (recall that m is the maximum non-negative integral exponent). In particular, Q_m has left upper submatrices $\{Q_j\}_{j=1}^m$ and right lower submatrices $\{M_i\}_{i=1}^m$,

$$Q_j = \begin{bmatrix} q_0(0) & & \\ \vdots & \ddots & \\ q_j(0) & \cdots & q_0(j) \end{bmatrix} \qquad M_i = \begin{bmatrix} q_0(i) & & \\ \vdots & \ddots & \\ q_m(i) & \cdots & q_0(m) \end{bmatrix}$$

and the following are immediately verified.

Lemma 1.3.10. (Comparison of type and cotype)

1. $j \in \mathrm{cotype}_\lambda(L)$ *if and only if* $\mathrm{rank}(Q_j) = \mathrm{rank}(Q_{j-1})$.
   $i \in \mathrm{type}_\lambda(L)$ *if and only if* $\mathrm{rank}(M_i) = \mathrm{rank}(M_{i+1})$.
2. $|\,\mathrm{type}_\lambda(L)| = |\,\mathrm{cotype}_\lambda(L)| = \lim_{m\to\infty} \dim(W_m) = \dim(W_\infty)$.
3. *If* $\dim(W_\infty) = d$ *(number of non-negative integral exponents), then* $\mathrm{type}_\lambda(L) = \mathrm{cotype}_\lambda(L) = \{$*non-negative integer exponents of* L *at* $x = \lambda\}$.
4. *If* $x = \lambda$ *is nonsingular, and* n *is the order of* L, *then* $\mathrm{type}_\lambda(L) = \mathrm{cotype}_\lambda(L) = \{0, 1, \ldots, n-1\} = \{$*all exponents of* L *at* $x = \lambda\}$.

Lemma 1.3.10, Part 3 shows that the type and cotype are the same when every non-negative integral exponent corresponds to a power series solution. When this is the case, by Corollary 1.3.3 and Theorem 1.2.7, we get a relation between $\mathrm{in}_{(0,1)}(\mathrm{Cl}(L))$ and the spaces of power series solutions of L at the singular points of L. In particular, we can derive the following applications, where it is clear that every non-negative integral exponent at a point of K corresponds to a power series solution. We state these applications without proof.

Corollary 1.3.11. (Entire Solutions) *Let $L \in D$ have order n. The following are equivalent:*

1. *L has a basis of n entire solutions.*
2. *L has a basis of n power series solutions for every point of K.*
3. $\xi^i \in \mathrm{in}_{(0,1)}(\mathrm{Cl}(L))$ *for some* $i \in \mathbb{N}$.

Moreover, Corollary 1.3.3 and Theorem 1.2.7 give a relation between $\mathrm{in}_{(0,1)}\mathrm{Cl}(L)$ *and the spaces of the power series solutions of L at each point of K.*

Corollary 1.3.12. (First Order Equations) *Let $L = p(x)\partial - q(x)$ with p, q relatively prime. Let $p(x) = c(x-\lambda_1)^{e_1}\cdots(x-\lambda_k)^{e_k}$ be the factorization of p. Then*

$$\mathrm{in}_{\lambda_i}(\mathrm{Cl}(L)) = \begin{cases} \langle (x-\lambda_i)^{e_i}\xi \rangle & \text{if } e_i > 1 \text{ or } \frac{q(\lambda_i)}{c} \notin \mathbb{N} \\ \langle (x-\lambda_i)\xi, \xi^{n+1} \rangle & \text{if } e_i = 1 \text{ and } \frac{q(\lambda_i)}{c} = n \end{cases}$$

$$\mathrm{in}_{(0,1)}(\mathrm{Cl}(L)) = \langle (\prod_{\{i \mid e_i=1, \frac{q(\lambda_i)}{c} \in \mathbb{Z}_{>n}\}} (x-\lambda_i)^{-1})p(x)\xi^n \mid n \in \mathbb{N}\rangle.$$

Thus, $\mathrm{in}_{(0,1)}(\mathrm{Cl}(L))$ *is combinatorially related to the zeros and their multiplicities as well as the singularities and their order of growth of* $\mathrm{Sol}(L) = e^{\int q/p}$.

The following corollaries follow from Bernstein's theorem and Lemma 1.3.10.

Corollary 1.3.13. *(Annihilators of Polynomials) Let* $\{f_1(x), \ldots, f_n(x)\} \subset K[x]$. *Then*

$$\mathrm{Ann}_D(f_1, \ldots, f_n) = \mathrm{Cl}(L) = D \cdot \{L, \partial^{\max_i\{\deg(f_i)\}+1}\}$$

where

$$L_i = \frac{f_i(x)\partial - f_i'(x)}{\gcd(f_i, f_i')}$$

$$L = LCLM(L_1, \ldots, L_n).$$

Here, LCLM stands for the least common left multiple in D. Again, Corollary 1.3.3 and Theorem 1.2.7 give a relation between $\mathrm{in}_{(0,1)}(\mathrm{Ann}_D(f_1, \ldots, f_n))$ *and the the analytic nature of the vector space* $\mathrm{Span}_K\{f_1, \ldots, f_n\}$ *of functions at each point of* K.

Proof. The proof uses an argument of Galligo in [21] on how to find the 2 generators of an ideal in the first Weyl algebra. The element L is of lowest order and thus its lead monomial with respect to the lexicographic order with $\partial > x$ will occur on the bottom wall of the staircase of $\mathrm{in}_<(\mathrm{Ann}_D(f_1, \ldots, f_n))$. On the other hand, the element $\partial^{\max_i\{\deg(f_i)\}+1}$ has lead monomial on the left hand wall of the staircase. It follows that $D \cdot \{L, \partial^{\max_i\{\deg(f_i)\}+1}\}/\mathrm{Ann}_D(f_1, \ldots, f_n)$ is 0-dimensional, hence is the 0-module by Bernstein's theorem. □

Corollary 1.3.14. (Annihilator of a rational function) *The annihilator ideal of a rational function* $p/q \in K(x)$ *where* $\gcd(p, q) = 1$ *is the ideal,*

$$\mathrm{Ann}_D\left(\frac{p}{q}\right) = D \cdot \left\{\frac{pq\partial - (qp' - pq')}{\gcd(p, p')\gcd(q, q')}, \partial^{\deg(p)+1}q\right\}$$

Proof. The initial form of the first generator with respect to the order filtration is the element $(pq/\gcd(p, p')\gcd(q, q'))\xi$ while the initial form of the second generator is $q\xi^{\deg(p)+1}$. By multiplying the first generator by $\partial^{\deg(p)}$, we get an operator whose initial form is the element $(pq/\gcd(p, p')\gcd(q, q'))\xi^{\deg(p)+1}$. A suitable $K[x]$-linear combination of this element and the second generator will produce an operator with initial form $(q/\gcd(q, q'))\xi^{\deg(p)+1}$. By Lemma 1.3.10, this new operator will have lead term with respect to $\partial > x$ on the left wall of the staircase of $\mathrm{in}_<(\mathrm{Ann}_D(p/q))$. As in the proof of the polynomial annihilator, Bernstein's theorem again implies that the elements generate. □

Example 1.3.15. Let us reexamine Example 1.0.2 in terms of the type and cotype. We have that $\mathrm{Cl}(L) = \mathrm{Ann}_D(x^4, x(x-1)^2)$, and hence $\mathrm{type}_\lambda(L) = \mathrm{cotype}_\lambda(L)$ for all λ since the solutions are entire. We saw earlier that $\mathrm{in}_{(0,1)}(\mathrm{Cl}(L)) = \langle x^2(x-1)(x-3)\xi^2, x\xi^3, \xi^5\rangle$, hence by Corollary 1.3.3,

$$\begin{aligned}
\mathrm{type}_0(L) &= \{1,4\} \\
\mathrm{type}_1(L) &= \{0,2\} \\
\mathrm{type}_3(L) &= \{0,2\} \\
\mathrm{type}_\lambda(L) &= \{0,1\} \text{ for all other } \lambda
\end{aligned}$$

By the definition of type, this means that the vector space $\mathrm{Span}_K\{x^4, x(x-1)^2\}$ contains functions of multiplicity 1 and 4 at $x=0$, multiplicity 0 and 2 at $x=1$ and at $x=3$, and multiplicity 0 and 1 elsewhere. Thus, we can determine the analytic nature of a vector space of polynomials by computing the characteristic ideal of its annihilator ideal. This is what is meant by Corollary 1.3.13.

Example 1.3.16. Our previous applications depended upon the type and cotype being the same. In this example, we show that when $\dim(W_\infty) < d$, i.e. not every non-negative integral exponent corresponds to a power series solution, then the type and cotype can contain different information. Consider the operator

$$\begin{aligned}
L &= x^2\partial^2 + (x^3 + x^2 - 3x)\partial + 3 \\
&= (\theta-1)(\theta-3) + x\theta + x^2\theta
\end{aligned}$$

which has matrix representation around the point $x=0$,

$$Q(L) = \begin{bmatrix} 3 & & & & & & \\ 0 & 0 & & & & & \\ 0 & 1 & -1 & & & & \\ 0 & 1 & 2 & 0 & & & \\ 0 & 0 & 2 & 3 & 3 & & \\ 0 & 0 & 0 & 3 & 4 & 8 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

By Lemma 1.3.10, $\mathrm{cotype}_0(L) = (1)$ while $\mathrm{type}_0(L) = (3)$. I particulare there are two non-negative integer exponents but one of them corresponds to a logarithmic solution. Moreover, the solution spaces $\{W_j\}$ are

$$0 \leftarrow \{0\} \leftarrow K\cdot\{x\} \leftarrow K\cdot\{x+x^2\} \overset{0}{\leftarrow} K\cdot\{x^3\} \leftarrow K\cdot\{x^3 - x^4\} \leftarrow \cdots$$

$$\cdots(\leftarrow W_\infty = K\cdot\{x^3 - x^4 + \frac{1}{8}x^5 + \cdots\})$$

which shows that the maps are not necessarily surjective.

1.4 Jordan-Hölder series

In this section, we give an algorithm to construct a Jordan-Hölder series for a holonomic D-module M, a formula for its length, and a partial characterization of the factors of M and their multiplicities.

By definition, a Jordan-Hölder series for M is a maximal chain of submodules

$$0 = N_0 \subset N_1 \subset \cdots \subset N_{r-1} \subset N_r = M$$

with N_i/N_{i-1} simple. When M is holonomic, such a series is finite and has a well-defined length r. Further, for any simple D-module N, the multiplicity of N in M, which is the number of times N_i/N_{i-1} is isomorphic to N, is also well-defined. We denote it by $\operatorname{mult}(N, M)$ and if it is greater than 0, we say that N is a factor of M.

It is well-known that a holonomic D-module is cyclic. A brute force algorithm to find a cyclic presentation would be to use parametric Gröbner bases and test elements with undetermined coefficients of higher and higher Bernstein degree until a generator is found. However as far as we know, an efficient algorithm to find a cyclic generator is unknown. Nevertheless, we shall assume that M is presented to us as D/I for some left ideal $I \subset D$. Note that a Jordan-Hölder series for D/I equivalently corresponds to a maximal chain of ideals containing I,

$$I = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = D.$$

Before going to the general case, let us first consider the Jordan-Hölder problem when $I = DL$, where L is irreducible as an operator in $R = K(x)\langle\partial\rangle$.

Theorem 1.4.1. *Let $L \in D$ be a balanced operator which is irreducible as an element of R, and let $\{\lambda_1, \ldots, \lambda_t\}$ be the singular points of L. Then*

$$\operatorname{length}\left(\frac{D}{DL}\right) = 1 + \sum_{k=1}^{t} \dim_K \left(\operatorname{Hom}_D\left(\frac{D}{DL}, K((x-\lambda_k))\right)\right).$$

Furthermore,

$$\operatorname{mult}\left(\frac{D}{D(x-\lambda_k)}, \frac{D}{DL}\right) = \dim_K \left(\operatorname{Hom}_D\left(\frac{D}{DL}, K((x-\lambda_k))\right)\right)$$

which accounts for all but one factor. The unaccounted factor is isomorphic to $D/\operatorname{Cl}(L)$ if $\dim_K(\operatorname{Hom}_D(\frac{D}{DL}, K((x-\lambda_k)))) = \dim_K(\operatorname{Hom}_D(\frac{D}{DL}, K[[x-\lambda_k]]))$ for all singular points. Otherwise the unaccounted factor is isomorphic to $D \cdot \{f(x), \operatorname{Cl}(L)\}/\operatorname{Cl}(L)$, where $f(x) \in K[x]$ is of least degree such that $f(\partial)$ annihilates all solutions of $\mathcal{F}(\operatorname{Cl}(L))$ in $\oplus_{k=1}^{t} K[x]e^{\lambda_k x}$. Here $\mathcal{F} : D \to D$ is the Fourier transform sending $x \mapsto \partial$ and $\partial \mapsto -x$.

Proof. Our strategy will be to place the closure in the middle,

$$DL \subset \operatorname{Cl}(L) \subset D$$

and to construct Jordan-Hölder series for $\operatorname{Cl}(L)/DL$ and for $D/\operatorname{Cl}(L)$.

Part 1 (Jordan-Hölder for $\operatorname{Cl}(L)/DL$): This part is an application of our theory for the Weyl closure. In particular, we have already shown

$$\frac{\operatorname{Cl}(L)}{DL} = \bigoplus_{k=1}^{t} H^0_{(x-\lambda_k)}\left(\frac{D}{DL}\right) = \bigoplus_{k=1}^{t} D \cdot \ker[x-\lambda_k].$$

Let $\{f_{k0}, \ldots, f_{ks_k}\}$ be a basis for $\ker[x-\lambda_k]$. By Kashiwara's equivalence,

$$D \cdot \ker[x-\lambda_k] = \bigoplus_{i=0}^{s_k} D f_{ki} \cong \bigoplus_{i=0}^{s_k} \frac{D}{D(x-\lambda_k)}.$$

Now consider the ideals obtained by successively adjoining elements of $\ker[x-\lambda_k]$, i.e. let $I_{10} = D \cdot \{L, f_{10}\}$ and define inductively

$$I_{ab} = \begin{cases} I_{a,b-1} + D f_{ab} & \text{if } b > 0 \\ I_{a-1,s_{a-1}} + D f_{a0} & \text{if } b = 0. \end{cases}$$

for each $1 \leq a \leq t$ and $0 \leq b \leq s_a$. Then the sequence of ideals

$$DL \subset \cdots \subset I_{ab} \subset \cdots \subset \mathrm{Cl}(L)$$

have as successive quotients

$$\frac{I_{ab}}{I_{a,b-1}} \cong \frac{D}{D(x-\lambda_a)}$$
$$\frac{I_{a0}}{I_{a-1,s_{a-1}}} \cong \frac{D}{D(x-\lambda_a)}.$$

Each quotient has multiplicity one and dimension one, and hence is simple. Thus, this sequence of ideals is maximal and corresponds to a Jordan-Hölder series for $\mathrm{Cl}(L)/DL$. Moreover, by Lemma 1.3.10, we see that

$$\mathrm{mult}\left(\frac{D}{D(x-\lambda_k)}, \frac{\mathrm{Cl}(L)}{DL}\right) = \dim_K(\ker[x-\lambda_k]) = \dim_K\left(\mathrm{Hom}_D\left(\frac{D}{DL}, K[[x-\lambda_k]]\right)\right)$$

$$\mathrm{length}\left(\frac{\mathrm{Cl}(L)}{DL}\right) = \sum_{k=1}^{t} \dim_K\left(\mathrm{Hom}_D\left(\frac{D}{DL}, K[[x-\lambda_k]]\right)\right).$$

Part 2 (Jordan-Hölder for $D/\mathrm{Cl}(L)$): In this part, we seek to construct a maximal chain of ideals between $\mathrm{Cl}(L)$ and D. The key step will be the following claim.

Claim 1.4.2. *Suppose that $x = \lambda$ is a singular point. Then the following are equivalent.*

1. $D \cdot \{(x-\lambda)^n, \mathrm{Cl}(L)\}$ *is proper such that* $(x-\lambda)^{n-1} \notin D \cdot \{(x-\lambda)^n, \mathrm{Cl}(L)\}$.
2. $D \cdot \{(\partial-\lambda)^n, \mathcal{F}(\mathrm{Cl}(L))\}$ *is proper such that* $(\partial-\lambda)^{n-1} \notin D \cdot \{(\partial-\lambda)^n, \mathcal{F}(\mathrm{Cl}(L))\}$.
3. $\mathcal{F}(\mathrm{Cl}(L))$ *has a solution* $h(x)e^{\lambda x}$ *with* $h(x) \in K[x]$ *of degree* $n-1$.
4. L *has a solution in* $K((x-\lambda))$ *with a pole of order* $n > 0$.

Proof. (Claim 1.4.2) The conditions (1),(2), and (3) are clearly equivalent. To prove the equivalence of (3) and (4), we will construct maps

$$\mathrm{Hom}_D\left(\frac{D}{DL}, K((x-\lambda))\right) \longrightarrow \mathrm{Hom}_D\left(\frac{D}{\mathrm{Cl}(L)}, \frac{K((x-\lambda))}{K[[x-\lambda]]}\right)$$
$$\xrightarrow{\cong} \mathrm{Hom}_D\left(\frac{D}{\mathcal{F}(\mathrm{Cl}(L))}, K[x]e^{\lambda x}\right).$$

In the following 2 steps, we show that the second map is an isomorphism and that the first map is surjective with kernel equal to $\mathrm{Hom}_D\left(\frac{D}{DL}, K[[x-\lambda]]\right)$. This proves the equivalence of (3) and (4). □

Step 1.4.3. *Given $T \in D$, there is an isomorphism*

$$\mathrm{Hom}_D\left(\frac{D}{DT}, \frac{K((x-\lambda))}{K[[x-\lambda]]}\right) \cong \mathrm{Hom}_D\left(\frac{D}{D\mathcal{F}(T)}, K[x]e^{\lambda x}\right)$$
$$\Psi\left[\sum_{i=-n}^{-1} a_i(x-\lambda)^i\right] \mapsto \Psi\left[\sum_{i=-n}^{-1} \frac{1}{(-i-1)!} a_i x^{-i-1} e^{\lambda x}\right]$$

where $\Psi[m]$ denotes the morphism of D-modules defined by sending 1 to m.

Proof. (Step 1.4.3) Let us write,

$$T = \sum_{i=r}^{s} \zeta_i p_i(\theta_\lambda) \qquad \zeta_i = \begin{cases} \partial^i & \text{if } i \leq 0 \\ (x-\lambda)^i & \text{if } i > 0 \end{cases}$$

where $p_r(\theta) \neq 0$. Then

$$T \bullet (x-\lambda)^j = \sum_{k<0} [j]_{-k} p_k(j)(x-\lambda)^{j+k} + \sum_{k \geq 0} p_k(j)(x-\lambda)^{j+k}.$$

Therefore, if we represent an element $\sum_{i<0} v_i(x-\lambda)^i \in K((x-\lambda))/K[[x-\lambda]]$ by the vector $[\cdots, v_{-3}, v_{-2}, v_{-1}]^t$, where $v_i = 0$ for $i \ll 0$, then the action of T on $K((x-\lambda))/K[[x-\lambda]]$ is represented by the matrix $[\overline{Q}(T)_{ij}]_{i,j<0}$ with entries

$$\overline{Q}(T)_{ij} = \begin{cases} p_{i-j}(j) & \text{if } i \geq j \\ [j]_{j-i} p_{i-j}(j) & \text{if } i < j. \end{cases} \tag{4.3}$$

On the other hand, for the Fourier transform, we have

$$\mathcal{F}(T) = \sum_{i=r}^{s} \widehat{\zeta}_i p_i(-\widehat{\theta}_\lambda - 1) \qquad \widehat{\theta}_\lambda = x(\partial - \lambda) \qquad \widehat{\zeta}_i = \begin{cases} (-x)^i & \text{if } i \leq 0 \\ (\partial-\lambda)^i & \text{if } i > 0 \end{cases}$$

$$\mathcal{F}(T) \bullet x^j e^{\lambda x} = \sum_{k \leq 0} (-1)^k p_k(-j-1) x^{j-k} e^{\lambda x} + \sum_{k>0} [j]_k p_k(-j-1) x^{j-k} e^{\lambda x}.$$

Therefore, if we represent an element $\sum_{i\in\mathbb{N}} w_i x^i e^{\lambda x} \in K[x]e^{\lambda x}$ by the corresponding vector $\vec{w} = [w_0, w_1, w_2, \ldots]^t$, where $w_i = 0$ for $i \gg 0$, then the action of $\mathcal{F}(T)$ on $K[x]e^{\lambda x}$ is represented by the matrix $[A(\mathcal{F}(T))_{ij}]_{i,j\in\mathbb{N}}$ with entries

$$A(\mathcal{F}(T))_{ij} = \begin{cases} (-1)^{j-i} p_{j-i}(-j-1) & \text{if } i \geq j \\ [j]_{j-i} p_{j-i}(-j-1) & \text{if } i < j \end{cases}$$

for $i, j \in \mathbb{N}$.

Finally, let us compare the matrices $\overline{Q}(T)$ and $A(\mathcal{F}(T))$. First, we flip $\overline{Q}(T)$ by considering the matrix $[\underline{Q}(T)_{ij}]_{i,j\in\mathbb{N}}$ defined by $\underline{Q}(T)_{ij} = \overline{Q}(T)_{-i-1,-j-1}$. Then a computation shows that $A(\mathcal{F}(T)) = S^{-1}\underline{Q}(T)S$ where S is the diagonal matrix with entries $S_{ii} = i!$. Thus we get our desired isomorphism,

$$\ker(\overline{Q}(T)) \to \ker(A(\mathcal{F}(T))) \qquad \sum_{i=-n}^{-1} a_i (x-\lambda)^i \mapsto \sum_{i=-n}^{-1} \frac{1}{(-i-1)!} a_i x^{-i-1} e^{\lambda x}.$$

$\square$

Step 1.4.4. *The map*

$$\mathrm{Hom}_D\left(\frac{D}{DL}, K((x-\lambda))\right) \longrightarrow \mathrm{Hom}_D\left(\frac{D}{\mathrm{Cl}(L)}, \frac{K((x-\lambda))}{K[[x-\lambda]]}\right)$$

is surjective with kernel equal to $\mathrm{Hom}_D\left(\frac{D}{DL}, K[[x-\lambda]]\right)$.

Proof. (Step 1.4.4) Suppose that $\phi = \sum_{i=-n}^{-1} a_i (x-\lambda)^i$ is a solution of $\mathrm{Cl}(L)$ in $K((x-\lambda))/K[[x-\lambda]]$. We wish to show that ϕ has a lift $\widetilde{\phi}$ in $K((x-\lambda))$ which is a also solution of L.

Let us represent $\sum_{i\in\mathbb{Z}} v_i (x-\lambda)^i \in K((x-\lambda))$ by the vector $[\cdots, v_{-1}, v_0, v_1, \cdots]^t$. Then using the notation of Step 1.4.3, the action of T on $K((x-\lambda))$ is represented by the matrix $[\widetilde{Q}(T)_{ij}]_{i,j\in\mathbb{Z}}$ with entries given by (4.3). In terms of $\widetilde{Q}(T)$, the assumption that ϕ is a solution of $\mathrm{Cl}(L)$ means that for every $T \in \mathrm{Cl}(L)$, the vector $\vec{a}(\phi) = [a_{-n}, \cdots, a_{-1}]^t$ is in the kernel of the submatrix $[\widetilde{Q}(T)_{ij}]_{i\leq -1, -n\leq j\leq -1}$.

The problem of lifting ϕ to $\widetilde{\phi}$ is equivalent to extending $\vec{a}(\phi)$ to a vector $\vec{a}(\widetilde{\phi}) = [a_{-n}, \cdots, a_{-1}, a_0, a_1, \cdots]^t$ in the kernel of the extended submatrix $[\widetilde{Q}(L)_{ij}]_{i\in\mathbf{Z}, j\geq -n}$. Let us decompose this submatrix as

$$[\widetilde{Q}(L)_{ij}]_{i\in\mathbb{Z}, j\geq -n} = \begin{bmatrix} \overline{Q} & 0 \\ Q_- & Q_+ \end{bmatrix} \qquad \begin{aligned} \overline{Q} &= [\widetilde{Q}(L)_{ij}]_{i\leq -1, -n\leq j\leq -1} \\ Q_- &= [\widetilde{Q}(L)_{ij}]_{i\geq 0, -n\leq j\leq -1} \\ Q_+ &= [\widetilde{Q}(L)_{ij}]_{i\geq 0, j\geq 0}. \end{aligned}$$

Since Q_+ is lower triangular, let us first show that $\vec{a}(\phi)$ can be extended to a vector $[a_{-n}, \cdots, a_{-1}, \cdots, a_k]^t$ in the kernel of $[\widetilde{Q}(L)_{ij}]_{i\leq k, -n\leq j\leq k}$. This is equivalent to showing that $(Q_-)_k \cdot \vec{a}(\phi) \in \mathrm{im}((Q_+)_k)$, where M_k denotes the matrix consisting of the first $k+1$

rows of M. Let us denote the row vectors of Q_- and Q_+ by $\vec{w}_i^{(-)}$ and $\vec{w}_i^{(+)}$ respectively. Now we claim that for any relation amongst the row vectors of Q_+, we have

$$\sum_{i=0}^{k} c_i \vec{w}_i^{(+)} = 0 \quad \Rightarrow \quad (\sum_{i=0}^{k} c_i \vec{w}_i^{(-)}) \circ \vec{a}(\phi) = \vec{c} \circ (Q_-)_k \vec{a}(\phi) = 0 \tag{4.4}$$

where $\circ$ denotes the dot product. Before proving (4.4), let us argue why it implies that $(Q_-)_k \cdot \vec{a}(\phi) \in \mathrm{im}((Q_+)_k)$. For $W \subset K^{k+1}$, let $W^\perp = \{\vec{v} \in K^{k+1} | \vec{v} \circ \vec{w} = 0 \; \forall \; \vec{w} \in W\}$. By the non-degeneracy of the dot product, $W^{\perp\perp} = W$. Now note that the hypothesis of (4.4) is $\vec{c} \in \mathrm{im}((Q_+)_k)^\perp$, and hence (4.4) shows that $(Q_-)_k \cdot \vec{a}(\phi) \in \mathrm{im}((Q_+)_k)^{\perp\perp} = \mathrm{im}((Q_+)_k)$.

To prove (4.4), suppose we have a relation $\sum_i c_i \vec{w}_i^{(+)} = 0$, and let $\vec{c} = [c_0, c_1, \ldots]^t$. Then $Q_+^t \vec{c} = 0$. By Corollary 1.3.3, recall that $Q_+^t = Q(L)^t = SR_\lambda(L)S^{-1}$ where S is diagonal with entries $S_{ii} = i!$. In particular, $R_\lambda(L)S^{-1}\vec{c} = 0$, so that by Algorithm 1.1.2, the element $T_c = (x-\lambda)^{-1}(\sum_i (c_i/i!)\partial^i)L \in \mathrm{Cl}(L)$. Now by a computation which we leave to the reader,

$$[\widetilde{Q}(T_c)_{-1,j}]_{-n \le j \le -1} = \sum_i c_i \vec{w}_i^{(-)}.$$

Since ϕ is a solution of $T_c \in \mathrm{Cl}(L)$, we noted earlier that $\vec{a}(\phi)$ is in the kernel of the matrix $[\widetilde{Q}(S_u)_{ij}]_{i \le -1, -n \le j \le -1}$. This means that $\sum_i c_i \vec{w}_i^{(-)} \circ \vec{a}(\phi) = 0$, as desired.

Thus, we have shown that $\vec{a}(\phi)$ can be extended to a vector $[a_{-n}, \cdots, a_{-1}, \cdots, a_k]^t$ in the kernel of $[\widetilde{Q}(L)_{ij}]_{i \le k, -n \le j \le k}$ for any k. To complete the proof, we should make sure that the extensions for increasing k can be chosen so as to converge to a meaningful vector. Since L is balanced, we can write $L = \sum_{i=0}^r x^i q_i(\theta)$. Let m be the maximum non-negative integer root of $q_0(\theta)$ if it exists, or -1 otherwise. By the above, we can extend $\vec{a}(\phi)$ to some $[a_{-n}, \cdots, a_{-1}, \cdots, a_m]^t$ in the kernel of $[\widetilde{Q}(L)_{ij}]_{i \le m, -n \le j \le m}$. Since Q_+ is lower triangular with diagonal entries $q_0(i)$, which are all nonzero for $i \ge m$, then $[a_{-n}, \cdots, a_{-1}, \cdots, a_m]^t$ extends uniquely to $\vec{a}(\phi) = [a_{-n}, \cdots, a_{-1}, \cdots, a_m, \cdots]^t$ in the kernel of $[\widetilde{Q}(L)_{ij}]_{i \in \mathbb{Z}, j \ge -n}$. □

(Continuation of Part 2) To complete the Jordan-Hölder problem for $D/\mathrm{Cl}(L)$, we consider all the singular points $\lambda_1, \ldots, \lambda_t$ of L. For each λ_a, we find a maximal set of solutions $\phi_{a1}, \ldots, \phi_{ar_a}$ to L in $K((x-\lambda_a)) \setminus K[[x-\lambda_a]]$ with the property that they have poles of order $0 < m_{a1} < m_{a2} < \cdots < m_{ar_a}$ respectively. By Claim 1.4.2, we get the ideal

$$J_{ab} = D(x-\lambda_a)^{m_{ab}} + \mathrm{Cl}(L) \qquad (x-\lambda_a)^{m_{ab}-1} \notin J_{ab}.$$

Using these ideals, we can construct a descending chain of ideals

$$D \supset H_{11} \supset H_{12} \supset \cdots \supset H_{ab} \supset \cdots \supset H_{tr_t} \supset \mathrm{Cl}(L)$$

where

$$\begin{aligned} H_{ab} &= J_{1r_1} \cap J_{2r_2} \cap \cdots \cap J_{a-1,r_{a-1}} \cap J_{ab} \\ &= D \cdot \{(x-\lambda)^{m_{ab}} \prod_{k=1}^{a-1} (x-\lambda_k)^{m_{kr_k}}\} + \mathrm{Cl}(L). \end{aligned}$$

We claim that this is a maximal chain of ideals between $\mathrm{Cl}(L)$ and D. To see this, we first note using Claim 1.4.2 that $\mathcal{F}(\mathrm{Cl}(L))$ has a basis $h_{a1}(x)e^{\lambda_a x}, \cdots, h_{ar_a}(x)e^{\lambda_a x}$ of solutions in $K[x]e^{\lambda_a x}$ such that $\deg(h_{ab}) = m_{ab} - 1$. Therefore,

$$\mathcal{F}(H_{ab}) = \mathrm{Ann}_D(h_{11}e^{\lambda_1 x}, \ldots, h_{1r_1}e^{\lambda_1 x}, \ldots, h_{a1}e^{\lambda_a x}, \ldots, h_{ab}e^{\lambda_a x})$$

and it follows that

$$\frac{\mathcal{F}(H_{a,b})}{\mathcal{F}(H_{a,b+1})} \cong \ker\left(\frac{D}{\mathrm{Ann}_D(h_{11}e^{\lambda_1 x}, \ldots, h_{a,b-1}e^{\lambda_a x})} \to \frac{D}{\mathrm{Ann}_D(h_{11}e^{\lambda_1 x}, \ldots, h_{ab}e^{\lambda_a x})}\right).$$

From the linear independence of $\{h_{ij}e^{\lambda_i x}\}$, we leave to the reader to show that the map

$$\frac{D}{\mathrm{Ann}(h_{11}e^{\lambda_1 x}, \ldots, h_{ab}e^{\lambda_a x})} \longrightarrow \frac{D}{\mathrm{Ann}(h_{11}e^{\lambda_1 x})} \oplus \cdots \oplus \frac{D}{\mathrm{Ann}(h_{ab}e^{\lambda_a x})}$$

$$1 \mapsto (1, \ldots, 1)$$

is an isomorphism of D-modules and that

$$\frac{\mathcal{F}(H_{ab})}{\mathcal{F}(H_{a,b+1})} \cong \frac{D}{\mathrm{Ann}(h_{ab}e^{\lambda x})} \cong Dh_{ab}e^{\lambda x} \cong K[x]e^{\lambda x} \cong \frac{D}{D(\partial - \lambda_a)}.$$

In other words,

$$\frac{H_{ab}}{H_{a,b+1}} \cong \frac{D}{D(x - \lambda_a)} \qquad \frac{H_{a1}}{H_{a-1,s_{a-1}}} \cong \frac{D}{D(x - \lambda_a)}.$$

Finally, we must show that H_{tr_t} is minimal over $\mathrm{Cl}(L)$. First, we claim that an ideal I properly containing $\mathrm{Cl}(L)$ also contains a polynomial $f(x)$. To see this, suppose we are given $T \notin \mathrm{Cl}(L)$. Then as elements of $R = K(x)\langle\partial\rangle$, L and T generate R because L is irreducible. This implies that the left ideal $D \cdot \{L, T\} \subset D$ contains some polynomial $f(x)$.

Second, we claim that an ideal I properly containing $\mathrm{Cl}(L)$ cannot contain a polynomial $f(x)$ having no singular points as roots. This claim is a simple application of Gröbner bases and Bernstein's theorem: Suppose $f(x)$ has no singular points as roots and that L has order n. Then for some pair of polynomials $g(x)$ and $h(x)$, the operator $T = g(x)\partial^n f(x) + h(x)L \in D \cdot \{f(x), L\}$ has initial term $\mathrm{in}_{(0,1)}(T) = \xi^n$. In particular, $\langle f(x), \xi^n\rangle \subset \mathrm{in}_{(0,1)}(D \cdot \{f(x), L\})$ which implies that $D/(Df(x) + \mathrm{Cl}(L))$ has dimension 0. By Bernstein's theorem, $Df(x) + \mathrm{Cl}(L) = D$.

Translating over to the Fourier transform, we conclude that any ideal properly containing $\mathcal{F}(\mathrm{Cl}(L))$ contains some constant coefficient operator $f(\partial)$. There is a unique minimal ideal of the form $Df(\partial) + \mathcal{F}(\mathrm{Cl}(L))$, and it is equal to the annihilating ideal of the solutions of $\mathcal{F}(\mathrm{Cl}(L))$ in $\oplus_{\lambda \in K} K[x]e^{\lambda x}$. But $\mathcal{F}(\mathrm{Cl}(L))$ has no solutions in $K[x]e^{\lambda x}$ when λ is not a singular point because $D(\partial - \lambda)^m + \mathcal{F}(\mathrm{Cl}(L)) = D$ for all m in this case. Therefore, a minimal ideal of the form $Df(\partial) + \mathcal{F}(\mathrm{Cl}(L))$ is equal to the annihilating ideal of the solutions of $\mathcal{F}(\mathrm{Cl}(L))$ in $\oplus_{k=1}^{t} K[x]e^{\lambda_k x}$, which is exactly $\mathcal{F}(H_{tr_t})$. □

Corollary 1.4.5. *An operator $L = p_n(x)\partial^n + \cdots + p_0(x) \in D$ generates a maximal ideal in D if and only if*

1. $\gcd(p_n(x), \ldots, p_0(x)) = 1$.

2. *L is irreducible as an operator in R.*

3. *For each singular point λ of L,*

a. $\text{cotype}_\lambda(L) = \emptyset$ *or* $\{0, 1, \ldots, n_\lambda\}$ *for some* $n_\lambda \in \mathbb{N}$.

b. Every solution of L in $K((x-\lambda))$ lies in $K[[x-\lambda]]$.

In particular, (3) *is satisfied if the exponents of L are non-integral at every singular point.*

Based upon our theorem, we shall now present an algorithm to compute a Jordan-Hölder series for a holonomic D-module. First however, we must address the role of the field K. In particular, the Jordan-Hölder series depends on the field K. For instance the D-module $D/D(x^2+1)$ has length 2 when $K = \mathbb{C}$ and length 1 when $K = \mathbb{Q}$. Moreover, when $K = \mathbb{C}$ in this example, the output of the algorithm cannot be defined in the subfield $\mathbb{Q}$ even though the input could be defined there.

To avoid this difficulty, we shall continue to assume that K is algebraically closed and that we can factor completely over K. In this sense, our algorithm is a theoretical one. However, if we would like to take $K = \mathbb{Q}$, then our algorithm can be turned into a practical algorithm by using the same techniques employed in Algorithm 1.2.4.

Algorithm 1.4.6. (Jordan-Hölder series for holonomic D-modules)
INPUT: $I \subset D$, a left ideal.
OUTPUT: Jordan-Hölder series of D/I.

1. Find an operator $L \in I$ of minimal order by computing a Gröbner basis of I with respect to the order filtration.

2. Factor L as an operator in R, e.g. by using algorithms in [50]. This factorization can be expressed as
$$L = \frac{1}{g(x)} L_1 \cdots L_m$$
where $g(x) \in K[x]$ and where $L_1, \ldots, L_m \in D$ are operators of positive order which are irreducible as operators of R. Replace L by $g(x)L = L_1 \ldots L_m$ which is also of minimal order in I. Here, we have assumed that we can factor over K.

3. For each j from 1 and m, find the singular points of L_j and label them $\{\lambda_{j1}, \ldots, \lambda_{jt_j}\}$. Again, we have assumed that we can factor over K.

4. For each j between 1 and m and each k between 1 and t_j, use Algorithm 1.1.2 to find a basis $\{f_{jk0}(\partial), \ldots, f_{jks_{jk}}(\partial)\}$ of $\ker(\frac{D}{(x-\lambda_{jk})D} \xrightarrow{\circ L_j} \frac{D}{(x-\lambda_{jk})D})$. Using Gröbner bases, find the indices $\{r_{jkl}\}_{l=1}^{u_{jk}}$ such that
$$(x-\lambda_{jk})^{-1} f_{jkr_{jkl}}(\partial) L_j L_{j+1} \cdots L_m \notin$$
$$I + \{(x-\lambda_{jk})^{-1} f_{jki}(\partial) L_j L_{j+1} \cdots L_m : 0 \leq i < r_{jkl}\}.$$

5. For each j from 1 and m and each k from 1 and t_j, find the set
$$\{n_{jkl} > 0 : \exists \phi \in K((x-\lambda_{jk})), L_j(\phi) = 0, \mathrm{ord}_{\lambda_{jk}}(\phi) = -n_{jkl}\}_{l=1}^{v_{jk}}.$$
This is done by solving the kernel of a truncated submatrix of $\widetilde{Q}(L_j)$, where the truncation is determined by computing the minimal and maximal integer exponents at λ_{jk}. Let us also assume that the set is ordered with $n_{jki} > n_{jk,i+1}$.

6. Construct ideals by successively adjoining elements,
$$I_{j0} = I + DL_jL_{j+1}\cdots L_m$$
$$I_{ja} = I_{ja-1} + D\chi_{ja}L_{j+1}\cdots L_m$$
where χ_{ja} is the a-th element of the ordered set
$$S_j = \{(x-\lambda_{jk})^{-1} f_{jkr_{jkl}}(\partial)L_j : 1 \leq k \leq t_j, 1 \leq l \leq u_{jk}\}$$
$$\cup \quad \{(x-\lambda_{jk})^{n_{jkl}} \prod_{h=1}^{t-k} (x-\lambda_{jh})^{n_{jh1}} : 1 \leq k \leq t_j, 1 \leq l \leq v_{jk}\}.$$
Here, both subsets are ordered from least to greatest under the lexicographic order on the indices (k,l). The ordering of the total set S_j is obtained by concatenating the second ordered subset onto the end of the first ordered subset.

7. Return $\{I_{ja}/I\}_{1\leq j\leq m, 1\leq a\leq m_j}$, a Jordan-Hölder series for D/I, where the inclusions follow the lexicographic order on the indices (j,a).

Proof. (Correctness of Algorithm 1.4.6): Consider the chain of increasing ideals
$$I = I + DL_1\cdots L_m \subset I + DL_2\cdots L_m \subset \cdots \subset I + DL_m \subset D.$$
To compute a Jordan-Hölder series, it suffices to refine this chain of ideals to a maximal chain of ideals. Let us now describe a refinement between $I + DL_i\cdots L_m$ and $I + DL_{i+1}\cdots L_m$. Note that the quotient is
$$\frac{I + DL_{i+1}\cdots L_m}{I + DL_i\cdots L_m} \cong \frac{DL_{i+1}\cdots L_m}{DL_i\cdots L_m + (I\cap DL_{i+1}\cdots L_M)} \cong \frac{D}{I_i}$$
where I_i is the kernel of the surjective morphism
$$D \to \frac{DL_{i+1}\cdots L_m}{DL_i\cdots L_m + (I\cap DL_{i+1}\cdots L_m)} \qquad 1 \mapsto L_{i+1}\cdots L_m.$$
Moreover, since $L = L_1\cdots L_m$ is of minimal order in I,
$$I \cap DL_{i+1}\cdots L_m \subset \mathrm{Cl}(L_1\cdots L_i)L_{i+1}\cdots L_m \subset \mathrm{Cl}(L_i)L_{i+1}\cdots L_m.$$
This implies that $DL_i \subset I_i \subset \mathrm{Cl}(L_i)$.

Since L_i is irreducible, we can use the methods developed in Theorem 1.4.1 to construct a Jordan-Hölder series for D/I_i. These methods lead exactly to steps 5, 6, and 7 of our algorithm, where a maximal chain of ideals between $I+DL_i\cdots L_m$ and $I+DL_{i+1}\cdots L_m$ is constructed. $\square$

Example 1.4.7. Let us compute a Jordan-Hölder series for D/DL where

$$L = (x^6 + 2x^4 - 3x^2)\partial^2 - (4x^5 - 4x^4 - 12x^2 - 12x)\partial + (6x^4 - 12x^3 - 6x^2 - 24x - 12)$$

For step 2, L has the factorization $L = L_1L_2$ where

$$\begin{aligned} L_1 &= (x^4 + x^3 + 3x^2 + 3x)\partial + (4x^3 + 3x^2 + 6x + 3) \\ L_2 &= (x^2 - x)\partial + (-2x + 4) \end{aligned}$$

For step 3, the singular points of L_1 are $\{0, -1, \sqrt{3}i, -\sqrt{3}i\}$, and the singular points of L_2 are $\{0, 1\}$. For step 4, the various kernels have bases,

Kernel	Basis
$\frac{D}{xD} \xrightarrow{\circ L_1} \frac{D}{xD}$	$\{\partial - 2\}$
$\frac{D}{(x+1)D} \xrightarrow{\circ L_1} \frac{D}{(x+1)D}$	$\{\partial + 3\}$
$\frac{D}{(x-\sqrt{3}i)D} \xrightarrow{\circ L_1} \frac{D}{(x-\sqrt{3}i)D}$	$\{2\partial - 1 + 3\sqrt{3}i\}$
$\frac{D}{(x+\sqrt{3}i)D} \xrightarrow{\circ L_1} \frac{D}{(x+\sqrt{3}i)D}$	$\{2\partial - 1 - 3\sqrt{3}i\}$
$\frac{D}{xD} \xrightarrow{\circ L_2} \frac{D}{xD}$	$\{\partial^4 - 4\partial^3\}$
$\frac{D}{(x-1)D} \xrightarrow{\circ L_2} \frac{D}{(x-1)D}$	$\{0\}$

and adding the following elements one by one to L creates strictly increasing ideals,

$$T_1 = \tfrac{\partial-2}{x}L_1L_2 = \left[\begin{array}{c} (x^3 + x^2 + 3x + 3)\partial^2 - 2(x^3 + x^2 + 3x + 3)\partial \\ +(8x^2 - 6x + 6) \end{array}\right] L_2$$

$$T_2 = \tfrac{\partial+3}{x+1}L_1L_2 = [(x^3 + 3x)\partial^2 + 3(x^3 + 3x)\partial - (12x^2 + 9x + 15)]L_2$$

$$T_3 = \tfrac{2\partial-1+3\sqrt{3}i}{x-\sqrt{3}i}L_1L_2 = \left[\begin{array}{c} 2(x^3 + (1+\sqrt{3}i)x^2 + \sqrt{3}ix)\partial^2 \\ -(1 - 3\sqrt{3}i)(x^3 + (1+\sqrt{3}i)x^2 + \sqrt{3}ix)\partial \\ +(4 - 12\sqrt{3}i)x^2 + (21 - 5\sqrt{3}i)x + (9 + 3\sqrt{3}i) \end{array}\right] L_2$$

$$T_4 = \tfrac{2\partial-1-3\sqrt{3}i}{x+\sqrt{3}i}L_1L_2 = \left[\begin{array}{c} 2(x^3 + (1-\sqrt{3}i)x^2 - \sqrt{3}ix)\partial^2 \\ -(1 + 3\sqrt{3}i)(x^3 + (1-\sqrt{3}i)x^2 - \sqrt{3}ix)\partial \\ +(4 + 12\sqrt{3}i)x^2 + (21 + 5\sqrt{3}i)x + (9 - 3\sqrt{3}i) \end{array}\right] L_2$$

$$T_5 = \tfrac{\partial^4-4\partial^3}{x}L_2 = (x-1)\partial^5 - (4x - 10)\partial^4 - 16\partial^3$$

For step 5, we note that the solution space of L_1 is spanned by $x^4 + x^3 + 3x^2 + 3x$ while the solution space of L_2 is spanned by $x^4/(x-1)^2$. Therefore, the only singular point where a solution has a pole occurs for L_2 at $x = 1$, which is a pole of order 2. For steps 6 and 7, it follows that a Jordan-Hölder series for D/DL is

$$0 \subset \frac{D\{L, T_1\}}{DL} \subset \frac{D\{L, T_1, T_2\}}{DL} \subset \frac{D\{L, T_1, T_2, T_3\}}{DL} \subset \frac{D\{L, T_1, T_2, T_3, T_4\}}{DL}$$
$$\subset \frac{D\{L_2\}}{DL} \subset \frac{D\{L_2, T_5\}}{DL} \subset \frac{D\{L_2, T_5, (x-1)^3\}}{DL} \subset \frac{D}{DL}.$$

1.5 Inverse problem of Gröbner basis theory

Given a term order or filtration of D, the inverse problem of Gröbner basis theory is to determine all initial ideals that occur under the term order or filtration. The inverse problem for the order filtration was solved by Strömbeck [44] who gave a set of inequalities which must be satisfied by the initial ideal, and who also proved the existence of initial ideals satisfying any such set of inequalities. As a corollary of Theorem 1.2.7, we obtain another proof of Strömbeck's inequality. At the level of linear algebra, our proof is probably ultimately the same as Strömbeck's. However, we hope that our use of the Weyl closure offers a clarification by organizing the linear algebra involved.

Let us also remark that Strömbeck's solution implies the solution of the inverse Gröbner basis problem with respect to lexicographic order $\partial > x$ or with respect to the order filtration refined by the V-filtration. The inverse Gröbner basis problem with respect to the V-filtration was solved by Briançon and Maisonobe [8]. As far as we know, the problem is open for the Bernstein filtration.

Theorem 1.5.1. (Strömbeck [44], Inverse problem for the order filtration)

1. *Let $I \subset D$ be a left ideal. Suppose that*

$$\mathrm{in}_{(0,1)}(I) = \left\langle \left(\prod_{k=1}^{t} (x-\lambda_k)^{j_{k,m}} \right) \xi^m : m \geq n \right\rangle \tag{5.5}$$

where for each k between 1 and t, $\{j_{k,m}\}_{m\geq n}$ is a decreasing sequence of non-negative integers with $\lim_{m\to\infty} j_{k,m} = \mu_k$. Then

$$j_{k,n+\mu_k} \leq n + \mu_k. \tag{5.6}$$

2. *Conversely suppose for each k between 1 and t, $\{j_{k,m}\}_{m\geq n}$ is a decreasing sequence of non-negative integers with limit μ_k such that $j_{k,n+\mu_k} \leq n+\mu_k$. Then there exists a left ideal $I \subset D$ such that $\mathrm{in}_{(0,1)}(I)$ is given by (5.5).*

Proof. PART 1: From the point of view of the closure, we think of I as sandwiched between $DL \subset I \subset \mathrm{Cl}(L)$, where L is an element of minimal order n in I (see proof of Corollary 1.2.8). We might as well choose L with $\mathrm{in}_{(0,1)}(L) = (\prod_{k=1}^{t}(x-\lambda_k)^{j_{k,n}})\xi^n$. We can now describe the initial ideal of I by Theorem 1.2.7. Let us review how this is done. By hypothesis, L has singular points $\{\lambda_1, \ldots, \lambda_t\}$. Then $I = I(V_1) + \cdots + I(V_t)$ for some subspaces $V_k \subset \ker(D/(x-\lambda_k)D \overset{\circ L}{\to} D/(x-\lambda_k)D)$. Finally, if $\{f_{k0}(\partial), \ldots, f_{ks_k}(\partial)\}$ is a basis of V_k with the property that $\deg(f_{ki}) < \deg(f_{k,i+1})$ for all i, then

$$\begin{array}{l} \mathrm{in}_{\lambda_k}(I) = \langle (x-\lambda_k)^{j_{k,n}-(i+1)} \xi^{n+\deg(f_{ki})-i} : 0 \leq i \leq s_k \rangle \\ \mathrm{in}_{(0,1)}(I) = \langle (\prod_{k=1}^{t}(x-\lambda_k)^{i_k})\xi^m : (x-\lambda_k)^{i_k}\xi^m \in \mathrm{in}_{\lambda_k}(I) \rangle. \end{array} \tag{5.7}$$

It only remains to translate this description into the inequality (5.6). For instance, $\mu_k = j_{k,n} - \dim(V_k)$. To go further, we need to understand the subspaces V_k. By Algorithm 1.1.2, V_k can be identified with a subspace of $\ker(R_{\lambda_k}(L))$, so let us examine the matrix

$R_{\lambda_k}(L)$ of (1.1). We can write $L = p_n(x)\partial^n + \cdots + p_0(x)$ where $p_n(x) = \prod_{k=1}^{t}(x-\lambda_k)^{j_{k,n}}$. We can also write L in the form

$$L = \sum_{i=r}^{s} \zeta_i q_i(\theta_{\lambda_k}) \qquad \zeta_i = \begin{cases} \partial^i & \text{if } i \leq 0 \\ (x-\lambda_k)^i & \text{if } i > 0 \end{cases}$$

such that $q_r(\theta) \neq 0$. Notice then that the term $p_n(x)\partial^n$ becomes a subsum of the above sum with the shape,

$$p_n(x)\partial^n = \sum_{i \geq j_{k,n}-n} \zeta_i g_i(\theta_{\lambda_k})$$

where $g_{j_{k,n}-n}(\theta) \neq 0$ and $\deg(g_{j_{k,n}-n}) = \min\{n, j_{k,n}\}$. Thus, $r \leq j_{k,n} - n$ and $\deg(q_r) \leq n$.

Now recall that $\ker(R_{\lambda_k}(L)) = \ker\left[R_{\lambda_k}(L)_{ij}\right]_{0\leq i\leq m, 0\leq j\leq m+r}$. This matrix is identically 0 below the rth diagonal while its rth diagonal consists of entries $\{q_r(i)\}_{i\geq \max\{0,-r\}}$. Here, the rth diagonal means the diagonal of entries in row i and column $i+r$.

Now we have 2 cases. First, if $r \leq 0$, then $\dim(\ker(R_{\lambda_k}(L))) \leq n$ since $\deg(q_r) \leq n$. We conclude that $j_{k,n} - \mu_k = \dim(V_k) \leq n$, which implies $j_{k,n+\mu_k} \leq j_{k,n} \leq n + \mu_k$.

Second, if $r > 0$, then the first r columns of $R_{\lambda_k}(L)$ are identically 0. So the subspace W spanned by $\{\partial^i\}_{i=0}^{r-1}$ is contained in $\ker(R_{\lambda_k}(L))$. Now let $W_k = V_k \cap W$ and suppose that $\dim(W_k) = r'$. Then by (5.7), $(x-\lambda_k)^{j_{k,n}-r'}\xi^{n+r-r'} \in \mathrm{in}_{\lambda_k}(I)$, which implies that $j_{k,n+r-r'} \leq j_{k,n} - r'$. We also claim that $n + \mu_k \geq n + r - r'$. This follows because $\mu_k = j_{k,n} - \dim(V_k)$, where $j_{k,n} \geq n + r$ and $\dim(V_k) \leq n + r'$. Finally, we obtain

$$j_{k,n+\mu_k} \leq j_{k,n+r-r'} \leq j_{k,n} - r' \leq j_{k,n} + n - \dim(V_k) = n + \mu_k$$

which completes the proof of Part 1.

Part 2: The strategy is to construct L with $\mathrm{in}_{(0,1)}(L) = (\prod_{k=1}^{t}(x-\lambda_k)^{j_{k,n}})\xi^n$ and such that $\ker(R_{\lambda_k}(L))$ contains the appropriate vector space V_k. To see what V_k should be, we compare (5.7) with (5.5). It follows that we simply need $\{\deg(f_{k0}), \ldots, \deg(f_{ks_k})\}$ to be equal to the set

$$S_k := \bigcup_{\{i>0: j_{k,n+i} < j_{k,n+i-1}\}} [k + j_{k,n} - j_{k,n+i-1}, i + j_{k,n} - j_{k,n+i} - 1] \subset \mathbb{N}$$

whose cardinality is $|S_k| = j_{k,n} - \mu_k$. Here, $[a,b]$ denotes the set of integers between a and b including endpoints. Also, let $t_k = \max_{i\in S_k}\{j_{k,n} - n, i\}$ and let $m = \max_k\{t_k\}$.

First, let us construct L such that $\ker(R_{\lambda_k}(L))$ contains V_1. There are two subcases to consider:

(i) If $j_{1,n} \leq n$, then define $L = \partial^{n-j_{1,n}}(\theta_{\lambda_1}+1)^{\mu_1}\prod_{i\in S_1}(\theta_{\lambda_1} - (i + n - j_{1,n}))$. Then $V_1 = \mathrm{Span}_K\{\partial^i : i \in S_1\} = \ker(R_{\lambda_1}(L))$, as desired.

(ii) If $j_{1,n} > n$, then $j_{1,n+\mu_1} \leq n + \mu_1$ implies that $j_{1,n} - j_{1,n+\mu_1} \geq j_{1,n} - n - \mu_1 = |S_1| - n$. It follows that $|S_1 \cap [0, j_{1,n} - n - 1]| \geq |S_1| - n$ and hence, $|S_1 \cap \mathbb{N}_{\geq j_{1,n}-n}| \leq n$. So we can define $L = (x-\lambda_k)^{j_{1,n}-n}(\theta_{\lambda_1}+1)^{n-|S_1\cap\mathbb{N}_{\geq j_{1,n}-n}|}\prod_{i\in S_1\cap\mathbb{N}_{\geq j_{1,n}-n}}(\theta_{\lambda_1} - (i + n - j_{1,n}))$. Then $V_1 = \mathrm{Span}_K\{\partial^i : i \in S_1\} \subset \ker(R_{\lambda_1}(L))$, as desired.

We now replace L by the operator $(x-\lambda_2)^{j_{2,n}}\cdots(x-\lambda_t)^{j_{t,n}}L$, which continues to have the property that $\ker(R_{\lambda_1}(L)) \supset \mathrm{Span}_K\{\partial^i : i \in S_1\} = V_1$ and also enjoys the property that $\mathrm{in}_{(0,1)}(L) = (\prod_{k=1}^t (x-\lambda_k)^{j_{kn}})\xi^n$. It now remains to further adjust L so that $\ker(R_{\lambda_k}(L))$ contains V_k as well. We shall give a brief sketch as to how this is done, and leave the details to the reader.

At the moment, the only thing we know about the matrix $R_{\lambda_k}(L)$ for $k \neq 1$ is that its $(j_{k,n}-n)$-th diagonal has entries coming from a polynomial of degree n and the matrix is zero below this diagonal. As usual, let ζ_i denote ∂^i if $i \leq 0$ and denote $(x-\lambda_k)^i$ if $i > 0$. Then by adding to L an operator $\prod_{i\neq k}(x-\lambda_i)^M \zeta_{j_{kn}-n} g_0(\theta_{\lambda_k})$ where $M \gg m$ and where $\deg(g_0) \leq n-1$, we can adjust the matrix $R_{\lambda_k}(L)$ so that the entries of the $(j_{k,n}-n)$-th diagonal come from an arbitrary polynomial of degree n and so that the matrix is still zero below this diagonal. In particular, we can force the entries indexed by $S_k \cap N_{\geq j_{k,n}-n}$ to be zero. Also, for M sufficiently large, the first m columns of the matrices $R_{\lambda_i}(L)$ for $i \neq k$ are unaffected.

Similarly, for $r > 0$, we can add to L operators $\prod_{i\neq k}(x-\lambda_i)^M (x-\lambda)^r \zeta_{j_{kn}-n} g_r(\theta_{\lambda_k})$ with $M \gg m$ and $\deg(g_r) \leq n-1$ so that the $(j_{k,n}-n+r)$-th diagonal of $R_{\lambda_k}(L)$ is given by an arbitrary polynomial either of degree n or degree $n-1$ and such that lesser diagonals are unchanged. So we can also force the entries indexed by $S_k \cap N_{\geq j_{k,n}-n} \setminus \{t_k\}$ of the $(j_{kn}-n+r)$-th diagonal to be zero. Again for M sufficiently large, the first m columns of the matrices $R_{\lambda_i}(L)$ for $i \neq k$ are unaffected. If we do this for all r less than some sufficiently large r', we eventually obtain L such that $\mathrm{in}_{(0,1)}(L) = \prod_{k=1}^t (x-\lambda_k)^{j_{k,n}}\xi^n$ and such that $\ker(R_{\lambda_k}(L))$ contains $\mathrm{Span}_K\{\partial^i : i \in S_k\} = V_k$. Continuing this procedure for all k, we produce the desired L. □

Corollary 1.5.2. (Inverse problem for the order filtration refined by the V-filtration) *There exists a left ideal $I \subset D$ such that $\mathrm{in}_\lambda(I) = \langle (x-\lambda)^{j_m}\xi^m : m \geq n\rangle$ where $\{j_m\}$ is a decreasing sequence of non-negative integers with limit μ if and only if $j_{n+\mu} \leq n+\mu$.*

Corollary 1.5.3. (Inverse problem for pure lexicographic order $\partial > x$) *The set of initial ideals with respect to the pure lexicographic term order $\partial > x$ is equal to the set of monomial ideals of $K[x,\xi]$ having dimension 1.*

Proof. Consider $I = \mathrm{Ann}_D(p(x))$ where $p(x) = (x-\lambda_1)^{e_1}\cdots(x-\lambda_r)^{e_r}$. By Lemma 1.3.10, the cotype of $L = ((x-\lambda_1)^{e_1}\cdots(x-\lambda_r)^{e_r})\partial - \sum_{i=1}^r e_i \prod_{j\neq i}(x-\lambda_j)$ at λ_i is $\{e_i\}$. By Theorem 1.2.7 and Corollary 1.3.3,

$$\mathrm{in}_<(I) = \langle x^r\xi, x^{r-|\{i:e_i\leq j\}|}\xi^{j+1} : j \geq 0\rangle$$

Note that $\xi^{\max_i\{e_i\}+1} \in \mathrm{in}_<(I)$. Since the $\{e_i\}$ are arbitrary, we can thus make an ideal corresponding to any staircase which intersects the ξ-axis and whose lower wall has ξ-exponent 1. By multiplying the generators of I on the right by $x^i\partial^j$, we can make an ideal corresponding to any staircase which does not intersect the x-axis. Since neither x or ∂ is preferred in the Weyl algebra, by symmetry we can also make ideals corresponding to any staircase not intersecting the ξ-axis. This exhaust all 1-dimensional ideals. □

Another set of interesting problems is to consider the inverse Gröbner basis problems for a restricted class of ideals. For example,

Corollary 1.5.4. (Inverse problem of closed ideals for the order filtration)
There exists a closed ideal I such that $\mathrm{in}_{(0,1)}(I)$ is given by (5.5) if and only if for each k, either (i.) $j_{kn} \leq n$, or (ii.) $j_{kn} > n$ and $\mu_k \geq j_{kn} - n + 1$.

Similarly in Corollary 1.3.13, we described the initial ideal of the annihilator of polynomials. It would be interesting to solve the inverse problem for this set of ideals. By Corollary 1.3.3 and Lemma 1.3.10, this problem is essentially equivalent to determining whether a vector space of polynomials can be found satisfying prescribed multiplicity conditions at various points. For instance, the ideal $\langle x^2(x-1)\xi^2, x\xi^3, \xi^4\rangle$ is the initial ideal of the annihilator of polynomials if and only if there exists a 2-dimensional vector space V of polynomials which have multiplicity 1 or 4 at $x = 0$, multiplicity 0 or 2 at $x = 1$, and multiplicity 0 or 1 everywhere else. We saw in Example 1.0.2 that the candidate $V = \mathrm{Span}_K\{x^4, x(x-1)^2\}$ does not work since it contains a polynomial $p(x) = 4x^4 - 27x(x-1)^2 = x(4x-3)(x-3)^2$ with multiplicity 2 at $x = 3$. More generally, we could ask the same questions for the set of annihilator ideals of rational functions.

1.6 Isomorphism classes of left ideals

The space of isomorphism classes of right ideals of D was described by Cannings and Holland as a special case of their work [9]. It was further studied by Le Bruyn [25]. In this section, we will reproduce the description given by Cannings and Holland from the point of view of the Weyl closure. One added benefit of this approach is that given generators of an ideal, we can determine the corresponding isomorphism class using variants of our closure algorithms. To remain consistent with the rest of this chapter, we shall consider left ideals rather than right ideals.

Theorem 1.6.1. *The space of isomorphism classes (as D-modules) of left ideals of D is*

$$\mathrm{Isom}(D) = \bigcup_{t\in\mathbb{N}} \bigcup_{\{\lambda_1,\ldots,\lambda_t\}\subset K} \{(\lambda_1, V_1), (\lambda_2, V_2), \cdots, (\lambda_t, V_t)\}$$

where $V_k \subset K[\partial]$ is a nonzero finite dimensional vector space associated to λ_k. To determine the isomorphism class of a left ideal I, there is a unique $L \in I$ of minimal order such that all other elements of minimal order in I are $K[x]$ multiples of L. Then I/DL is supported on a finite subset $\{\lambda_1, \ldots, \lambda_t\} \subset K$, and $V_k = \{f(\partial) \in K[\partial] : f(\partial)L = (x-\lambda_k)T$ for some $T \in I\}$.

Proof. Given a left ideal I and an operator $T \in I$, the D-module I/DT is annihilated by some $p(x) \in K[x]$ if and only if T is of minimal order in I. Moreover, the operator L defined in the theorem is unique since $R \cdot I$ is principal in $R = K(x)\langle\partial\rangle$. Thus I/DL is the unique smallest D-module of the form I/DT which is totally supported on a finite subset of K. This makes I/DL an isomorphism invariant. Similarly, the spaces V_k are isomorphism invariants. Furthermore by Kashiwara's equivalence, $I = D \cdot \{L, (x-\lambda_k)^{-1} f(\partial)L : f(\partial) \in V_k, k = 1, \ldots, t\}$ which shows that the spaces V_k also determine I as a D-module up to isomorphism. Finally, the existence of a representative ideal for each isomorphism class can be obtained using the methods of Theorem 1.5.1, Part 2. ∎

Algorithm 1.6.2. (Computing the isomorphism class of a left ideal I)
INPUT: $\{L_1, \ldots, L_m\}$, generators of a left ideal $I \subset D$.
OUTPUT: $\{(\lambda_k, V_k) : V_k \neq 0\}$, a point of $\mathrm{Isom}(D)$.

1. Compute a reduced Gröbner basis of I with respect to the order filtration. Set L equal to the element of lowest order in I.

2. Suppose $L = p_n(x)\partial^n + \cdots + p_0(x)$. We assume we have the factorization $p_n(x) = (x - \lambda_1)^{e_1} \cdots (x - \lambda_t)^{e_t}$. If we are not able to obtain the factorization, then we need to employ methods similar to Algorithm 1.2.4, and we leave the details to the reader.

3. For each k between 1 and t, compute $\ker(R_{\lambda_k}(L))$ using Algorithm 1.1.2. Set $V_k = \{f(\partial) \in \ker(R_{\lambda_k}(L)) : (x - \lambda_k)^{-1} f(\partial) L \in I\}$, which we can compute using Gröbner bases and undetermined coefficients.

4. Return $\{(\lambda_k, V_k) : V_k \neq 0\}$.

Chapter 2

Weyl closure, torsion, and local cohomology

In Chapter 1, we introduced the Weyl closure operation for ideals of the first Weyl algebra (see Definition 1.0.1). In this chapter, we study the analogous operation in the n-th Weyl algebra, $K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$, which will henceforth be denoted by D or D_n. Similarly, let R denote the ring of differential operators with rational function coefficients, $K(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle$. We may now extend Definition 1.0.1 and formulate the Weyl closure operation for arbitrary left submodules of D^r.

Definition 2.0.3. *Let $N \subset D^r$ be a left D-submodule. The Weyl closurEe of N, denoted* $\mathrm{Cl}(N)$*, is the submodule*

$$\mathrm{Cl}(N) = R \cdot N \cap D^r.$$

The question of computing the Weyl closure for finite rank ideals of D (see Definition 2.1.2 for a definition of rank) was posed in [14] by Chyzak and Salvy, who call this question the "extension-contraction problem" and consider it more generally for left ideals of Ore algebras. Their motivation to compute Weyl closure is for non-commutative elimination and its application to symbolic integration. Namely, given a left ideal I in $D_{n+d} = D_n\langle t_1, \ldots, t_d, \partial_{t_1}, \ldots, \partial_{t_d}\rangle$ consisting of operators which annihilate a function $f = f(x_1, \ldots, x_n, t_1, \ldots, t_d)$, then the left ideal of D_n defined by

$$J(I) = (I + \partial_{t_1} D_{n+d} + \cdots + \partial_{t_d} D_{n+d}) \cap D_n$$

consists of operators which annihilate the integral

$$g(x_1, \ldots, x_n) = \int_C f dt_1 \cdots dt_d,$$

where C is a suitable homology cycle (see [40, Theorem 5.5.1] for a proof).

The intersection ideal $J(I)$ for an ideal I which is holonomic can be computed by an algorithm due to Takayama [46] and refined in [40, Algorithm 5.5.4]. Thus, given a function f, the first step in Chyzak and Salvy's technique for symbolic integration is to obtain enough operators annihilating f so that the ideal I they generate is holonomic.

Moreover, it is also clear that if we can make the ideal I even larger, then we will obtain an ideal $J(I)$ which is possibly also larger and therefore a better description of the integral of f. Thus ideally, we would like to obtain the full annihilating ideal of f in D, that is $I = \mathrm{Ann}_D(f)$. As we shall see, in many situations it is easy to obtain generators for a slightly different ideal, $\mathrm{Ann}_R(f)$, which is the annihilating ideal of f in R. In this case, the annihilating ideal $\mathrm{Ann}_D(f)$ is the contraction $\mathrm{Ann}_R(f) \cap D$, i.e. the Weyl closure.

Example 2.0.4. We give an example taken from Stanley's book [43] and which is the subject of recent work of Pemantle [37]. Given a rational function $F(x,t)$ in two variables and a power series expansion $F(x,t) = \sum_{r,s\in\mathbb{N}} a_{rs}x^r t^s$, then the diagonal function $\xi(x) = \sum_{r\in\mathbb{N}} a_{rr}x^r$ is known to be an algebraic function. It can be computed using residues as $\int_C t^{-1}F(t,x/t)dt$, where C is an appropriate homology cycle. For instance, consider $F(x,t) = (1-x-t-xt)^{-1}$, for which a_{rs} counts the number of ways to reach (r,s) from $(0,0)$ using steps of size $(1,0)$, $(0,1)$, or $(1,1)$. Then we would like to compute,

$$\int_C \frac{dt}{t-x-t^2-xt}.$$

Let us compute differential equations describing the integral. Namely, first note that the integrand $f = (t-x-t^2-xt)^{-1}$ is a solution of the differential operators $L_1 = \partial_t(t-x-t^2-xt)$ and $L_2 = \partial_x(t-x-t^2-xt)$. The operators L_1 and L_2 generate the annihilating ideal of f in R since they form a rank 1 system. However, the ideal $I = D\cdot\{L_1, L_2\}$ is not the annihilating ideal of f in D and is not even holonomic. In fact,

$$J(I) = (I + \partial_t D) \cap K\langle x, \partial_x\rangle = 0,$$

hence the ideal I does not produce any information about the integral. On the other hand, using our implementation in Macaulay 2, we can find the annihilator over D by using the Weyl closure.

```
i2 : (W = QQ[x,t,Dx,Dt, WeylAlgebra => {x=>Dx, t=>Dt}];
      f = t-x-t^2-x*t;
      I = ideal(Dx*f, Dt*f);
      ClI = WeylClosure(I))

                                                         2
o2 = ideal (x*Dx + 2t*Dx - t*Dt - Dx - Dt, x*t*Dx + t Dx + x*Dx - ...
```

We can test whether or not the Weyl closure $\mathrm{Cl}(I)$ is equal to I.

```
i3 : ClI == I
o3 = false
```

By some experimentation, we find that $\mathrm{Cl}(I) = I + D\cdot\{(x+2t-1)\partial_x - (t+1)\partial_t\}$.

```
i4 : ClI == I + ideal((x+2*t-1)*Dx - (t+1)*Dt)
o4 = true
```

Moreover, the integration ideal $J(\mathrm{Cl}(I))$ is,

```
i5 : DintegrationIdeal(ClI, {0,1})
              2
o5 = ideal(x Dx - 6x*Dx + x + Dx - 3)

o5 : Ideal of QQ [x, Dx, WeylAlgebra => {x => Dx}]
```

Here the parameter $\{0,1\}$ of `DintegrationIdeal` indicates that t is to be integrated out as opposed to x. The integration ideal $J(\mathrm{Cl}(I))$ is a rank one ordinary differential equation whose solution space is spanned by the function $p(x) = (x^2 - 6x + 1)^{-1/2}$. Hence the integral $\int_C f(x,t)dt$ a scalar multiple of $p(x)$, and since $a_{1,1} = 3$, we conclude that $\xi(x) = (x^2 - 6x + 1)^{-1/2}$.

In this chapter, we present algorithms to compute the Weyl closure. In Section 2.1, we recall basic facts of D-module theory, in particular the notion of singular locus and the relation between finite rank and holonomic D-modules. In Section 2.2, we provide an algorithm to compute Weyl closure for finite rank modules. The theoretical basis is a lemma which states that $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$ for any polynomial f vanishing on the singular locus of D^r/N, and the algorithm then becomes a direct application of the localization algorithm due to Oaku, Takayama, and Walther [36]. In Section 2.3, we provide an algorithm to compute the Weyl closure for arbitrary modules. The details of the algorithm are given in Section 2.4, where we present an algorithm to compute torsion of an arbitrary finitely generated D-module with respect to an ideal of polynomials. The algorithm is a slight extension of Oaku's algorithm for computing torsion of a holonomic D-module [32]. In Section 2.5, we give an algorithm to find the primes ideals of the polynomial ring which are associated to a D-module M. In Section 2.6, we discuss a construction for local cohomology due to Oaku and Takayama and also independently to Adolphson and Sperber. This section is expository and serves to relate the work of both sets of researchers.

Example 2.0.5. We mention here a class of ideals where the Weyl closure can be explicitly described. Let $\theta_i = x_i\partial_i$. A torus invariant ideal $I \subset D$ is an ideal generated by elements of the form $\boldsymbol{x}^\alpha p(\boldsymbol{\theta})\boldsymbol{\partial}^\beta$, where $\alpha, \beta \in \mathbb{N}^n$ and $p(\boldsymbol{\theta}) \in K[\boldsymbol{\theta}]$. For a discussion of torus invariant ideals and their solution spaces, see [40, Section 2.3].

In the expression $\boldsymbol{x}^\alpha p(\boldsymbol{\theta})\boldsymbol{\partial}^\beta$ of I, we may assume that α and β have disjoint support. Now given any set of generators $G = \{\boldsymbol{x}^{\alpha_1} p_1(\boldsymbol{\theta})\boldsymbol{\partial}^{\beta_1}, \ldots, \boldsymbol{x}^{\alpha_m} p_m(\boldsymbol{\theta})\boldsymbol{\partial}^{\beta_m}\}$ of I, consider the ideal $J = \langle q_1(\boldsymbol{\theta}), \ldots, q_m(\boldsymbol{\theta})\rangle \subset K[\boldsymbol{\theta}]$ where $q_i(\boldsymbol{\theta}) = \boldsymbol{x}^{\beta_i} p_i(\boldsymbol{\theta})\boldsymbol{\partial}^{\beta_i}$. Then J has the property that $R \cdot I = R \cdot J$, hence it suffices to describe the Weyl closure of ideals $J \subset K[\boldsymbol{\theta}]$, which are called *Frobenius ideals*. This is accomplished by the following lemma, whose proof is straightforward.

Lemma 2.0.6. *Let $J \subset K[\theta_1, \ldots, \theta_n]$ be an ideal. Then*

$$\mathrm{Cl}(D \cdot J) = \bigoplus_{\alpha \in \mathbf{N}^n} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} (J : [\theta_1]_{\alpha_1} \cdots [\theta_n]_{\alpha_n})$$

where $[\theta_i]_{\alpha_i} = \theta_i(\theta_i - 1) \cdots (\theta_i - \alpha_i + 1)$. Moreover, the unique maximal torus invariant ideal I in D with the property that $I \cap K[\theta_1, \ldots, \theta_n] = J$ is the ideal denoted

$$\mathrm{Sat}(J) = \bigoplus_{\beta \in Z^n} u_1^{\beta_1} \cdots u_n^{\beta_n} (J : [\theta_1]_{\beta_1} \cdots [\theta_n]_{\beta_n})$$

where

$$u_i = \begin{cases} \partial_i^{\beta_i} & \text{if } \beta_i \geq 0 \\ x_i^{-\beta_i} & \text{if } \beta_i < 0 \end{cases}$$

$$[\theta_i]_{\beta_i} = \begin{cases} \theta_i(\theta_i - 1)\cdots(\theta - \beta_i + 1) & \text{if } \beta_i \geq 0 \\ (\theta_i + 1)(\theta_i + 2)\cdots(\theta + |\beta_i|) & \text{if } \beta_i < 0 \\ 1 & \text{if } \beta_i = 0 \end{cases}$$

The above lemma can be compared with a lemma of Briançon and Maisonobe on torus invariant ideals of the first Weyl algebra in [8]. Namely, the analysis there can be extended to understanding the inclusion $J \subset \mathrm{Sat}(J)$ in terms of holomorphic and micro-function solutions of J. For future reference, we also give the following lemma on dimension. Here the dimension of a commutative ideal $J \subset K[\theta]$ is the usual Krull dimension while the dimension of an ideal $I \subset D$ is defined in Section 2.1.

Lemma 2.0.7. *Let $J \subset K[\theta_1, \ldots, \theta_n]$ be an ideal, and let $I \subset D$ be a torus invariant ideal with the property that $I \cap K[\boldsymbol{\theta}] = J$. Then* $\dim(I) = \dim(J) + n$.

Proof. First, if J has dimension k, then the Frobenius ideal $D \cdot J$ has dimension $n + k$. Since $D \cdot J \subset I \subset \mathrm{Sat}(J)$, it suffices to show that $\dim(\mathrm{Sat}(J)) = n + k$. We know already that $\dim(\mathrm{Sat}(J)) \leq n + k$ and we are left to show the inequality $\dim(\mathrm{Sat}(J)) \geq n + k$. The proof now follows the approach of [40, Theorem 5.1.3]. Let X be an irreducible component of dimension k of the variety $V(J) \subset K^n$ so that $J \subset I(X)$. Then for each i from 1 to n, X is contained in at most one hyperplane $\{\theta_i - j = 0\}$ for all $j \in \mathbb{Z}$. By relabeling coordinates, we may assume that X is contained in the hyperplanes $\{\theta_1 - n_1 = 0, \cdots, \theta_r - n_r = 0\}$ with $n_i \geq 0$, the hyperplanes $\{\theta_{r+1} + m_{r+1} = 0, \cdots, \theta_s + m_s = 0\}$ with $m_j > 0$, and in none of the hyperplanes $\{\theta_i - j = 0\}$ for $i > s, j \in \mathbb{Z}$. Then

$$I(X) = \langle \theta_1 - n_1, \cdots, \theta_r - n_r, \theta_{r+1} + m_{r+1}, \cdots, \theta_s + m_s \rangle + K$$

where $K \subset K[\theta_{s+1}, \ldots, \theta_n]$ is an ideal of dimension $k - s$ which is not supported on any of the special hyperplanes. In this case,

$$\mathrm{Sat}(I(X)) = \langle \partial_1^{n_1+1}, \ldots, \partial_r^{n_r+1}, x_{r+1}^{m_{r+1}}, \ldots, x_s^{m_s} \rangle +$$
$$\langle \theta_1 - n_1, \cdots, \theta_r - n_r, \theta_{r+1} + m_{r+1}, \cdots, \theta_s + m_s \rangle + K$$

so that

$$\mathrm{in}_{(0,e)}(\mathrm{Sat}(I(X))) = \langle \xi_1^{n_1+1}, \ldots, \xi_r^{n_r+1}, x_{r+1}^{m_{r+1}}, \ldots, x_s^{m_s} \rangle +$$
$$\langle x_1\xi_1, \cdots, x_s\xi_s \rangle + \mathrm{in}_{(0,e)}(K)$$

and has dimension $n + k$. Since $\mathrm{Sat}(J) \subset \mathrm{Sat}(I(X))$, this shows $\dim(\mathrm{Sat}(J)) \geq n + k$. $\square$

2.1 Singular locus

In this section, we collect some basic facts of D-module theory which will be useful in our discussion of Weyl closure. In particular, we recall the notion of singular locus, establish the relation between finite rank and holonomic modules, and provide an analytic interpretation of the Weyl closure. The material in this section is based upon the book [40, Section 1.4] and Kashiwara's paper [27].

Let us first recall a number of standard definitions. The *order* of an operator $L = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta$ in D is by definition equal to $\max_{\{\alpha,\beta:c_{\alpha\beta}\neq 0\}}|\beta|$. The ring D is naturally filtered by order, i.e $D = \cup_{j=0}^\infty D(j)$, where $D(j)$ consists of operators of order less than or equal to j. For a finitely generated left D-module M, a filtration $\cup_{k=0}^\infty F_k(M)$ of M is said to be *good* (with respect to the order filtration) if $F_k(M)$ is finitely generated over $K[\boldsymbol{x}]$ and if there exists $k_0 \in \mathbb{N}$ such that $D(j)\cdot F_{k_0}(M) = F_{k_0+j}(M)$ for all $j\in\mathbb{N}$. Given a presentation D^r/M_0 of M with generators $\{e_i\}_{i=1}^r$, there is a corresponding *standard good filtration*,

$$F_0(M) = \mathrm{Span}_K\{e_i\}_{i=1}^r \qquad F_j(M) = D(j)\cdot F_0(M).$$

For a good filtration of M, we denote the *associated graded module* by $\mathrm{gr}M$. It is a module over the coordinate ring $\mathrm{gr}D = K[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$ of the cotangent bundle T^*K^n. The *characteristic variety* of M, denoted $\mathrm{char}(M)$, is the support of $\mathrm{gr}M$ on T^*K^n, i.e. the zero locus of $\mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M)$. The *local dimension* of M at a point $p\in T^*K^n$ is equal to the dimension of the characteristic variety at p. The local dimension is either zero or greater than or equal to n by the Fundamental Theorem of Algebraic Analysis (see e.g. [40, Theorem 1.4.6]). The set of points where the local dimension is strictly greater than n is called the *non-holonomic locus.* The *dimension* of M is equal to the dimension of the characteristic variety, or in other words the maximum of the local dimensions. The set of D-modules of dimension exactly n form a special subcategory.

Definition 2.1.1. *A D-module M is holonomic if its dimension is equal to n.*

Holonomic D-modules have finite rank, whose definition we now recall.

Definition 2.1.2. *The rank of a D-module M, denoted* $\mathrm{rank}(M)$ *is equal to* $\dim_{K(x)} R\cdot M$. *We say that M has finite rank if* $\mathrm{rank}(M)<\infty$.

As we shall see, the notions of holonomic and finite rank are closely related. In order to understand their connections, it will be useful to introduce the notion of singular locus. The canonical projection of the cotangent bundle to the base space is denoted $\pi : T^*K^n \to K^n$.

Definition 2.1.3. *Let M be a finitely generated D-module. Then the singular locus of M, denoted* $\mathrm{Sing}(M)$, *is the Zariski closure of the projection of the characteristic variety minus the zero section from the cotangent bundle to K^n, i.e.* $\mathrm{Sing}(M) = Z(\pi(\mathrm{char}(M)\setminus V(\xi_1,\dots,\xi_n)))$. *Algebraically, for any good filtration of M with respect to the order filtration, the singular locus is the zero set*

$$V((\mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M) : \langle\xi_1,\dots,\xi_n\rangle^\infty)\cap K[x_1,\dots,x_n])\subset K^n.$$

The singular locus can be computed by using Gröbner bases in the Weyl algebra D and in the polynomial ring $K[\boldsymbol{x},\boldsymbol{\xi}]$. Given an element $L=\sum_{i=1}^{r}\sum_{\alpha\in\mathbb{N}^n} p_{i\alpha}(\boldsymbol{x})\boldsymbol{\partial}^{\alpha}e_i \in D^r$, let us put

$$\mathrm{in}_{(0,e)}(L) = \sum_{\{i,\alpha\,:\,p_{i\alpha}\neq 0,\ |\alpha|\ maximal\}} p_{i\alpha}(\boldsymbol{x})\boldsymbol{\xi}^{\alpha}e_i \in K[\boldsymbol{x},\boldsymbol{\xi}]^r.$$

Algorithm 2.1.4. (Singular locus of M [40, Section 1.4])
Input: a presentation $M \simeq D^r/D\cdot\{T_1,\dots,T_a\}$ of a left D-module.
Output: $\{f_1,\dots,f_b\}\subset K[\boldsymbol{x}]$ such that $\mathrm{Sing}(M)=V(f_1,\dots,f_b)$.

1. Set $N = D\cdot\{T_1,\dots,T_a\}$ and compute generators for $\mathrm{in}_{(0,e)}(N)$.
2. Compute $\mathrm{ann}_{K[\boldsymbol{x},\boldsymbol{\xi}]}(K[\boldsymbol{x},\boldsymbol{\xi}]^r/\,\mathrm{in}_{(0,e)}(N))$, e.g. by intersecting the kernels of the component maps $\phi_i : K[\boldsymbol{x},\boldsymbol{\xi}] \to K[\boldsymbol{x},\boldsymbol{\xi}]^r/\,\mathrm{in}_{<}(N)$ such that $\phi_i(1)=e_i$ for $i=1,\dots,r$.
3. Using commutative Gröbner bases methods for saturation and intersection, compute generators $\{f_1,\dots,f_b\}$ of $\left(\mathrm{ann}_{K[\boldsymbol{x},\boldsymbol{\xi}]}(K[\boldsymbol{x},\boldsymbol{\xi}]^r/\,\mathrm{in}_{<}(N)) : \langle\xi_1,\dots,\xi_n\rangle^{\infty}\right)\cap K[\boldsymbol{x}]$.
4. Return $\{f_1,\dots,f_b\}$.

The singular locus contains useful information when M has finite rank. For instance, we have the following lemmas.

Lemma 2.1.5. $\mathrm{Sing}(M)$ *is a proper subvariety of* K^n *if and only if* M *has finite rank.*

Proof. Suppose M has finite rank. Any filtration of M compatible with the order filtration extends to a filtration of $R\cdot M$. Hence by standard arguments, $\dim_{K(\boldsymbol{x})}(K(\boldsymbol{x})[\boldsymbol{\xi}]\cdot \mathrm{gr}M) = \dim_{K(\boldsymbol{x})}(R\cdot M)<\infty$, which implies that for each i, there exists some $f_i(\boldsymbol{x})\xi_i^j \in \mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M)$ for $j\gg 0$. It follows that $(\mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M) : \langle\xi_1,\dots,\xi_n\rangle^{\infty})$ contains the polynomial $\prod_i f_i(\boldsymbol{x})$ and is nonzero. Conversely, assume that $\mathrm{Sing}(M)$ is proper. Then some $f(\boldsymbol{x})\neq 0$ vanishes on $\mathrm{Sing}(M)$ or in other words there exists $m,N\gg 0$ such that $f(\boldsymbol{x})^m\boldsymbol{\xi}^{\alpha}\in \mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M)$ for all $|\alpha|\geq N$. Moreover, since the filtration F is good, $R\cdot M$ is generated as an R-module by the finite-dimensional $K(\boldsymbol{x})$-vector space $F_{k_0}(R\cdot M)$ for some $k_0\in\mathbb{N}$. It follows that $R\cdot M$ is spanned by $\{\boldsymbol{\xi}^{\alpha}F_{k_0}(R\cdot M) : |\alpha|<N\}$ and is finite-dimensional. ☐

Lemma 2.1.6. *If* M *has finite rank, then the non-holonomic locus of* M *is contained inside* $\pi^{-1}(\mathrm{Sing}(M))$, *the subvariety of the cotangent bundle defined by the inverse image of the singular locus.*

Proof. : Suppose the polynomial f vanishes on $\mathrm{Sing}(M)$. Then there exists $m,N\gg 0$ such that $f^m\xi_i^N\in \mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M)$ for all i. This implies that $\mathrm{char}(M)\subset V(f)\cup V(\xi_1,\dots,\xi_n)$, or in other words, the non-holonomic locus is contained inside $f=0$ on the cotangent bundle. ☐

These lemmas were used by Kashiwara to establish the relation between finite rank and holonomic D-modules [27]. Let us phrase the relation in terms of the Weyl closure, a formulation first made by Takayama in [47].

Proposition 2.1.7. *A left D-module $M = D^r/N$ is of finite rank if and only if $D^r/\mathrm{Cl}(N)$ is holonomic.*

Proof. If M has finite rank, then by the lemmas above the singular locus is proper and its inverse image in the cotangent bundle contains the non-holonomic locus. Let f be any polynomial vanishing on the singular locus. Since $f = 0$ contains the non-holonomic locus, it follows from Kashiwara [27] that $M[f^{-1}]$ is holonomic. The module $D^r/\mathrm{Cl}(N)$ is a subquotient of $M[f^{-1}]$ and hence is holonomic as well.

Conversely, suppose that $D^r/\mathrm{Cl}(N)$ is holonomic. Since $R \cdot M = R^r/R \cdot \mathrm{Cl}(N)$, it suffices to show that $D^r/\mathrm{Cl}(N)$ has finite rank. The following argument from [40, Proposition 1.4.9] shows that any holonomic module M' has finite rank. If M' is holonomic, then by definition $\mathrm{char}(M') = V(\mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M'))$ has dimension n. The projection of $\mathrm{char}(M)$ onto the $n+1$-dimensional subspace with coordinates $\{x_1, \ldots, x_n, \xi_i\}$ therefore has dimension $\leq n$ so that its defining ideal is nonempty. Since this projection equals $V(\mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M') \cap K[\boldsymbol{x}, \xi_i])$, this implies that there exists some nonzero $f(\boldsymbol{x})\xi_i^{N_i} \in \mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}M)$. By definition of good filtration, there exists some $k_0 \in \mathbb{N}$ such that $M = D \cdot F_{k_0}(M)$, where $F_{k_0}(M)$ is finitely generated over $K[\boldsymbol{x}]$. It follows that $R \cdot M$ is finite-dimensional over $K(\boldsymbol{x})$ because it is spanned over $K(\boldsymbol{x})$ by the elements in the finite dimensional $K(\boldsymbol{x})$-vector spaces $\{\boldsymbol{\xi}^\alpha F_{k_0}(M) : \alpha_i \leq N_i\}$. □

From the analytic perspective, the singular locus of a finite rank module generalizes the notion of singular points of a linear ordinary differential equation. In particular, let us state a special case of the famous theorem of Cauchy-Kovalevskii-Kashiwara.

Theorem 2.1.8. (Cauchy-Kovalevskii-Kashiwara, see e.g. [40, Theorem 1.4.19]) *Let $M = D^r/N$ be a module of finite rank and let U be a simply connected domain in $\mathbb{C}^n \setminus Sing(M)$. Consider the system of vector-valued linear partial differential equations,*

$$L \bullet \vec{u} = 0, \qquad L \in N, \qquad \textit{for vectors } \vec{u} \textit{ of holomorphic functions on } U$$

Then the dimension of the complex vector space of holomorphic solutions on U, denoted $\mathrm{Sol}_U(N)$*, is equal to* $\mathrm{rank}(I)$.

Using Theorem 2.1.8, we arrive at the following analytic interpretation of the Weyl closure.

Proposition 2.1.9. *Let $M = D^r/N$ and U be as in Theorem 2.1.8. Then*

$$\mathrm{Cl}(N) = \mathrm{Ann}_D(\mathrm{Sol}_U(N))$$

where $\mathrm{Ann}_D(\mathrm{Sol}_U(N))$ *denotes the set of all differential operators in D^r which annihilate the functions in* $\mathrm{Sol}_U(N)$.

Proof. Since $\mathrm{Sol}_U(N) = \mathrm{Sol}_U(\mathrm{Cl}(N))$, it is clear that $N \subset \mathrm{Cl}(N) \subset \mathrm{Ann}_D(\mathrm{Sol}_U(N))$. By Theorem 2.1.8, the ideal $\mathrm{Ann}_D(\mathrm{Sol}_U(N))$ must have the same rank as N. Therefore, the surjection of finite dimensional $K(\boldsymbol{x})$-vector spaces,

$$\frac{R^r}{R \cdot N} \to \frac{R^r}{R \cdot \mathrm{Ann}_D(\mathrm{Sol}_U(N))}$$

is an isomorphism. It follows that $R \cdot N = R \cdot \mathrm{Ann}_D(\mathrm{Sol}_U(N))$ and hence $\mathrm{Ann}_D(\mathrm{Sol}_U(N)) \subset R \cdot N \cap D = \mathrm{Cl}(N)$. $\square$

Using Proposition 2.1.9, the Weyl closure operation of an ideal I in D can be loosely regarded as analogous to the radical operation of an ideal J in $K[\boldsymbol{x}]$. That is, the Weyl closure of I is the ideal of operators which annihilate the common solutions of I whereas the radical of J is the ideal of functions which vanish on the common zeroes of J.

2.2 Finite rank algorithm

In this section, we provide an algorithm to compute the Weyl closure of a submodule $N \subset D^r$ such that the quotient module $M = D^r/N$ has finite rank. This solves the extension-contraction problem posed by Chyzak and Salvy in [14] for the case of the Weyl algebra. To obtain the algorithm, we first identify polynomials f such that for a given submodule $N \subset D^r$, we have $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$. The algorithm is then a direct application of the localization algorithm of Oaku-Takayama-Walther [36]. In the next section, we will provide an algorithm to compute the Weyl closure of arbitrary submodules.

Theorem 2.2.1. *Let $N \subset D^r$ be a left D-submodule such that D^r/N is finite rank. Then* $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$ *for any polynomial f vanishing on the singular locus* $\mathrm{Sing}(D^r/N)$.

Proof. By definition of singular locus, we have $f \in \sqrt{(\mathrm{ann}_{\mathrm{gr}D}(\mathrm{gr}(D^r/N)) : \langle \xi_1, \ldots, \xi_n \rangle^\infty)}$, or in other words, $\mathrm{gr}(D^r/N)$ is annihilated by elements $f^c \xi_1^{d_1}, \ldots, f^c \xi_n^{d_n}$ for some $c, d_1, \ldots, d_n \in \mathbb{N}$. Thus the finitely generated $K[\boldsymbol{x}, \xi][f^{-1}]$-module $\mathrm{gr}((D^r/N)[f^{-1}])$ is annihilated by $\xi_1^{d_1}, \ldots, \xi_n^{d_n}$, and in particular is finitely generated over $K[\boldsymbol{x}][f^{-1}]$ as well. It follows that the $D[f^{-1}]$-module $(D^r/N)[f^{-1}]$ is finitely generated over $K[\boldsymbol{x}][f^{-1}]$. It is a basic fact of D-module theory that a $\mathcal{D}_X$-module which is also coherent as an $\mathcal{O}_X$-module is locally free over $\mathcal{O}_X$ (see e.g. [22, Lemma 5, Lemma 6]). Here, we take X to be the nonsingular variety $\mathbb{A}^n \setminus V(f)$, whose ring of differential operators is precisely $D[f^{-1}]$. In particular, $(D^r/N)[f^{-1}]$ is thus torsion-free with respect to $K[\boldsymbol{x}][f^{-1}]$. From this fact, it follows that the D-submodule,

$$\frac{D^r}{D[f^{-1}] \cdot N \cap D^r} \subset \frac{D[f^{-1}]^r}{D[f^{-1}] \cdot N} = \frac{D^r}{N}[f^{-1}]$$

is torsion-free with respect to $K[\boldsymbol{x}]$. In other words, let $L \in \mathrm{Cl}(N)$, which means that $gL \in N \subset D[f^{-1}] \cdot N \cap D^r$ for some $g \in K[\boldsymbol{x}]$. Then the image of L in $D^r/D[f^{-1}] \cdot N \cap D^r$ has torsion with respect to g, and hence it must already be the case that $L \in D[f^{-1}] \cdot N \cap D^r$. We conclude that $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$. $\square$

Remark 2.2.2. Theorem 2.2.1 suggests the possibility of the statement "If f vanishes on $\mathrm{Sing}(M)$, then M has torsion with respect to f". However, this statement is easily dismissed for holonomic and finite rank modules by considering the family of ideals $I = D_1 \cdot \{x\partial - \alpha\}$, whose quotients modules $D_1/D_1 \cdot \{x\partial - \alpha\}$ have singular locus $x = 0$. As we can verify by the methods of Chapter 1, if $\alpha \in \mathbb{N}$ then $\mathrm{Cl}(I) = D_1 \cdot \{\partial^{\alpha+1}, x\partial - \alpha\}$ and D/I is not torsion-free while if $\alpha \notin \mathbb{N}$ then $\mathrm{Cl}(I) = I$ and D/I is torsion-free. Finding a finer description of the Weyl closure which resolves families such as $D_1/D_1 \cdot \{x\partial - \alpha\}$ seems a difficult problem.

Investigating b-functions with respect to the singular locus (see Definition 3.2.1) might be a good place to start. Similarly, we can also dismiss the converse statement "If M has torsion with respect to f, then f vanishes on $\mathrm{Sing}(M)$". This statement is a little more subtle, as it is even true for holonomic D_1-modules. To see that it is false in general, consider the holonomic module $M = D_2/D_2 \cdot \{x_1\partial_1, x_2\partial_1, x_2\partial_2 + 1\}$. Then $\mathrm{Sing}(M) = \{x_2 = 0\}$, while the element $\partial_1 \in D_2/I$ has (x_1, x_2)-torsion.

For the reader's convenience, we now summarize the localization algorithm of Oaku, Takayama, and Walther [36], which allows us to compute $M[f^{-1}]$ for a D-module M that is holonomic away from the zero locus of f. Let $D_v = D\langle v, \partial_v\rangle$ and let us define a ring map

$$\phi : D \to D_v \qquad x_i \mapsto x_i \ \ , \ \ \partial_j \mapsto \partial_j - v^2(\partial f/\partial x_i)\partial_v.$$

We also denote by ϕ the componentwise extension of ϕ to a map $D^r \to D_v^r$.

Algorithm 2.2.3. (Localization of D-modules [36])
INPUT: $f \in K[\boldsymbol{x}]$ and generators $\{T_1, \ldots, T_a\}$ of $N \subset D^r$ such that D^r/N is holonomic on $U = K^n \setminus V(f)$.
OUTPUT: $k_1, \ldots, k_m \in \mathbb{N}$ and $\{S_1, \ldots, S_b\} \subset D^r$, where $D^r/D\cdot\{S_1, \ldots, S_b\} \simeq (D^r/N)[f^{-1}]$ and where the localization map $\varphi : D^r/N \to D^r/D\cdot\{S_1, \ldots, S_b\}$ is defined by $\varphi(e_i) = f^{k_i}e_i'$.

1. Compute $\{\phi(T_1), \ldots, \phi(T_a)\}$.

2. For each $i = 1, \ldots, r$, compute the b-function $b_i(s)$ for integration of $D_v(e_i+\phi(N))/D_v\cdot \{\phi(N), (1-vf)e_i\}$ along ∂_v. That is, find the monic generator $b_i(s)e_i$ to the intersection $K[v\partial_v]e_i \cap \mathrm{in}_w(D_v \cdot \{\phi(N), (1-vf)e_i\})$, where w is the weight vector assigning weight 1 to v, -1 to ∂_v, and 0 to all other variables. Replace $v\partial_v$ by $-s-1$.

3. For each $i = 1, \ldots, r$, let k_i be the largest non-negative root of $b_i(s)$. If there is no such root put $u_i = 0$. Otherwise, put $u_i = v^k e_i$.

4. Compute the kernel of the map $D^r \to (D_v^r/\partial_v \cdot D^r + D_v\cdot\{(1-fv)e_j, \phi(N)\}_{j=1}^r)$ defined by sending $e_i \mapsto \overline{u}_i$. Let these generators be $\{S_1, \ldots, S_b\}$.

5. Return $\{k_1, \ldots, k_r\} \subset \mathbb{N}$ and $\{S_1, \ldots, S_b\} \subset D^r$.

We can now formulate the Weyl closure algorithm for finite rank modules, and give some examples using the implementation in Macaulay 2.

Algorithm 2.2.4. (Weyl closure of finite rank modules)
INPUT: generators $\{T_1, \ldots, T_a\}$ of $N \subset D^r$ such that D^r/N is finite rank.
OUTPUT: G, a generating set for $\mathrm{Cl}(N)$.

1. Compute a polynomial $f \in K[\mathbf{x}]$ vanishing on the singular locus of D^r/N.

2. Compute the localization map $\varphi : (D^r/N) \to (D^r/N)[f^{-1}]$ using Algorithm 2.2.3.

3. Compute generators G for the kernel of $D^r \xrightarrow{\pi} (D^r/N) \xrightarrow{\varphi} (D^r/N)[f^{-1}]$.

4. Return G.

Proof. By Lemma 2.1.6, the non-holonomic locus of M is contained inside the singular locus of M, hence we may apply Algorithm 2.2.3 to obtain the localization of M at f. By Theorem 2.2.1, the Weyl closure of N is the kernel of $\varphi \circ \pi$. □

Example 2.2.5. The annihilator of $e^{1/(x^3-y^2z^2)}$ in R is the rank 1 ideal generated by

$$\mathcal{G} = \{(x^3 - y^2z^2)^2\partial_x + 3x^2, (x^3 - y^2z^2)^2\partial_y - 2yz^2, (x^3 - y^2z^2)^2\partial_x - 2y^2z\}.$$

Then $\mathrm{Ann}_D(e^{1/(x^3-y^2z^2)})$ equals the Weyl closure $R \cdot \mathcal{G} \cap D$, and using Macaulay 2, we find that it is generated by the elements $\mathcal{G} \cup \{y\partial_y - z\partial_z,\ y^2z^3\partial_z - \frac{2}{3}x^4\partial_x - 2x^3z\partial_z - 2\}$.

```
i1 : (W = QQ[x,y,z,Dx,Dy,Dz, WeylAlgebra => {x=>Dx, y=>Dy, z=>Dz}];
      f = (x^3-y^2*z^2);
      I = ideal(f^2*Dx+3*x^2, f^2*Dy-2*y*z^2, f^2*Dz-2*y^2*z);
      ClI = WeylClosure(I))
                                  2     3  2     2        3  2     ...
o1 = ideal (y*Dy - z*Dz, y*z Dx + -*x Dy, y z*Dx + -*x Dz, ...
                                  2                 2              ...
i2 : ClI == I
o2 = false

i3 : ClI == I + ideal(y*Dy-z*Dz, y^2*z^3*Dz-(2/3)*x^4*Dx-2*x^3*z*Dz-2)
o3 = true
```

Example 2.2.6. In this example, we see that a Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric ideal can be Weyl closed for some parameter vectors and not Weyl closed for others. Under the notation of [40, Section 3.1], we consider the GKZ ideal $I = H_A(\beta)$ associated to the matrix A and parameter vector β given by,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \beta = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In Macaulay 2, we type,

```
i1 : (A = matrix{{1,1,1},{0,1,2}};
      I = gkz(A, {0,0}))
               2
o1 = ideal (D   - D D , x D  + x D  + x D , x D  + 2x D )
             2     1 3   1 1    2 2    3 3   2 2     3 3
```

The singular locus of I is known to be $x_1x_3(x_2^2 - 4x_1x_2)$.

```
i2 : singLocus I
                2       2 2
o2 = ideal(x x x  - 4x x )
            1 2 3     1 3
```

To compute the Weyl closure, it turns out that the localization map with respect to the singular locus is computationally too intensive. It is more efficient to compute the localization maps iteratively with respect to factors of the singular locus and then compose them together. This is theoretically possible as long as the module is specializable along the factors. Using Macaulay 2, we find that I is already Weyl closed:

```
i3 : (W = ring I;
      F0 = map(W^1/I, W^1, matrix{{1_W}});
      F1 = DlocalizeMap(I, x_2^2-4*x_1*x_3);
      F2 = DlocalizeMap(target F1, x_1);
      F3 = DlocalizeMap(target F2, x_3);
      ClI = ideal kernel (F3*F2*F1*F0))
                 2      2 2  1    2 2   1        2   3         ...
o3 = ideal (- D D D  + D D , -*x D D  + -*x D D D  + -*D D D , ...
               1 2 3    1 3  8  2 1 3   4  3 1 2 3   8  1 2 3  ...
i4 : ClI == I
o4 = true
```

Now consider the GKZ ideal I' corresonding to the different parameter vector $\beta = [-1, -2]^T$. Using Macaulay 2, we now find,

```
i5 : (I' = gkz(A, {-1,-2});
      W = ring I';
      F0 = map(W^1/I', W^1, matrix{{1_W}});
      F1 = DlocalizeMap(I', x_2^2-4*x_1*x_3);
      F2 = DlocalizeMap(target F1, x_1);
      F3 = DlocalizeMap(target F2, x_3);
      ClI' = ideal kernel (F3*F2*F1*F0))
               1    2      1  2 2   1     2 2    1        2   ...
o5 = ideal (- -*D D D  + -*D D , --*x D D  + --*x D D D  + ...
              3  1 2 3   3  1 3  24  2 1 3   12  3 1 2 3   ...
i6 : ClI' == I'
o6 = false
```

After some experimentation, we find that the Weyl closure has the extra generator given below,

```
i7 : ClI' == I' + ideal(x_2^2*D_1^2-4*x_3^2*D_1*D_3-6*x_3*D_1)
o7 = true
```

Moreover this extra generator L has the property that $x_1 L \in I'$ and we can determine its representation by the original generators.

```
i8 : (L = (x_2^2*D_1^2-4*x_3^2*D_1*D_3-6*x_3*D_1);
     (x_1*L) % I')
o8 = 0
```

```
i9 : (x_1*L) // (gens I')
o9 = {2} | -1x_2^2x_3                           |
     {2} | x_2^2D_1-4x_3^2D_3-6x_3              |
     {2} | -1x_2^2D_1+x_2x_3D_2+2x_3^2D_3+3x_3  |
```

In other words, we find that we can write,

$$L = \frac{1}{x_1}\begin{pmatrix} -x_2^2x_3(\partial_2^2 - \partial_1\partial_3) \\ +(x_2^2\partial_1 - 4x_3^2\partial_3 - 6x_3)(\theta_1 + \theta_2 + \theta_3 + 1) \\ +(-x_2^2\partial_1 + x_2x_3\partial_2 + 2x_3^2\partial_3 + 3x_3)(\theta_2 + 2\theta_3 + 2) \end{pmatrix}$$

The question of whether GKZ hypergeometric systems are generically Weyl closed is currently being investigated by Laura Matusevich.

2.3 General algorithm

In this section we give an algorithm to compute the Weyl closure of an arbitrary submodule $N \subset D^r$. To obtain the algorithm, we again first identify polynomials f such that $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$. Then we develop an algorithm to compute the torsion submodule $H_f^0(D^r/N) = \{L \in D^r/N : f^iL = 0 \text{ for some } i > 0\}$. From these ingredients, the Weyl closure algorithm can be summarized as follows.

Algorithm 2.3.1. (Weyl closure of $N \subset D^r$)
INPUT: $\{T_1, \ldots, T_a\}$, generators of $N \subset D^r$.
OUTPUT: $\{U_1, \ldots, U_c\} \subset D^r$, generators of $\mathrm{Cl}(N)$.

1. Compute a polynomial f such that $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$ using Algorithm 2.3.3.
2. Compute $\{R_1, \ldots, R_b\} \subset D^r$ whose images in D^r/N generate the torsion module $H_f^0(D^r/N)$ using Algorithm 2.4.3.
3. Return $\{T_1, \ldots, T_a, R_1, \ldots, R_b\}$.

The first step is thus to identify polynomials f such that $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$. An equivalent way to formulate this statement is to say that f is contained in every nonzero prime ideal of $K[\boldsymbol{x}]$ which is associated to $M = D^r/N$. When M has finite rank we were able to show in Section 2.2 that this is the case for f vanishing on the singular locus. Since the singular locus is the Zariski closure of the projection of the characteristic variety minus the zero section, this suggests a connection between associated primes and characteristic varieties. For finitely generated D-modules, we will give such a connection in Proposition 2.5.4 of Section 2.5. Using it, we can make the following analog to the singular locus. For $i = n, \cdots, 2n$, let $\mathrm{char}(M_i)$ be the characteristic variety of the maximal submodule M_i of M with dimension i. Now let V be the union of the projections of those irreducible components of each $\mathrm{char}(M_i)$ which do not project onto the entire K^n. Then any polynomial f vanishing on V will be contained in every nonzero prime ideal of $K[\boldsymbol{x}]$ which is associated to M and hence will have the property that $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$. Note that when M has finite rank, then by Lemma 2.1.6 the variety V is exactly the singular locus. We remark however that the variety V is quite intensive to compute as it requires primary decomposition as well as an algorithm to find the canonical filtration of M by dimension.

Another approach to identifying polynomials f satisfying $\mathrm{Cl}(N) = D[f^{-1}] \cdot N \cap D^r$ which is considerably less intensive computationally is to use the following basic fact of D-module theory (see e.g. [6, Chapter VII, Lemma 9.3]).

Lemma 2.3.2. *Let M be a finitely generated D-module. Then there exists $f \in K[\mathrm{x}]$ such that $M[f^{-1}]$ is a free $K[\mathrm{x}][f^{-1}]$-module.*

Moreover, the polynomial f guaranteed by Lemma 2.3.2 can be computed using Gröbner bases.

Algorithm 2.3.3. (Computing f so that $M[f^{-1}]$ is a free $K[\mathrm{x}][f^{-1}]$-module)
INPUT: a presentation $M \simeq D^r/D \cdot \{T_1, \ldots, T_a\}$ of a left D-module.
OUTPUT: $f \in K[\boldsymbol{x}]$ such that $M[f^{-1}]$ is free over $K[\boldsymbol{x}][f^{-1}]$.

1. Put $N = D \cdot \{T_1, \ldots, T_a\}$. Choose any term order $<$ on R^r and compute a Gröbner basis $\mathcal{G}$ of $R \cdot N$ with respect to $<$ which also has the property that $\mathcal{G} \subset N$.

2. For each element $g_j \in \mathcal{G}$, suppose that the lead term of g_j is equal to $f_j(\boldsymbol{x})\boldsymbol{\partial}^{\alpha_j} e_{i_j}$.

3. Return $f = \prod_j f_j(\boldsymbol{x})$.

Proof. Let $S = \{(\beta, i) \in \mathbb{N}^n \times \mathbb{N} : \boldsymbol{\partial}^\beta e_i \notin \mathrm{in}(R \cdot N)\}$, and let us denote $D[f^{-1}] \cdot N$ by $N[f^{-1}]$. We claim that $D[f^{-1}]/N[f^{-1}] = \oplus_{(\beta,i)\in S} K[\boldsymbol{x}][f^{-1}]\boldsymbol{\partial}^\beta e_i$, i.e. $(D^r/N)[f^{-1}]$ is free as a $K[\boldsymbol{x}][f^{-1}]$-module. To see this, let $\mathcal{G}'$ be the result of normalizing each element in $\mathcal{G}$ to have lead coefficient 1. Note then that the elements of $\mathcal{G}'$ are also elements of $N[f^{-1}]$. Therefore, for any $(\alpha, j) \notin S$, then $\boldsymbol{\partial}^\alpha e_j \in \mathrm{in}(R \cdot N)$ and $(\boldsymbol{\partial}^\alpha e_j \bmod (N[f^{-1}])$ is in $\oplus_{(\beta,i)\in S} K[\mathrm{x}][f^{-1}]\boldsymbol{\partial}^\beta e_i$ by "reduction". Therefore $\boldsymbol{\partial}^\beta e_i$ for $(\beta, i) \in S$ span. Now suppose there were some relation $\sum_k f_k(\boldsymbol{x})\boldsymbol{\partial}^{\alpha_k} e_{i_k} \in N[f^{-1}]$ with $f_i \in K[\boldsymbol{x}][f^{-1}]$ and $(\alpha_k, i_k) \in S$. Then by multiplying by a high enough power of f we get a relation $\sum_k g_k(\boldsymbol{x})\boldsymbol{\partial}^{\alpha_k} e_{i_k} \in N$ with $g_k \in K[\boldsymbol{x}]$ and $(\alpha_k, i_k) \in S$. This is a contradiction to the definition of S. It follows that $\boldsymbol{\partial}^\beta e_i$ for $(\beta, i) \in S$ are also independent as required. □

Remark 2.3.4. For the case of finite rank modules, we note that the singular locus provides a better candidate for Weyl closure than the output of Algorithm 2.3.3. For example, let $I = D_1 \cdot \{x\partial - 1, \partial^2\}$. Then $\mathrm{Sing}(D_1/I) = \emptyset$, hence D_1/I is locally free over $K[x]$ (and hence free) according to Theorem 2.2.1. On the other hand, the generators $\{x\partial - 1, \partial^2\}$ form a Gröbner basis with respect to any term order, hence Algorithm 2.3.3 always produces $f = x$, and we are only allowed to conclude from this that $(D_1/I)[x^{-1}]$ is free over $K[x, x^{-1}]$. However, in this case the module D_1/I is already the free module $D_1 \bullet x = K[x]$, only Algorithm 2.3.3 is not able to detect this.

2.4 Computing torsion

In this section, we give an algorithm to compute the torsion of a finitely generated D-module M with respect to a polynomial f, or more generally with respect to an ideal $I \subset K[\boldsymbol{x}]$. This completes the ingredients of Algorithm 2.3.1 for computing the general Weyl closure.

Our algorithm is a slight extension of Oaku's algorithm to compute torsion of a holonomic D-module [32]. Let us start by recalling the theoretical basis for this algorithm.

Theorem 2.4.1. [32] *Let M be a left D-module and $f \in K[\boldsymbol{x}]$. Then*

$$H^0_f(M) \simeq \ker \left(\frac{D_{n+1}}{J} \otimes_{K[\boldsymbol{x}]} M \xrightarrow{t\cdot} \frac{D_{n+1}}{J} \otimes_{K[\boldsymbol{x}]} M \right) \tag{4.1}$$

where $D_{n+1} := D\langle t, \partial_t\rangle$, J is the ideal $D_{n+1} \cdot \{t - f, \partial_{x_i} + \frac{\partial f}{\partial x_i}\partial_t\}_{i=1}^n$, and $(D_{n+1}/J) \otimes M$ is given the structure of a left D_{n+1}-module by

$$\begin{array}{ll} x_i(L \otimes m) = x_i L \otimes m & t(L \otimes m) = tL \otimes m \\ \partial_{x_i}(L \otimes m) = \partial_{x_i} L \otimes m + L \otimes \partial_{x_i} m & \partial_t(L \otimes m) = \partial_t L \otimes m. \end{array}$$

Proof. We first observe that D_{n+1}/J is free as a $K[\boldsymbol{x}]$-module with the decomposition $D_{n+1}/J = \oplus_{i=0}^{\infty} K[\boldsymbol{x}]\partial_t^i$; this can be seen from the isomorphism of left D_{n+1}-modules ψ : $(D_{n+1}/D_{n+1} \cdot \{t, \partial_{x_1}, \dots, \partial_{x_n}\}) \to (D_{n+1}/J)$ given by sending $x_i \mapsto x_i$, $t \mapsto t - f$, $\partial_{x_i} \mapsto \partial_{x_i} + (\partial f/\partial x_i)\partial_t$, and $\partial_t \mapsto \partial_t$. It follows that $(D_{n+1}/J) \otimes_{K[\boldsymbol{x}]} M = \oplus_{i=0}^{\infty}(\partial_t^i \otimes_K M)$. We now claim that the desired isomorphism (4.1) is given by the map

$$\ker \xrightarrow{\varphi} H^0_f(M) \qquad \sum_{i=0}^{\infty} \partial_t^i \otimes m_i \mapsto m_0.$$

To see this, suppose $m = \sum_{i=0}^{\infty} \partial_t^i \otimes m_i \in (D_{n+1}/J \otimes M)$, where $m_i = 0$ for all but finitely many i. Then

$$\begin{array}{rcl} tm & = & \sum_{i\geq 0} t\partial_t^i \otimes m_i \\ & = & \sum_{i\geq 0}(\partial_t^i t - i\partial_t^{i-1}) \otimes m_i \\ & = & \sum_{i\geq 0}(f\partial_t^i - i\partial_t^{i-1}) \otimes m_i \\ & = & \sum_{i\geq 0} \partial_t^i \otimes (fm_i - (i+1)m_{i+1}). \end{array}$$

Therefore, $tm = 0$ if and only if $(i+1)m_{i+1} = fm_i$ for all i, or equivalently $m_i = (1/i!)f^i m_0$ for all i. Since also $m_i = 0$ for all but finitely many i, this last condition also means $f^i m_0 = 0$ for $i \gg 0$, thus establishing that (4.1) is an isomorphism. □

More generally, the complex in (4.1) computes the local cohomology of M with respect to the principal ideal generated by f. This construction can also be extended to a complex which computes the local cohomology modules of M with respect to any ideal of the polynomial ring. We will discuss this construction in Section 2.6.

A presentation of the module $(D_{n+1}/J) \otimes_{K[\boldsymbol{x}]} M$ which appears in Theorem 2.4.1 was given by Walther in [51]. Put $\vartheta_i = \partial_{x_i} + (\partial f/\partial x_i)\partial_t$ and for an element $P = \sum_j p_j(x_1, \dots, x_n, t, \partial_1, \dots, \partial_n, \partial_t)e_j \in D^r_{n+1}$, define

$$\psi(P) := \sum_j p_j(x_1, \dots, x_n, t - f, \vartheta_1, \dots, \vartheta_n, \partial_t)e_j \in D^r_{n+1}.$$

Lemma 2.4.2. [51] *Given a presentation $M \simeq D^r/N$, then we have the presentation $(D_{n+1}/J) \otimes_{K[\boldsymbol{x}]} (D^r/N) \simeq D_{n+1}/K(N)$ as left D_{n+1}-modules, where*

$$K(N) := D_{n+1} \cdot \{(t-f)e_j\}_{j=1}^r + D_{n+1} \cdot \psi(N).$$

Proof. Consider the map $\phi : D_{n+1}^r \to (D_{n+1}/J) \otimes_{K[\boldsymbol{x}]} (D^r/N)$ of left D_{n+1}-modules defined by $\phi(e_j) = 1 \otimes e_j$. We claim that ϕ is surjective with kernel equal to $K(N)$. To show surjectivity, note that $(D_{n+1}/J) \otimes_{K[\boldsymbol{x}]} (D^r/N)$ is spanned as a K-vector space by the images of "monomials" $\partial_t^i \otimes x^\alpha \partial_x^\beta e_j$. A computation shows $\phi(\partial_t^i x^\alpha \vartheta^\beta) = \partial_t^i x^\alpha \vartheta^\beta \bullet (1 \otimes e_j) = \partial_t^i \otimes x^\alpha \partial_x^\beta e_j$ and therefore ϕ is surjective.

To determine the kernel of ϕ, a computation shows that $K(N) \subset \ker(\phi)$. To obtain the opposite inclusion, let m be an arbitrary element of $\ker(\phi)$. Note that the subalgebras $K\langle \boldsymbol{x}, \boldsymbol{\partial_x}, \partial_t\rangle$ and $K\langle \boldsymbol{x}, \boldsymbol{\vartheta}, \partial_t\rangle$ are equal. Thus modulo the relations $\{(t-f)e_j\}_{j=1}^r$, we can write $m = \sum c_{ij\alpha\beta}\partial_t^i x^\alpha \vartheta^\beta e_j$. Since $m \in \ker(\phi)$, this implies $\phi(m) = \sum (c_{ij\alpha\beta}\partial_t^i x^\alpha \vartheta^\beta) \bullet (1 \otimes e_j) = \sum \partial_t^i \otimes c_{ij\alpha\beta} x^\alpha {\partial_x}^\beta e_j = 0$ in $(D_{n+1}/J) \otimes_{K[\boldsymbol{x}]} (D^r/N)$. Recall that D_{n+1}/J is free as a $K[\boldsymbol{x}]$-module with basis $\{\partial_t^i\}$, and therefore we must have $\sum_{\alpha,\beta,j} c_{ij\alpha\beta} x^\alpha {\partial_x}^\beta e_j \in N$ for all i. It follows that $\sum c_{ij\alpha\beta}\partial_t^i x^\alpha \vartheta^\beta e_j \in D_{n+1} \cdot \psi(N)$ and hence $m \in K(N)$. $\square$

Now we may give an adapted version of Oaku's torsion algorithm by combining and extending the techniques of [32], [33], and [51]. The algorithm computes torsion for an arbitrary finitely generated D-module. One ingredient of the algorithm and its proof is the notion of V-filtration and V-adapted resolution. We explain these notions in the Appendix, where they play a role more generally for the restriction algorithm. For the reader's convenience, we recall the definitions here as well. The *V-filtration* F_Y of a shifted free module $D_{n+1}^r[\vec{m}]$ with respect to the hyperplane $Y = \{t = 0\}$ is defined by

$$F_Y^i(D_{n+1}^r[\vec{m}]) = \mathrm{Span}_K\{\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta t^j \partial_t^k e_l : \alpha, \beta \in \mathbb{N}^n, k - j \leq m_l + i\}$$

A free resolution $X^\bullet$ of the form,

$$X^\bullet : \cdots \longrightarrow D_{n+1}^{r_{j+1}}[\vec{m}_{j+1}] \xrightarrow{\psi_{j+1}} D_{n+1}^{r_j}[\vec{m}_j] \longrightarrow \cdots$$

is said to be *V-adapted* if

$$\psi_{j+1}(F_Y^i(D_{n+1}^{r_{j+1}}[\vec{m}_{j+1}])) \subset F_Y^i(D_{n+1}^{r_j}[\vec{m}_j]))$$

for all i and all j, and if a resolution is also induced on the level of associated graded modules,

$$\mathrm{gr}(X^\bullet) : \cdots \longrightarrow \mathrm{gr}(D_{n+1}^{r_{j+1}}[\vec{m}_{j+1}]) \xrightarrow{\mathrm{gr}(\psi_{j+1})} \mathrm{gr}(D_{n+1}^{r_j}[\vec{m}_j]) \longrightarrow \cdots.$$

The *b-function* of a module D_{n+1}^r/N with respect to the V-filtration F_Y is the monic polynomial $b(s) \in K[s]$ of least degree such that $b(\theta)\mathrm{gr}_0(D_{n+1}^r/P) = 0$ where $\theta = t\partial_t$.

Algorithm 2.4.3. (Torsion module $H_f^0(M)$)
INPUT: a presentation $M \simeq D^r/D \cdot \{T_1, \ldots, T_a\}$ of a left D-module.
OUTPUT: $\{R_1, \ldots, R_b\} \subset D^r$ whose images in $M \simeq D^r/D \cdot \{T_1, \ldots, T_a\}$ generate $H_f^0(M)$.

1. Compute the generators $\{t-f, \psi(T_1), \ldots, \psi(T_a)\}$ of $K(N)$.

2. Compute a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ of $K(N)$ with respect to the weight vector $(-w, w)$ where $w = (0, \ldots, 0, 1)$. (Here $(-w, w)$ gives the elements $\{te_j\}_{j=1}^r$ weight -1, the elements $\{\partial_t e_j\}_{j=1}^r$ weight 1, and the elements $\{x_i e_j, \partial_i e_j\}_{i,j}$ weight 0). Note that $\mathcal{G}$ generates $\mathrm{gr}(K(N))$ with respect to the induced V-filtration.

3. Compute $d \in \mathbb{Z}$ such that $gr_{k+1}(D^r_{n+1}/K(N)) \xrightarrow{t\cdot} gr_k(D^r_{n+1}/K(N))$ is injective for $k > d$. If D^r/N is holonomic, then d can be taken to be the maximum integer root of the b-function $b(s)$ of $D^r_{n+1}/K(N)$ with respect to V-filtration. If D^r/N is not holonomic, then use Algorithm 2.4.5.

4. Let $\phi_{\mathcal{G}} : D^s_{n+1}[\vec{m}] \to D^r_{n+1}[\vec{0}]$ be the map of (filtered) left D_{n+1}-modules (with respect to the V-filtration F_Y) defined by sending $e_j \mapsto g_j$, and where $\vec{m} \in \mathbb{Z}^s$ is such that m_j equals the $(-w, w)$-weight of g_j. Express the map $\overline{\phi}_{\mathcal{G}} : F^d_Y(\Lambda^s[\vec{m}]) \to F^d_Y(\Lambda^r[\vec{0}])$ induced by $\phi_{\mathcal{G}}$ as a map of finitely generated D-modules, where $\Lambda := (D_{n+1}/t \cdot D_{n+1})$.

5. Compute elements $\{P_i = \sum_{j=1}^s P_{ij} e_j\}_{i=1}^b \subset D^s_{n+1}$ whose images in $F^m_Y(\Lambda^r[\vec{0}])$ generate the kernel of $\overline{\phi}_{\mathcal{G}}$.

6. Compute elements $\{Q_i\}_{i=1}^b \subset D^r_{n+1}$ such that $tQ_i = \sum_{j=1}^r P_{ij} g_j$ for all i.

7. Reduce each Q_i modulo $D_{n+1} \cdot \{(t-f)e_j\}_{j=1}^r$ to an element $Q'_i \in K[\boldsymbol{x}, \partial_x, \partial_t]$. Let $R_i \in D^r$ be the element obtained by substituting $\partial_t = 0$ in Q'_i.

8. Return $\{R_1, \ldots, R_b\}$.

Proof. Let the kernel of $(D^r_{n+1}/K(N)) \xrightarrow{t\cdot} (D^r_{n+1}/K(N))$ be denoted by W. From Theorem 2.4.1 and Lemma 2.4.2 and using the notation there, we have that $H^0_f(D^r/N) = \varphi \circ \phi(W)$. Step 3 implies that $W \subset F^d_Y(D^r_{n+1}[-\vec{1}]/K(N))$. To see why d may be taken to be the maximum integer root of $b(s)$ when D^r/N is holonomic, note that $D^{r+1}/K(N)$ is also holonomic, hence $b(s) \neq 0$. By definition $b(\theta_t + k) \cdot \mathrm{gr}_k(D^r_{n+1}/K(N)) = 0$. To show injectivity, suppose $m \in \mathrm{gr}_{k+1}(D^r_{n+1}/K(N))$ with $tm = 0$. Then $0 = b(\theta_t + k + 1)m = b(\partial_t t + k)m = b(k)m$. Thus if $k > d$, then $b(k) \neq 0$ and $m = 0$.

Let now $X^\bullet$ be a V-adapted free resolution of $D^r_{n+1}/K(N)$ extending the map $\phi_{\mathcal{G}}$ of Step 4, and let $Z^\bullet$ denote the complex $0 \to D^r_{n+1}[-\vec{1}] \xrightarrow{t\cdot} D^r_{n+1} \to 0$. Then the total complex of $X^\bullet \otimes_{D_{n+1}} Z^\bullet$ is quasi-isomorphic to both $\Lambda \otimes_{D_{n+1}} X^\bullet$ and $Z^\bullet \otimes_{D_{n+1}} (D^r_{n+1}/K(N))$. Moreover, these quasi-isomorphisms induce quasi-isomorphisms on the graded level with respect to the V-filtration F_Y. Since the kernel W, which is the -1-th cohomology of $Z^\bullet \otimes_{D_{n+1}} (D^r_{n+1}/K(N))$, is contained in $F^d_Y(D^r_{n+1}[-\vec{1}]/K(N))$, therefore also the -1-th cohomology of $\Lambda \otimes_{D_{n+1}} X^\bullet$ may be generated by cycles in $F^d_Y(\Lambda^s[\vec{m}])$, or in other words by the kernel of $\overline{\phi}_{\mathcal{G}}$.

Note that the maps in $\Lambda \otimes_{D_{n+1}} X^\bullet$ are maps of left D-modules, and moreover since $F^k_Y(\Lambda) = \oplus_{i=0}^k D \cdot \partial_t^i$, both the source and target of $\overline{\phi}_{\mathcal{G}}$ are free D-modules of finite rank. Thus the kernel of $\overline{\phi}_{\mathcal{G}}$ can be computed in Step 5 by Gröbner bases. The cycles $\{\overline{P}_i\}_{i=1}^b$ can now be lifted to cycles $\{P_i \oplus Q_i\}_{i=1}^b$ which generate the -1-th cohomology of

$\mathrm{Tot}^\bullet(X^\bullet \otimes_D Z^\bullet)$. These are in turn projected to cycles $\{\overline{Q}_i\}_{i=1}^b$ which generate the kernel W, which is the process of Step 6.

Finally, for each Q_i, we need to compute $\varphi \circ \phi(\overline{Q}_i)$. To do this, we can assume that Q_i is reduced modulo $D_{n+1} \cdot \{(t-f)e_j\}_{j=1}^r$. Now we should express Q_i in the form $Q_i = \sum_{jk\alpha\beta} c_{ijk\alpha\beta}\boldsymbol{x}^\alpha \vartheta^\beta \partial_t^k e_j$. Then $\phi(\overline{Q}_i) = \sum_{jk\alpha\beta} \partial_t^k \otimes c_{ijk\alpha\beta}\boldsymbol{x}^\alpha \partial_{\boldsymbol{x}}^\beta e_j$ and $\varphi \circ \phi(\overline{Q}_i) = \sum_{j0\alpha\beta} c_{ij0\alpha\beta}\boldsymbol{x}^\alpha \partial_{\boldsymbol{x}}^\beta e_j$. Note however that given the standard expression $Q_i = \sum_{jk\alpha\beta} c'_{ijk\alpha\beta}\boldsymbol{x}^\alpha \partial_{\boldsymbol{x}}^\beta \partial_t^k e_j$, then $c'_{ij0\alpha\beta} = c_{ij0\alpha\beta}$. Thus it suffices to substitute $\partial_t = 0$ in Q_i to get $\varphi \circ \phi(\overline{Q}_i)$, as we do in Step 7. ☐

To achieve Step 3 of Algorithm 2.4.3 when D^r/N is not holonomic, we shall use the following proposition.

Proposition 2.4.4. *Let $B \subset D_{n+1}^r$ be a left submodule, let $w = (0,\ldots,0,1)$, and let $H = \mathrm{in}_{(-w,w)}(B) \cap D[\theta_t]^r$ where $\theta_t = t\partial_t$. If $\langle H : \theta_t - k\rangle = H$, then the map $\mathrm{gr}_{k+1}(D_{n+1}^r/B) \xrightarrow{t\cdot} \mathrm{gr}_k(D_{n+1}^r/B)$ is injective. Moreover, for any left $D[\theta_t]$-submodule $H' \subset D[\theta_t]^r$, the condition $\langle H' : \theta_t - k\rangle = H'$ is generic.*

Proof. We shall prove the contrapositive, that is, if there exists a nonzero element $T \in \ker(\mathrm{gr}_{k+1}(D_{n+1}^r/B) \xrightarrow{t\cdot} \mathrm{gr}_k(D_{n+1}^r/B))$, then $\langle H : \theta_t - k\rangle \neq H$. So assume the existence of T. Then T lifts to a $(-w,w)$-homogeneous element $T' \in D_{n+1}^r$ of weight $k+1$ such that $T' \notin \mathrm{in}_{(-w,w)}(B)$ while $tT' \in \mathrm{in}_{(-w,w)}(B)$.

Assume now that $k \geq 0$. The argument is similar for $k < 0$ by interchanging the roles of t and ∂_t. We can now write $T' = \partial_t^{k+1}P$ where $P \in D[\theta_t]^r$. Then $t^{k+1}T' = \theta_t \cdots (\theta_t - k)P \in H$. We shall now show $t^k\partial_t^k P = \theta_t \cdots (\theta_t - k + 1)P \notin \mathrm{in}_{(-w,w)}(B)$, which proves that $\langle H : \theta_t - k\rangle \neq H$. In fact, we claim that $t^i\partial_t^j P \notin \mathrm{in}_{(-w,w)}(B)$ for all $0 \leq i \leq j \leq k$. This is true for all j when $i = 0$ because $T' = \partial_t^{k+1}P \notin \mathrm{in}_{(-w,w)}(B)$ by assumption. For general (i,j) with $i \geq 1$, we argue by contradiction. If $t^i\partial_t^j P \in \mathrm{in}_{(-w,w)}(B)$, then $(\theta_t + k - j + i)\cdots(\theta_t + k - j + 1)\partial_t^k P = \partial_t^{k-j+i}t^i\partial_t^j P \in \mathrm{in}_{(-w,w)}(B)$. We also know that $\theta_t\partial_t^k P = t\partial_t^{k+1}P \in \mathrm{in}_{(-w,w)}(B)$. Since θ_t and $(\theta_t + k - j + i)\cdots(\theta_t + k - j + 1)$ are relatively prime, this implies $\partial_t^k P \in \mathrm{in}_{(-w,w)}(B)$, which contradicts the assumption $\partial_t^{k+1}P \notin \mathrm{in}_{(-w,w)}(B)$.

To prove the statement about submodules $H' \subset D[\theta_t]^r$, let F denote the standard good filtration of $D[\theta_t]^r$ coming from the order filtration of $D[\theta_t]$, where θ_t gets order 0. For each $k \in \mathbb{Z}$ such that $\langle H' : \theta_t - k\rangle \neq H$, there exists an element $T_k \notin H'$ such that $(\theta_t - k)T_k \in H'$. We can further assume that $\mathrm{in}_{(0,e)}(T_k) \notin \mathrm{in}_{(0,e)}(H')$ while $(\theta_t - k)\,\mathrm{in}_{(0,e)}(T_k) \in \mathrm{in}_{(0,e)}(H')$. In other words, there is a nonzero element of $K[\boldsymbol{x}, \xi_{\boldsymbol{x}}, \theta]^r/\mathrm{in}_{(0,e)}(H)$ having torsion with respect to $(\theta_t - k)$. Since $K[\boldsymbol{x}, \xi_{\boldsymbol{x}}, \theta]^r/\mathrm{in}_{(0,e)}(H')$ is finitely generated over the polynomial ring $K[\boldsymbol{x}, \xi_{\boldsymbol{x}}, \theta]$, this can only occur for finitely many integers k by primary decomposition. ☐

We can now make Proposition 2.4.4 algorithmic.

Algorithm 2.4.5. (Injectivity of $\mathrm{gr}_{k+1}(D_{n+1}^r/B) \xrightarrow{t\cdot} \mathrm{gr}_k(D_{n+1}^r/B)$)

INPUT: generators $\{S_1, \ldots, S_a\}$ of $B \subset D^r_{n+1}$.
OUTPUT: $d \in \mathbb{Z}$ with $\mathrm{gr}_{k+1}(D^r_{n+1}/B) \xrightarrow{t\cdot} \mathrm{gr}_k(D^r_{n+1}/B)$ injective for all $k > d$.

1. Compute $(-w, w)$-homogeneous generators $\{L_1, \ldots, L_b\} \subset D^r_{n+1}$ of the initial ideal $\mathrm{in}_{(-w,w)}(B)$ where $w = (0, \ldots, 0, 1)$. Let d_i be the $(-w, w)$-weight of L_i, and let H be the left $D[\theta_t]$-submodule of $D[\theta_t]^r$ generated by $\{\zeta_1 L_1, \ldots, \zeta_c L_c\}$ where $\zeta_i = t^{d_i}$ if $d_i \geq 0$ and $\zeta_i = \partial_t^{d_i}$ if $d_i < 0$.

2. Compute generators $\{h_1, \ldots, h_c\} \subset K[\boldsymbol{x}, \boldsymbol{\xi}, \theta_t]^r$ of $\mathrm{in}_{(0,e)}(H)$ where $(0, e)$ is the weight vector giving the element $\boldsymbol{x}^\alpha \boldsymbol{\partial}^\beta \theta_t^i e_j$ weight $|\beta|$.

3. Compute a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_u\}$ of the $K(\lambda)[\boldsymbol{x}, \boldsymbol{\xi}, \theta_t, s]$-module generated by $\{(1-s)h_i, s(\theta_t - \lambda)e_j\}_{i,j} \subset K(\lambda)[\boldsymbol{x}, \boldsymbol{\xi}, \theta_t, s]^r$ using any elimination order having $\{se_i\}_{i=1}^r > \{\boldsymbol{x}e_j, \boldsymbol{\xi}e_j, \theta_t e_j\}_{j=1}^r$. In the process of computing $\mathcal{G}$, keep all computations in $K[\lambda]$, that is never perform any division in the field $K(\lambda)$ but rather cross-multiply lead coefficients in forming S-pairs.

4. Compute the greatest integer root d which occurs in any of the leading coefficients $p_i(\lambda)$ of g_i.

5. Return d.

Proof. The submodule H computed in Step 1 is the intersection $H = \mathrm{in}_{(-w,w)}(B) \cap D[\theta_t]^r$. Step 2 computes $\mathrm{in}_{(0,e)}(H)$. By the proof of Proposition 2.4.4, if $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - k\rangle = \mathrm{in}_{(0,e)}(H)$, then $\langle H : \theta_t - k\rangle = H$, which implies by the same proposition that the map $\mathrm{gr}_{k+1}(D^r_{n+1}/B) \xrightarrow{t\cdot} \mathrm{gr}_k(D^r_{n+1}/B)$ is injective. So we are reduced to analyzing the condition $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - k\rangle = \mathrm{in}_{(0,e)}(H)$. Step 3 is a Gröbner basis method to compute the saturation $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - k\rangle$ (see e.g. [16]). Namely, put $\mathcal{P} = \{g/(\theta_t - \lambda) : g \in \mathcal{G} \cap K(\lambda)[\boldsymbol{x}, \xi_{\boldsymbol{x}}, \theta_t]\}$. Then $\mathcal{P}$ is a Gröbner basis for the saturation $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - \lambda\rangle$. Moreover, since $\mathrm{in}_{(0,e)}(H)$ is defined over K, it follows that $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - \lambda\rangle = \mathrm{in}_{(0,e)}(H)$. Now suppose that $p_i(\lambda_0) \neq 0$ for all i. Then the leading monomials of $\mathcal{G}|_{\lambda \mapsto \lambda_0}$ are the same as those of $\mathcal{G}$, and hence a straightening relation for any S-pair of $\mathcal{G}|_{\lambda \mapsto \lambda_0}$ can be obtained from the straightening relation of the corresponding S-pair of $\mathcal{G}$ by substitution $\lambda \mapsto \lambda_0$. It follows that $\mathcal{G}|_{\lambda \mapsto \lambda_0}$ remains a Gröbner basis for the submodule it generates. Put $\mathcal{P}_{\lambda_0} = \{g/(\theta_t - \lambda_0) : g \in \mathcal{G}_{\lambda \mapsto \lambda_0} \cap K[\boldsymbol{x}, \xi_{\boldsymbol{x}}, \theta_t]\}$. Then $\mathcal{P}_{\lambda_0}$ is now a Gröbner basis for the saturation $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - \lambda_0\rangle$. Note also that $\mathcal{P}_{\lambda_0} = \mathcal{P}|_{\lambda \mapsto \lambda_0}$ because the lead term of any element in $\mathcal{G} \setminus \mathcal{P}$ contains s and has lead coefficient $p(\lambda)$ with $p(\lambda_0) \neq 0$. Therefore $\mathcal{P}_{\lambda_0}$ also remains a Gröbner basis for $\mathrm{in}_{(0,e)}(H)$, hence $\langle \mathrm{in}_{(0,e)}(H) : \theta_t - \lambda_0\rangle = \mathrm{in}_{(0,e)}(H)$. This establishes the validity of the integer d of Step 4 and completes the proof. □

This completes the algorithm for computing torsion modules $H^0_f(M)$ of an arbitrary D-module M with respect to a polynomial f. By intersecting, we may similarly compute torsion modules with respect to arbitray ideals $I \subset K[\boldsymbol{x}]$.

Algorithm 2.4.6. (Torsion module $H^0_I(M)$)
INPUT: a presentation $M \simeq D^r/D \cdot \{T_1, \ldots, T_a\}$ and generators $\{f_1, \ldots, f_m\}$ of $I \subset K[\boldsymbol{x}]$.
OUTPUT: $\{R_1, \ldots, R_b\} \subset D^r$ whose images in $M \simeq D^r/D \cdot \{T_1, \ldots, T_a\}$ generate $H^0_I(M)$.

1. Compute the torsion submodules $H_{f_i}(M) \subset M$ using Algorithm 2.4.3.

2. Compute the intersection of all $H_{f_i}(M)$ using Gröbner bases.

3. Return generators of the intersection.

Remark 2.4.7. Algorithm 2.4.5 combined with Steps 2 to 6 of Algorithm 2.4.3 computes the kernel of the map $M \xrightarrow{t\cdot} M$ where M is any finitely generated left D_{n+1}-module. This kernel is equivalently the 1-th restriction module of M to the hyperplane $t = 0$. By computing intersections as in Algorithm 2.4.6, we can similarly compute the i-th restriction of M to any codimension i subspace.

Remark 2.4.8. Both Algorithm 2.4.3 and Algorithm 2.4.5 have yet to be implemented in Macaulay 2. The applications of computing torsion of D-modules which are not finite rank are not yet clear.

2.5 Associated primes of D-modules

In this section, we give an algorithm to find the prime ideals of the polynomial ring $K[x_1, \ldots, x_n]$ which are associated to a D-module M. For instance, this might be interesting for finding the associated primes of a local cohomology module $H_I^j(K[\boldsymbol{x}])$ where I is an ideal of $K[\boldsymbol{x}]$. The algorithm is based upon the following basic fact in D-module theory.

Lemma 2.5.1. *Let $K = \mathbb{C}$. The support of a D-module M is equal to the projection $\pi(\mathrm{char}(M))$ of its characteristic variety.*

Algorithm 2.5.2. (Associated primes of a D-module)
INPUT: a presentation $M \simeq D^r/D \cdot \{T_1, \ldots, T_m\}$ of a left D-module.
OUTPUT: prime ideals of $K[\boldsymbol{x}]$ associated to M.

1. Compute the characteristic module $\mathrm{in}_{(0,e)}(N) \subset K[\boldsymbol{x}, \boldsymbol{\xi}]^r$ of N.

2. Compute a primary decomposition of $K[\boldsymbol{x}, \boldsymbol{\xi}]^r / \mathrm{in}_{(0,e)}(N)$.

3. For each prime $q \subset K[\boldsymbol{x}, \boldsymbol{\xi}]$ associated to the characteristic module, compute the intersection $p_q = q \cap K[\boldsymbol{x}]$.

4. Compute $H^0_{p_q}(M)$ using Algorithm 2.4.6, and collect the set $\mathcal{P}$ of primes p_q such that $\pi(\mathrm{char}(H^0_{p_q}(M))) = V(p_q)$.

5. Return $\mathcal{P}$.

Proof. Suppose the ideal $p \subset K[\boldsymbol{x}]$ is associated to M. Then there exists $m \in M$ whose annihilator is $\mathrm{ann}_{K[\boldsymbol{x}]}(m) = p$. Similarly, every element of $D \cdot m \subset M$ is annihilated by some power of p. In other words, the support of the submodule $D \cdot m$ is equal to $V(p)$, hence by Lemma 2.5.1 the projection $\pi(\mathrm{char}(D \cdot m))$ of the characteristic variety of $D \cdot m$

is also equal to $V(p)$. This means that $\mathrm{gr}_{(0,e)}(D \cdot m)$ contains an associated prime q whose intersection $p_q = q \cap K[\boldsymbol{x}]$ equals p. Now the exact sequence

$$0 \to D \cdot m \to M \to M/D \cdot m \to 0$$

leads to an exact sequence of associated graded modules,

$$0 \to \mathrm{gr}_{(0,e)}(D \cdot m) \to \mathrm{gr}_{(0,e)}(M) \to \mathrm{gr}_{(0,e)}(M/D \cdot m) \to 0.$$

It follows that the associated primes of $\mathrm{gr}_{(0,e)}(D \cdot m)$ as a $K[\boldsymbol{x}, \xi]$-module are contained inside the associated primes of $\mathrm{gr}_{(0,e)}(M)$. Thus by computing the associated primes of $\mathrm{gr}_{(0,e)}(M)$ in Step 3, we produce the prime q. $\square$

From the theoretical point of view, the candidates for associated primes are related to irreducible components of various characteristic varieties. For instance, the singular locus can be regarded as the union of the projections of all irreducible components of $\mathrm{char}(M)$ except for the irreducible component corresponding to the zero section of the cotangent bundle (if it occurs). The general relationship follows from work of Kashiwara [26] and more recently Smith [42] on the relation between submodules and irreducible components of the characteristic variety.

Theorem 2.5.3. (Smith [42]) *If M is a finitely generated D-module whose nonzero submodules all have dimension $\geq p$, then every irreducible component of the characteristic variety of M also has dimension $\geq p$.*

The following description of associated primes is a direct consequence of Theorem 2.5.3.

Proposition 2.5.4. *Let M be a finitely generated D-module. For i from n to $2n-1$ let M_i be the maximal submodule of M of dimension i, and let $\{p_{ij}\}_j \subset K[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$ be the collection of prime ideals corresponding to the dimension i irreducible components of the characteristic variety of M_i. Then the associated primes of M are contained in the set of prime ideals $\{p_{ij} \cap K[x_1, \ldots, x_n]\}_{i,j}$.*

Proof. Suppose that a prime ideal $p \subset K[\boldsymbol{x}]$ is associated to M. Then there is an element $m \in M$ such that $\mathrm{ann}_{K[\boldsymbol{x}]}(m) = p$. Again, by Lemma 2.5.1 the projection $\pi(\mathrm{char}(D \cdot m))$ of the characteristic variety of $D \cdot m$ equals $V(p)$, thus there is a component ζ which projects onto $V(p)$. Let i be the dimension of this component, and consider the maximum submodule $(D \cdot m)_i$ of $D \cdot m$ of dimension i. We have an exact sequence,

$$0 \to (D \cdot m)_i \to D \cdot m \to \frac{D \cdot m}{(D \cdot m)_i} \to 0$$

so that $\mathrm{char}(D \cdot m) = \mathrm{char}((D \cdot m)_i) \cup \mathrm{char}(D \cdot m/(D \cdot m)_i)$. Moreover, $D \cdot m/(D \cdot m)_i$ contains no submodules of dimension $\leq i$, hence the components of its characteristic variety are all strictly bigger than i. It follows that ζ is a component of $(D \cdot m)_i$. Now consider the maximum submodule M_i of M of dimension i so that $(D \cdot m)_i \subset M_i$. Since the characteristic variety of M_i has dimension i and $\mathrm{char}(M_i) = \mathrm{char}((D \cdot m)_i) \cup \mathrm{char}(M_i/(D \cdot m)_i)$, we see that ζ is also a component of the characteristic variety of M_i. $\square$

Corollary 2.5.5. *Let M be a holonomic D-module. Then the associated primes of M are contained in the set of prime ideals corresponding to the projections of irreducible components of the characteristic variety of M.*

Example 2.5.6. Consider again the holonomic module $M = D_2/D_2\{x_1\partial_1, x_2\partial_1, x_2\partial_2 + 1\}$ of Remark 2.2.2. Its characteristic variety is equal to

$$V(x_1\xi_1, x_2\xi_1, x_2\xi_2) = V(\xi_1, \xi_2) \cup V(\xi_1, x_2) \cup V(x_1, x_2)$$

It follows that the associated primes are contained in the set of prime ideals $\{\langle x_2\rangle, \langle x_1, x_2\rangle\}$. In this case, $\langle x_1, x_2\rangle$ is the annihilator of ∂_1 and is the only prime ideal of $K[x_1, x_2]$ associated to M.

Remark 2.5.7. Algorithm 2.5.2 has yet to be implemented in Macaulay 2, even for the case of holonomic D-modules. It would be interesting to have this algorithm available, for instance to investigate the associated primes of local cohomology modules. Mustaţă has given an example based upon [29] where a local cohomology module of the polynomial ring with respect to a monomial ideal has an embedded associated prime [30]. This example suggests that local cohomology modules can have mysterious structures.

We also remark that Proposition 2.5.4 can be made algorithmic as well. The first step is to compute the canonical filtration $M_n \subset M_{n+1} \subset \cdots \subset M$, which can be achieved in principle by an implementation of the dualizing complex methods of [7, Chapter 2]. This would make an interesting project in computational homological algebra.

2.6 Local cohomology

In Section 2.4, we realized the f-torsion of a finitely generated D-module M as the cohomology of a complex of finitely generated D_{n+1}-modules (Theorem 4.1). In this section, we discuss an extension of this construction to computing local cohomology due to Oaku and Takayama [33]. The complex which they use has also been studied by Adolphson and Sperber [1]. We will bring together these points of view here.

Let $S = K[x_1, \ldots, x_n]$ denote the polynomial ring and $I = \langle f_1, \ldots, f_d\rangle \subset S$ an ideal. By definition, $H^i_I(-)$ is the i-th derived functor in the category of S-modules of the left exact functor for torsion,

$$H^0_I(M) = \{m \in M : I^k m = 0 \text{ for some } k > 0\}.$$

The usual method to realize the local cohomology modules is via the Cech complex, which we now describe. Let $C^\bullet(f_i)$ denote the complex of S-modules $0 \to S \to S[\frac{1}{f_i}] \to 0$ where the map is defined by sending $1 \mapsto \frac{1}{1}$. Then $H^i_I(M)$ is the i-th cohomology module of the Cech complex $C^\bullet(M; f_1, \ldots, f_d) := \bigotimes_{i=1}^d C^\bullet(f_i) \otimes M$, where the tensor products are all over S. When M is a holonomic D-module, the Cech complex is additionally a complex of holonomic D-modules. In this situation, Walther has given an algorithm to compute presentations of the cohomology modules as D-modules [51].

In [33], Oaku and Takayama introduce another complex, which we shall call the twisted Koszul complex, to compute local cohomology. They prove the following theorem, which specializes to Theorem 4.1 of Section 5 when $i = 0$ and $d = 1$.

Theorem 2.6.1. (Oaku-Takayama [33]) *Let M be a holonomic D-module. Then for any $i \geq 0$, we have an isomorphism of left D-modules,*

$$H_I^i(M) \cong H^{i-d}\left(\left(\frac{D_{n+d}}{J} \otimes_{K[x]} M\right)^{\bullet}_{\{t_1=\cdots=t_d=0\}}\right) \tag{6.2}$$

where D_{n+d} denotes the $(n+d)$th Weyl algebra $D\langle t_1, \ldots, t_d, \partial_{t_1}, \ldots, \partial_{t_d}\rangle$, where the right hand side denotes the $(i-d)$-th derived restriction module of the D_{n+d}-module $(D_{n+d}/J) \otimes M$ to the subspace $\{t_1 = \cdots = t_d = 0\}$, and where

$$J = D_{n+d} \cdot \{t_j - f_j(x), \partial_{x_i} + \sum_{k=1}^{d} \frac{\partial f_k}{\partial x_i} \partial_{t_k} | 1 \leq j \leq d, 1 \leq i \leq n\}.$$

Oaku and Takayama also provide in [33] an algorithm to compute the cohomology modules of a derived restriction complex, thus obtaining another algorithm to compute local cohomology of holonomic D-modules. The proof of Theorem 2.6.1 in [33] consists of an elegant equivalence in the derived category based upon ideas of Kashiwara [27]. On the other hand, a proof "in coordinates" follows from an earlier paper of Adolphson and Sperber [1], who consider the complex of Theorem 2.6.1 on the level of S-modules and establish a quasi-isomorphism to the Cech complex. More generally, Adolphson and Sperber make a similar construction of a twisted Koszul complex for A-modules M, where A is a commutative ring with identity. In trying to understand Oaku and Takayama's algorithm, we also arrived at a proof "in coordinates" which is essentially the same as the proof of Adolphson and Sperber. We shall provide an exposition here which brings together the work of Oaku and Takayama and the work of Adolphson and Sperber. We also give a presentation of $D_{n+d}/J \otimes M$ following [51], which facilitates the algorithmic computation of (6.2).

Let us first describe the twisted Koszul complex $T^\bullet(M; f_1, \ldots, f_d)$ on the level of S-modules where M is an S-module and $f_1, \ldots, f_d \in S$. We will return to the D-module structure and Theorem 2.6.1 afterwards. Let $S[t] := S[t_1, \ldots, t_d]$.

Definition 2.6.2. *The twisted Koszul module $M[\partial_{t_1}, \ldots, \partial_{t_d}]^{\{f_1,\ldots,f_d\}}$ of an S-module M with respect to the polynomials $f_1, \ldots, f_d \in S$ is the $S[t]$-module,*

$$M[\partial_{t_1}, \ldots, \partial_{t_d}]^{\{f_1,\ldots,f_d\}} := M \otimes_K K[\partial_{t_1}, \ldots, \partial_{t_d}] = \oplus_{\alpha \in \mathbb{N}^d} M \partial_t^\alpha,$$

where the action of $S[t]$ is as follows:

$$\begin{aligned} x_i \bullet (m_\alpha \otimes \partial_t^\alpha) &= x_i m_\alpha \otimes \partial_t^\alpha \\ t_j \bullet (m_\alpha \otimes \partial_t^\alpha) &= f_i m_\alpha \otimes \partial_t^\alpha - \alpha_j m \otimes \partial_t^{\alpha - e_j}. \end{aligned}$$

We will usually suppress notation and write $M[\partial_t]$ when $f_1, \ldots, f_d$ are clear.

Definition 2.6.3. *The twisted Koszul complex of an S-module M with respect to the polynomials $f_1, \ldots, f_d \in S$ is the Koszul complex $T^\bullet(M; f_1, \ldots, f_d) := K^\bullet(M[\partial_t]; t_1, \ldots, t_d)$.*

Let us consider the 0-th cohomology of the twisted Koszul complex. It equals

$$\begin{aligned}
H^0(T^\bullet) &= \{\vec{m} := \sum m_\alpha \otimes \partial^\alpha \in M[\partial_t] \mid t_j\vec{m} = 0 \ \forall\ j = 1, \dots, d\} \\
&= \{\vec{m} \mid \textstyle\sum_\alpha f_j m_\alpha \otimes \partial_t^\alpha - \alpha_j m_\alpha \otimes \partial_t^{\alpha - e_j} = 0 \ \forall j\} \\
&= \{\vec{m} \mid \textstyle\sum_\alpha (f_j m_\alpha - (\alpha_j + 1) m_{\alpha + e_j}) \otimes \partial_t^\alpha = 0 \ \forall j\} \\
&= \{\vec{m} \mid f_j m_\alpha = (\alpha_j + 1) m_{\alpha + e_j} \ \forall \alpha, \forall j\} \\
&= \{\vec{m} \mid m_\alpha = (1/\alpha!) f_1^{\alpha_1} \cdots f_d^{\alpha_d} m_{\vec{0}} \ \forall \alpha\}.
\end{aligned}$$

Since $m_\alpha = 0$ for all but finitely many α, the last line implies also that $f_j^N m_{\vec{0}} = 0$ for N sufficiently large for all j. It follows that the map $\varphi^0 : M[\partial_t] \to M$ sending $\vec{m} \mapsto m_{\vec{0}}$ induces an isomorphism $H^0(T^\bullet) \xrightarrow{\simeq} H_I^0(M)$.

Theorem 2.6.4. (Adolphson-Sperber [1]) *The map φ^0 extends to a quasi-isomorphism,*

$$\varphi^\bullet : T^\bullet(M; f_1, \dots, f_d) \to C^\bullet(M; f_1, \dots, f_d)$$

between the twisted Koszul complex and the Cech complex,

$$\begin{array}{ccccccccc}
T^\bullet : 0 \to M[\partial_t] & \longrightarrow & \bigoplus_{1\le i\le d} M[\partial_t]\vec{e}_i & \longrightarrow & \bigoplus_{1\le i<j\le d} M[\partial_t]\vec{e}_{ij} & \longrightarrow \cdots \longrightarrow & M[\partial_t]\vec{e}_{1\cdots d} \to 0 \\
\downarrow \varphi_0 & & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 \\
C^\bullet : 0 \to M & \longrightarrow & \bigoplus_{1\le i\le d} M[\frac{1}{f_i}] & \longrightarrow & \bigoplus_{1\le i<j\le d} M[\frac{1}{f_i f_j}] & \longrightarrow \cdots \longrightarrow & M[\frac{1}{f_1\cdots f_d}] \to 0
\end{array}$$

is defined by,

$$\varphi^{i_1\cdots i_s} : M[\partial_t]\vec{e}_{i_1\cdots i_s} \to M[\tfrac{1}{f_{i_1}\cdots f_{i_s}}]$$

$$\sum_\alpha (m_\alpha \otimes \partial_t^\alpha)\vec{e}_{i_1\cdots i_s} \mapsto \sum_{\left\{\substack{\alpha\ :\ \alpha_i = 0\ for \\ i\notin(i_1,\dots,i_s)}\right\}} \frac{\alpha! m_\alpha}{f_{i_1}^{\alpha_{i_1}+1} \cdots f_{i_s}^{\alpha_{i_s}+1}}.$$

Proof. The twisted Koszul module $M[\partial_t]$ has increasing $S[t]$-submodules,

$$M[\partial_t]_k = \oplus_{\{\alpha \in \mathbb{N}^d : \alpha_1, \dots, \alpha_d \le k\}} M \otimes \partial_t^\alpha.$$

Since $M[\partial_t] = \cup_{k=0}^\infty M[\partial_t]_k$, the twisted Koszul complex is the limit of its subcomplexes, $\lim_\to K^\bullet(M[\partial_t]_k; t_1, \dots, t_d)$, where the maps

$$\iota_k^\bullet : K^\bullet(M[\partial_t]_k; t_1, \dots, t_d) \longrightarrow K^\bullet(M[\partial_t]_{k+1}; t_1, \dots, t_d)$$

are induced from the inclusions $M[\partial_t]_k \hookrightarrow M[\partial_t]_{k+1}$. Similarly, the Cech complex can be defined as the limit of Koszul complexes, $\lim_\to K^\bullet(M; f_1^{k+1}, \dots, f_d^{k+1})$, where the maps

$$\gamma_k^\bullet : K^\bullet(M; f_1^{k+1}, \dots, f_d^{k+1}) \longrightarrow K^\bullet(M; f_1^{k+2}, \dots, f_d^{k+2})$$

are induced from the maps $M\vec{e}_{i_1\cdots i_s} \xrightarrow{(f_{i_1}\cdots f_{i_s})\cdot} M\vec{e}_{i_1\cdots i_s}$.

Let us now construct a family of chain-maps $\varphi_k^\bullet$ between the above sub-complexes $K^\bullet(M[\partial_t]_k; t_1, \ldots, t_d)$ and the Koszul complexes $K^\bullet(M; f_1^{k+1}, \ldots, f_d^{k+1})$. Define $\varphi_k^\bullet = \oplus_{s=0}^d \varphi_k^s$ by $\varphi_k^0 : M[\partial_t]_k \longrightarrow M$ where $\varphi_k^0(\sum_{\{\alpha\,:\,\alpha_1,\ldots,\alpha_d \leq k\}}(m_\alpha \otimes \partial_t^\alpha)) = m_{\vec{0}}$, and for $s \geq 1$,

$$\varphi_k^s = \bigoplus_{0 \leq i_1 < \cdots < i_s \leq d} \varphi_k^{i_1 \cdots i_s}, \qquad \varphi_k^{i_1 \cdots i_s} : M[\partial_t]_k \vec{e}_{i_1 \cdots i_s} \longrightarrow M \vec{e}_{i_1 \cdots i_s}$$

$$\sum_{\{\alpha\,:\,\alpha_1,\ldots,\alpha_d \leq k\}} (m_\alpha \otimes \partial_t^\alpha)\vec{e}_{i_1 \cdots i_s} \mapsto \sum_{\left\{\substack{\alpha\,:\,\alpha_i = 0 \text{ for} \\ i \notin (i_1,\ldots,i_s)}\right\}} (\alpha! f_{i_1}^{k-\alpha_{i_1}} \cdots f_{i_s}^{k-\alpha_{i_s}} m_\alpha)\vec{e}_{i_1 \ldots i_s}.$$

A computation shows that $\varphi_{k+1}^\bullet \circ \iota_k^\bullet = \gamma_{k+1}^\bullet \circ \varphi_k^\bullet$, so that the family of chain maps $\{\varphi_k^\bullet\}$ forms a map of towers of complexes. We will next show that $\varphi_k^\bullet$ is also a quasi-isomorphism. Since (see e.g. [54] for the first equality),

$$\begin{aligned} H_I^i(M) &= \lim_{\to} H^i(K^\bullet(M; f_1^{k+1}, \ldots, f_d^{k+1})) \\ &= \lim_{\to} H^i(K^\bullet(M[\partial_t]_k; t_1, \ldots, t_d)) \\ &= H^i(K^\bullet(M[\partial_t]; t_1, \ldots, t_d)), \end{aligned}$$

it would then follow that the twisted Koszul complex computes local cohomology and also that $\varphi^\bullet$ is a quasi-isomorphism. So it only remains to prove the following claim. □

Claim 2.6.5. $\varphi_k^\bullet : K^\bullet(M[\partial_t]_k; t_1, \ldots, t_d) \to K^\bullet(M; f_1^{k+1}, \ldots, f_d^{k+1})$ *is a chain map quasi-isomorphism.*

Proof. One checks by a computation that $\varphi^\bullet$ is a chain map. To prove that $\varphi^\bullet$ is a quasi-isomorphism, we induct on the length $d+1$ of the twisted Koszul complex. The base case $d = 1$ is outlined by Oaku in [32].

Base case. For $d = 1$, we are to show that the following chain map is a quasi-isomorphism,

$$\begin{array}{ccccccc} 0 \longrightarrow & M[\partial_t]_k & \xrightarrow{t_1 \cdot} & M[\partial_t]_k & \longrightarrow 0 \\ & \Big\downarrow{\scriptstyle \varphi_k^0} & & \Big\downarrow{\scriptstyle \varphi_k^1} & \\ 0 \longrightarrow & M & \xrightarrow{f_1^{k+1} \cdot} & M_s & \longrightarrow 0 \end{array}$$

where the maps are

$$\sum_{i=0}^k a_i \otimes \partial_{t_1}^i \overset{\varphi_k^0}{\mapsto} a_0 \qquad \sum_{i=0}^k b_i \otimes \partial_{t_1}^i \overset{\varphi_k^1}{\mapsto} \sum_{i=0}^k i! f_1^{k-i} b_i.$$

We have already seen that φ_k^0 induces an isomorphism on the level of homology, so we are left to show that φ_k^1 does as well.

For surjectivity of φ_k^0 on the level of homology, suppose $m \in M$ such that $f_1^{k+1} m = 0$. Then the element $\sum_{i=1}^k \frac{1}{i!} f_1^i m \otimes \partial_{t_1}^i \in M[\partial_t]_k$ is in the kernel of $[t_1 \cdot]$ and maps under φ_k^0 to m. For injectivity of φ_k^0 on the level of homology, suppose that $\vec{a} = \sum_{i=1}^k a_i \otimes \partial_{t_1}^i \in \ker[t_1 \cdot]$.

Then $t_1\vec{a} = \sum_{i=1}^{k}(a_i - (i+1)a_{i+1}) \otimes \partial_{t_1}^i = 0$, or $a_i = \frac{1}{i!}a_0$ for all i. Thus if $\varphi_k^0(\vec{a}) = a_0 = 0$, then $\vec{a} = 0$.

For surjectivity of φ_k^1 on the level of homology, given $m \in M$, the element $m \otimes \partial_{t_1}^k$ maps under φ_k^1 to m. For injectivity of φ_k^1 on the level of homology, let $\vec{b} = \sum_{i=0}^{k} b_i \otimes \partial_{t_1}^i$, and suppose that $\varphi_k^1(\vec{b}) \in \mathrm{im}[f_1^{k+1}]$. This means there exists $m \in M$ such that $\sum_{i=0}^{k} i! f_1^{k-i} b_i = f_1^{k+1} m$. We wish to show that $\vec{b} \in \mathrm{im}[t_1]$. Set $a_0 = 0$, and inductively set $a_i = \frac{-b_{i-1} + f a_{i-1}}{i}$ for $1 \le i \le k$. Then,

$$f_1 a_k = -\frac{1}{k} f_1 b_{k-1} + \frac{1}{k} f_1^2 a_{k-1} = \cdots = -\sum_{i=0}^{k-1} \frac{i! f_1^{k-i} b_i}{k!} = -\frac{f_1^{k+1} m}{k!}.$$

Now let $\vec{a} = \sum_{i=0}^{k} a_i \otimes \partial_{t_1}^i$. It follows that,

$$t_1\vec{a} = \sum_{i=0}^{k}(f_1 a_i - (i+1)a_{i+1}) \otimes \partial_{t_1}^i = \sum_{i=0}^{k-1} b_i \otimes \partial_{t_1}^i + f_1 a_k \otimes \partial_{t_1}^k = \sum_{i=0}^{k-1} b_i \otimes \partial_{t_1}^i - \frac{f_1^{k+1} m}{k!}.$$

Finally, set $\vec{a'} = \vec{a} + \sum_{i=0}^{k} \frac{f_1^i m}{i!} \otimes \partial_{t_1}^i$ so that $t_1\vec{a'} = \vec{b}$, as desired.

General case. The key tool for the induction step is a certain long exact sequence of Koszul complexes. In particular, consider the following three double complexes:

1. Rows are the Koszul complex $K^\bullet(M[\partial_t]_{d,k}; t_1, \ldots, t_{d-1})$ and columns are induced by the multiplication map $t_d : M[\partial_t]_{d,k} \to M[\partial_t]_{d,k}$, where $M[\partial_t]_{d,k}$ denotes the twisted Koszul module $(M[\partial_{t_1}, \ldots, \partial_{t_d}]^{\{f_1,\ldots,f_d\}})_k$

$$\begin{tikzcd}
0 \arrow[r] & M[\partial_t]_{d,k} \arrow[r] \arrow[d, "t_d"] & M[\partial_t]_{d,k}^{d-1} \arrow[r] \arrow[d, "t_d"] & \cdots \arrow[r] & M[\partial_t]_{d,k} \arrow[r] \arrow[d, "t_d"] & 0 \\
0 \arrow[r] & M[\partial_t]_{d,k} \arrow[r] & M[\partial_t]_{d,k}^{d-1} \arrow[r] & \cdots \arrow[r] & M[\partial_t]_{d,k} \arrow[r] & 0,
\end{tikzcd}$$

2. Rows are the Koszul complex $K^\bullet(M[\partial_t]_{d-1,k}; t_1, \ldots, t_{d-1})$ and columns are induced by the multiplication map $f_d^{k+1} : M[\partial_t]_{d-1,k} \to M[\partial_t]_{d-1,k}$, where $M[\partial_t]_{d-1,k}$ denotes the twisted Koszul module $(M[\partial_{t_1}, \ldots, \partial_{t_{d-1}}]^{\{f_1,\ldots,f_{d-1}\}})_k$.

$$\begin{tikzcd}
0 \arrow[r] & M[\partial_t]_{d-1,k} \arrow[r] \arrow[d, "f_d^{k+1}"] & M[\partial_t]_{d-1,k}^{d-1} \arrow[r] \arrow[d, "f_d^{k+1}"] & \cdots \arrow[r] & M[\partial_t]_{d-1,k} \arrow[r] \arrow[d, "f_d^{k+1}"] & 0 \\
0 \arrow[r] & M[\partial_t]_{d-1,k} \arrow[r] & M[\partial_t]_{d-1,k}^{d-1} \arrow[r] & \cdots \arrow[r] & M[\partial_t]_{d-1,k} \arrow[r] & 0,
\end{tikzcd}$$

3. Rows are the Koszul complex $K^\bullet(M; f_1^{k+1}, \ldots, f_{d-1}^{k+1})$ and columns are induced by the multiplication map $f_d^{k+1} : M \to M$,

$$\begin{array}{ccccccccc}
0 \longrightarrow & M & \longrightarrow & M^{d-1} & \longrightarrow \cdots \longrightarrow & M & \longrightarrow 0 \\
& \downarrow{\scriptstyle f_d^{k+1}} & & \downarrow{\scriptstyle f_d^{k+1}} & & \downarrow{\scriptstyle f_d^{k+1}} & \\
0 \longrightarrow & M & \longrightarrow & M^{d-1} & \longrightarrow \cdots \longrightarrow & M & \longrightarrow 0,
\end{array}$$

The Koszul complexes $K^\bullet(M[\partial_t]_k; t_1, \ldots, t_d)$ and $K^\bullet(M; f_1^{k+1}, \ldots, f_d^{k+1})$ can be identified with the total complexes of the double complexes (1) and (3). To complete the induction step, we construct maps of double complexes $\mu^{\bullet\bullet}$ from (1) to (2) and $\nu^{\bullet\bullet}$ from (2) to (3) such that on the level of total complexes, $\mathrm{Tot}(\mu^{\bullet\bullet})$ and $\mathrm{Tot}(\nu^{\bullet\bullet})$ are quasi-isomorphisms and $\varphi_k^\bullet = \mathrm{Tot}(\mu^{\bullet\bullet}) \circ \mathrm{Tot}(\nu^{\bullet\bullet})$.

To define $\mu^{\bullet\bullet}$, note that $M_{d,k} = (M_{d-1,k}[\partial_{t_d}]^{\{f_d\}})_k$. Thus by the base case ($d = 1$) applied to $M_{d-1,k}$ and the polynomial f_d, we get a chain map quasi-isomorphism

$$\mu_k^\bullet : K^\bullet(M_{d-1,k}[\partial_{t_d}]_k; t_d) \longrightarrow K^\bullet(M_{d-1,k}; f_d^{k+1})$$

$$\sum_{i=0}^k a_i \otimes \partial_{t_d}^i \overset{\mu_k^0}{\mapsto} a_0 \qquad \sum_{i=0}^k b_i \otimes \partial_{t_d}^i \overset{\mu_k^1}{\mapsto} \sum_{i=0}^k i! f_d^{k-i} b_i.$$

Since the columns of the double complexes (1) and (2) are copies of the Koszul complexes $K^\bullet(M_{d-1,k}[\partial_{t_d}]_k; t_d)$ and $K^\bullet(M_{d-1,k}; f_d^{k+1})$ respectively, $\mu_k^\bullet$ induces a map of double complexes $\mu^{\bullet\bullet}$ from (1) to (2). To define $\nu^{\bullet\bullet}$, we use the chain maps,

$$\nu_k^\bullet : K^\bullet(M_{d-1,k}; t_1, \ldots, t_{d-1}) \longrightarrow K^\bullet(M; f_1^{k+1}, \ldots, f_{d-1}^{k+1}),$$

which are quasi-isomorphisms by induction. Then using $\nu_k^\bullet$ for both the top and bottom rows defines a map of double complexes $\nu^{\bullet\bullet}$ between (2) and (3).

To show that $\mu^{\bullet\bullet}$ and $\nu^{\bullet\bullet}$ are quasi-isomorphisms, we can use the long-exact sequence of Koszul complexes coming from the row-filtration spectral sequence of (1), (2), or (3). Namely, these are long exact sequences of the form,

$$\cdots \to H^{i-2}(P_{(j)}^\bullet) \to H^i(N_{(j)}^\bullet) \to H^\bullet(S_{(j)}^\bullet) \to H^{i-1}(P_{(j)}^\bullet) \to \cdots,$$

where $N_{(j)}^\bullet$ are the induced complexes at the top row of $j = (1), (2)$, or (3) on the cohomology of the column maps, $P_{(j)}^\bullet$ are similarly the induced complexes at the bottom row on the cohomology of the column maps, and $S_{(j)}^\bullet$ are the total complexes. In particular, we have

(j)	$N_{(j)}^\bullet$	$P_{(j)}^\bullet$
(1)	$K^\bullet(\ker(M_{d,k} : t_d); t_1, \ldots, t_{d-1})$	$K^\bullet(\mathrm{cok}(M_{d,k} : t_d); t_1, \ldots, t_{d-1})$
(2)	$K^\bullet(\ker(M_{d-1,k} : f_d^{k+1}); t_1, \ldots, t_{d-1})$	$K^\bullet(\mathrm{cok}(M_{d-1,k} : f_d^{k+1}); t_1, \ldots, t_{d-1})$
(3)	$K^\bullet(\ker(M : f_d^{k+1}); f_1^{k+1}, \ldots, f_{d-1}^{k+1})$	$K^\bullet(\mathrm{cok}(M : f_d^{k+1}); f_1^{k+1}, \ldots, f_{d-1}^{k+1})$

Since $\mu_k^\bullet$ is a quasi-isomorphism, it induces isomorphisms

$$\ker(M_{d,k};t_d) \xrightarrow{\simeq} \ker(M_{d-1,k};f_d^{k+1}) \qquad \mathrm{cok}(M_{d,k};t_d) \xrightarrow{\simeq} \mathrm{cok}(M_{d-1,k};f_d^{k+1}).$$

It follows that $\mu^{\bullet\bullet}$ induces quasi-isomorphisms,

$$K^\bullet(\ker(M_{d,k}:t_d);t_1,\dots,t_{d-1}) \xrightarrow{\simeq} K^\bullet(\ker(M_{d-1,k}:f_d^{k+1});t_1,\dots,t_{d-1})$$

$$K^\bullet(\mathrm{cok}(M_{d,k}:t_d);t_1,\dots,t_{d-1}) \xrightarrow{\simeq} K^\bullet(\mathrm{cok}(M_{d-1,k}:f_d^{k+1});t_1,\dots,t_{d-1})$$

Similarly, a quick computation shows,

$$\ker(M_{d-1,k}:f_d^{k+1}) = \ker(M;f_d^{k+1})[\partial_{t_1},\dots,\partial_{t_{d-1}}]^{\{f_1,\dots,f_{d-1}\}}$$

$$\mathrm{cok}(M_{d-1,k}:f_d^{k+1}) = \mathrm{cok}(M;f_d^{k+1})[\partial_{t_1},\dots,\partial_{t_{d-1}}]^{\{f_1,\dots,f_{d-1}\}}.$$

Hence $\nu^{\bullet\bullet}$ induces chain maps having the form $\varphi_k^\bullet$ which are quasi-isomorphisms by induction,

$$K^\bullet(\ker(M_{d-1,k}:f_d^{k+1});t_1,\dots,t_{d-1}) \xrightarrow{\simeq} K^\bullet(\ker(M:f_d^{k+1});f_1^{k+1},\dots,f_{d-1}^{k+1}$$

$$K^\bullet(\mathrm{cok}(M_{d-1,k}:f_d^{k+1});t_1,\dots,t_{d-1}) \xrightarrow{\simeq} K^\bullet(\mathrm{cok}(M:f_d^{k+1});f_1^{k+1},\dots,f_{d-1}^{k+1}).$$

Thus, $\mu^{\bullet\bullet}$ and $\nu^{\bullet\bullet}$ induce isomorphisms

$$H^i(N_{(1)}^\bullet) \xrightarrow{\simeq} H^i(N_{(2)}^\bullet) \qquad H^i(P_{(1)}^\bullet) \xrightarrow{\simeq} H^i(P_{(2)}^\bullet)$$

$$H^i(N_{(1)}^\bullet) \xrightarrow{\simeq} H^i(N_{(2)}^\bullet) \qquad H^i(P_{(1)}^\bullet) \xrightarrow{\simeq} H^i(P_{(2)}^\bullet)$$

From the long exact sequence and the five-lemma, we conclude that $\mathrm{Tot}(\mu^{\bullet\bullet})$ and $\mathrm{Tot}(\nu^{\bullet\bullet})$ are quasi-isomorphisms. Thus, the composition $\mathrm{Tot}(\mu^{\bullet\bullet}) \circ \mathrm{Tot}(\nu^{\bullet\bullet})$ is a quasi-isomorphism, and a computation shows that it equals $\varphi_k^\bullet$. $\square$

Remark 2.6.6. There is a chain-map quasi-isomorphism

$$\psi_k^\bullet : K^\bullet(M;f_1^{k+1},\dots,f_d^{k+1}) \to K^\bullet(M[\partial_t]_k;f_1^{k+1},\dots,f_d^{k+1})$$

in the opposite direction from $\varphi^\bullet$ defined by,

$$\psi_k^0(m) = \sum_{\{\alpha_1,\dots,\alpha_d\le k\}} \frac{1}{\alpha!} f_1^{\alpha_1}\cdots f_d^{\alpha_d} m \otimes \partial_t^\alpha$$

$$\psi_k^{i_1\cdots i_s}(m\vec{e}_{i_1\cdots i_s}) = \sum_{\left\{\substack{\alpha\,:\,\alpha_i\le k\ \forall i \\ \alpha_i = k \text{ for } i\in(i_1,\dots,i_s)}\right\}} \Big(\prod_{i\notin(i_1,\dots,i_s)} f_i^{\alpha_i}\Big) m \otimes \partial_t^\alpha \vec{e}_{i_1\cdots i_s}$$

However, these maps do not agree for increasing k, and hence do not define a map in the opposite direction from the Cech complex to the twisted Koszul complex.

Now suppose M is a D-module, and let $D_{n+d} = D\langle t_1,\dots,t_d,\partial_{t_1},\dots,\partial_{t_d}\rangle$. Then Oaku and Takayama define a D_{n+d}-module structure on $M[\partial_t]$ as follows.

Definition 2.6.7. *Let M be a D-module. Then the twisted Koszul module $M[\partial_t]$ of M with respect to $f_1, \ldots, f_d$ has a D_{n+d}-module structure given by*

$$\begin{aligned} x_i \bullet (m_\alpha \otimes \partial_t^\alpha) &= x_i m_\alpha \otimes \partial_t^\alpha \\ t_j \bullet (m_\alpha \otimes \partial_t^\alpha) &= f_i m_\alpha \otimes \partial_t^\alpha - \alpha_j m \otimes \partial_t^{\alpha - e_j} \\ \partial_{x_i} \bullet (m_\alpha \otimes \partial_t^\alpha) &= \partial_i m_\alpha \otimes \partial_t^\alpha + \textstyle\sum_{j=1}^d \frac{\partial f_j}{\partial x_i} m_\alpha \otimes \partial_t^{\alpha+e_j} \\ \partial_{t_j} \bullet (m_\alpha \otimes \partial_t^\alpha) &= m_\alpha \otimes \partial_t^{\alpha+e_j} \end{aligned}$$

Lemma 2.6.8. *If M is a D-module, then the twisted Koszul module $M[\partial_t]$ of M with respect to $f_1, \ldots, f_d$ is isomorphic as D_{n+d}-module to $(D_{n+d}/J) \otimes_{K[\mathbf{x}]} M$, where $J = D_{n+d} \cdot \{t_j - f_j, \partial_{x_i} + \sum_{j=1}^d \frac{\partial f_j}{\partial x_i} \partial_{t_j}\}_{1 \le j \le d, 1 \le i \le n}$*

Proof. Given an element $\partial_t^\alpha \otimes m \in (D_{n+d}/J) \otimes_{k[x]} M$, we have

$$\begin{aligned} x_i \bullet (\partial_t^\alpha \otimes m) &= x_i \partial_t^\alpha \otimes m \\ &= \partial_t^\alpha \otimes x_i m \\ \partial_{x_i} \bullet (\partial_t^\alpha \otimes m) &= \partial_{x_i} \partial_t^\alpha \otimes m + \partial_t^\alpha \otimes \partial_{x_i} m \\ &= \partial_t^\alpha (-\textstyle\sum_{j=1^d} \frac{\partial f_j}{\partial x_i} \partial_{t_j}) \otimes x_i m + \partial_t^\alpha \otimes \partial_{x_i} m \\ &= -\textstyle\sum_{j=1}^d \partial_t^{\alpha+e_j} \otimes \frac{\partial f_j}{\partial x_i} m + \partial_t^\alpha \otimes \partial_{x_i} m \\ t_j \bullet (\partial_t^\alpha \otimes m) &= t_j \partial_t^\alpha \otimes m \\ &= \partial_t^\alpha t_j \otimes m - \alpha_j \partial_t^{\alpha-e_j} \otimes m \\ &= \partial_t^\alpha \otimes f_j m - \alpha_j \partial_t^{\alpha-e_j} \otimes m \\ \partial_{t_j} \bullet (\partial_t^\alpha \otimes m) &= \partial_t^{\alpha+e_j} \otimes m, \end{aligned}$$

This agrees with the previously defined D_{n+d}-structure of the twisted Koszul module. Moreover, the map of left D_{n+d}-modules

$$K[\mathbf{x}][\partial_t] = \frac{D_{n+d}}{D_{n+d} \cdot \{t_1, \ldots, t_d, \partial_{x_1}, \ldots, \partial_{x_n}\}} \longrightarrow \frac{D_{n+d}}{J}$$

$$x_i \mapsto x_i \qquad \partial_{t_j} \mapsto \partial_{t_j} \qquad t_j \mapsto t_j - f_j(x) \qquad \partial_{x_i} \mapsto \partial_{x_i} + \sum_{j=1}^d \frac{\partial f_j}{\partial x_i} \partial_{t_j}$$

is an isomorphism, so that $\frac{D_{n+d}}{J} \otimes_{K[\mathbf{x}]} M \simeq K[\mathbf{x}][\partial_t] \otimes_{K[\mathbf{x}]} M = K[\partial_t] \otimes_K M$. $\square$

The maps in the twisted Koszul complex are maps of D-modules although not maps of D_{n+d}-modules. Moreover, a quick computation shows that $\varphi^\bullet$ is a chain map of D-modules.

Proposition 2.6.9. *The chain map $\varphi^\bullet : T^\bullet(M; t_1, \ldots, t_d) \longrightarrow C^\bullet(M; f_1, \ldots, f_d)$ is a quasi-isomorphism of complexes of D-modules.*

We now give a presentation of the twisted Koszul module $M[\partial_t]$ in terms of a presentation of M by extending Lemma 2.4.2. As corollaries, we see that if M is finitely generated as a D-module, then $M[\partial_t]$ is finitely generated as a D_{n+d}-module. Similarly, if M is holonomic, then $M[\partial_t]$ is holonomic. As in Section 2.4, redefine $\vartheta_i = \partial_{x_i} + \sum_{j=1}^{d} (\partial f_j / \partial x_i) \partial_{t_j}$ and put

$$\psi(P) := \sum_j p_j(x_1, \ldots, x_n, t_1 - f_1, \ldots, t_d - f_d, \vartheta_1, \ldots, \vartheta_n, \partial_{t_1}, \ldots, \partial_{t_d}) e_j \in D^r_{n+1}$$

for $P = \sum_j p_j(x_1, \ldots, x_n, t_1, \ldots, t_d, \partial_1, \ldots, \partial_n, \partial_{t_1}, \ldots, \partial_{t_d}) e_j \in D^r_{n+d}$,

Lemma 2.6.10. *Given a presentation $M \simeq D^r/N$, then we have the presentation of left D_{n+d}-modules $(D_{n+d}/J) \otimes_{K[\boldsymbol{x}]} (D^r/N) \simeq D_{n+d}/K(N)$, where*

$$K(N) := D_{n+d} \cdot \{(t_i - f_i)e_j\}_{i,j=1}^{d,r} + D_{n+d} \cdot \psi(N).$$

Proof. Consider the map $\phi : D^r_{n+d} \longrightarrow (D_{n+d}/J) \otimes_{K[\boldsymbol{x}]} (D^r/N)$ of left D_{n+d}-modules defined by $\phi(e_j) = 1 \otimes e_j$. By the same arguments as in the proof of Lemma 2.4.2, we have that ϕ is surjective with kernel equal to $K(N)$. $\square$

Algorithms for local cohomology of D-modules. It now follows that the local cohomology of a D-module $M = D^r/N$ at the ideal $I = (f_1, \ldots, f_d)$ is equal to the cohomology of the Koszul complex $T^\bullet = K^\bullet((D^r_{n+d}/K(N)); t_1, \ldots, t_d)$. This Koszul complex equivalently computes the derived restriction of the D_{n+d}-module $D^r_{n+d}/K(N)$ to the subspace $\{t_1 = \cdots = t_d = 0\}$. When D^r/N and thus $D^r_{n+d}/K(N)$ are holonomic, an algorithm to compute derived restriction is given in [33] and which we summarize in the appendix. The method is to produce a complex of finitely generated D-modules (whose cohomology can be computed using Gröbner bases) which is quasi-isomorphic to $T^\bullet$. We have seen that the Cech complex is itself a complex of finitely generated D-modules which is quasi-isomorphic to $T^\bullet$. However the Cech complex is not the complex constructed by the restriction algorithm. In particular, the algorithm produces a complex consisting of free D-modules of finite rank. The maps between them can be quite complicated and are induced from Gröbner bases coming from Gröbner deformations. In contrast, the modules which appear in the Cech complex have the form $M[\frac{1}{f_{i_1} \cdots f_{i_s}}]$ and typically have complicated presentations as D-modules whereas the maps between them are now easy to understand. It would be interesting to investigate which method is more efficient and under what circumstances. These algorithms have been implemented in Macaulay 2 as the scripts

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localCohom( ..., Strategy => OT)
localCohom( ..., Strategy => Walther)
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Non-singular affine varieties. When $j : Y \hookrightarrow X$ is a closed embedding of nonsingular varieties, a theorem due to Kashiwara states that the category of $\mathcal{D}_Y$-modules is equivalent to the category of $\mathcal{D}_X$-modules supported on Y. For an ideal $I \subset K[\boldsymbol{x}]$ such that $j : V(I) \hookrightarrow K^n$ is nonsingular, one consequence of this equivalence is a relationship between the local cohomology of a D-module M with respect to I and the derived restriction of M to the variety $V(I)$.

Proposition 2.6.11. (see e.g. Bernstein [5]) *Let M be a left D-module, $I \subset K[\boldsymbol{x}]$ an ideal corresponding to a nonsingular variety, and $j : V(I) \hookrightarrow K^n$ the inclusion. Then*

$$H^i_I(M) = j_*\big(\operatorname{Tor}^{K[\boldsymbol{x}]}_{\dim(V(I))-i}(K[\boldsymbol{x}]/I, M) \otimes_{K[\boldsymbol{x}]} \omega_{K[\boldsymbol{x}]/I}\big)$$

where $\operatorname{Tor}^{K[\boldsymbol{x}]}_{\dim(V(I))-i}(K[\boldsymbol{x}]/I, M))$ is also the $(\dim(V(I)) - i)$-th derived restriction of M to $V(I)$ and j_* denotes the direct image.

Proof. A formulation of Kashiwara's equivalence for the inclusion $j : V(I) \hookrightarrow K^n$ is that $H^0_I(M) = j_* \circ \operatorname{Hom}_{K[\boldsymbol{x}]}(K[\boldsymbol{x}]/I, M)$ where we note that $\operatorname{Hom}_{K[\boldsymbol{x}]}(K[\boldsymbol{x}]/I, M)$ has the structure of a left $D_{V(I)}$-module. The functor j_* is additionally exact, hence

$$H^i_I(M) \simeq j_* \circ \operatorname{Ext}^i_{K[\boldsymbol{x}]}(K[\boldsymbol{x}]/I, M).$$

Now $\operatorname{Ext}^i_{K[\boldsymbol{x}]}(K[\boldsymbol{x}]/I, M) \simeq H^i(\operatorname{Hom}_{K[\boldsymbol{x}]}(F^\bullet, M)) \simeq H^i(\operatorname{Hom}_{K[\boldsymbol{x}]}(F^\bullet, K[\boldsymbol{x}]) \otimes_{K[\boldsymbol{x}]} M)$ where $F^\bullet$ is a free resolution of $K[\boldsymbol{x}]/I$. Since $V(I)$ is nonsingular, the complex $\operatorname{Hom}_{K[\boldsymbol{x}]}(F^\bullet, K[\boldsymbol{x}])$ is exact except in cohomological degree $\dim(V(I))$ where its cohomology is isomorphic to the canonical module $\omega_{K[\boldsymbol{x}]/I}$ which is also locally isomorphic to $K[\boldsymbol{x}]/I$. This shows that $H^i(\operatorname{Hom}_{K[\boldsymbol{x}]}(F^\bullet, K[\boldsymbol{x}]) \otimes_{K[\boldsymbol{x}]} M) \simeq \operatorname{Tor}^{K[\boldsymbol{x}]}_{\dim(V(I))-i}(K[\boldsymbol{x}]/I, M) \otimes \omega_{K[\boldsymbol{x}]/I}$. ☐

When I is a nonsingular complete intersection, then $\omega_{K[\boldsymbol{x}]/I} \simeq K[\boldsymbol{x}]/I$ as $D_{K[\boldsymbol{x}]/I}$-modules. Moreover, when I is generated by linear forms $f_1, \ldots, f_d$, then the derived restriction modules can be computed directly by the algorithm of Oaku and Takayama. Since direct images are easy to compute for inclusions, this gives an algorithm for local cohomology with respect to a linear subspace. In other words, we do not need to pass to the twisted Koszul complex or the Cech complex, which are computationally more intensive, to compute local cohomology with respect to $V(f_1, \ldots, f_d)$.

The above observation is useful for instance when $f_1, \ldots, f_d$ generate the maximal ideal $m = (x_1, \ldots, x_n)$ and M is holonomic. In this case, the derived restriction modules of M to the origin are finite-dimensional K-vector spaces, and the local cohomology modules $H^i_m(M)$ are the direct images of these vector spaces, which by Kashiwara's equivalence are their injective hulls. This gives a method to compute the Lyubeznik numbers $\lambda_{i,n-j}(S/I)$, which by definition are the socle dimensions of the local cohomology modules $H^i_m(H^j_I(S))$ and hence are equal to the dimension of the derived restriction modules of $H^j_I(S)$ to the origin. We remark that an algorithm to compute Lyubeznik numbers was first given by Walther in [51].

Algorithm 2.6.12. (Computing Lyubeznik numbers)
INPUT*: a left ideal $I \subset K[\boldsymbol{x}]$.*
OUTPUT*: the Lyubeznik numbers $\lambda_{i,n-j}(K[\boldsymbol{x}]/I)$.*

1. *Compute the local cohomology modules $H^j_I(K[\boldsymbol{x}])$ using the algorithm of [51] or the algorithm of [33].*

2. *Compute the dimensions $\lambda_{i,n-j} = \dim_K H^{-i}(K[\boldsymbol{x}]/(\boldsymbol{x}) \otimes^L_{K[\boldsymbol{x}]} H^j_I(K[\boldsymbol{x}]))$ of the derived restriction modules of $H^j_I(M)$ to the origin.*

3. Return the $\lambda_{i,n-j}$.

Example 2.6.13. As a simple example, let us compute the Lyubeznik numbers of the ring $K[x,y,z]/I$ where I is the ideal $(x(y-z), xyz)$. First, we compute using Walther's algorithm the local cohomology modules of polynomial ring $K[x,y,z]$ with respect to I.

```
i1 : (W = QQ[x,y,z,Dx,Dy,Dz, WeylAlgebra => {x=>Dx, y=>Dy, z=>Dz}];
      I = ideal (x*(y-z), x*y*z);
      HI = localCohom I)

o1 = HashTable{0 => 0                                       }
               1 => cokernel {0} | Dz Dy xDx+2 x2 |
               2 => cokernel {0} | y-1z zDy+zDz+2 xDx+2 z2 |
```

Second, we compute the derived restrictions to the origin of each nonzero local cohomology module. In the above table, $H^j_I(K[\boldsymbol{x}])$ is accessed by `HI#j`.

```
i2 : Drestriction (HI#1, {1,1,1})
o2 = HashTable{0 => 0  }
               2 => 0
               3 => 0
                      1
               1 => QQ

i3 : Drestriction (HI#2, {1,1,1})
o3 = HashTable{0 => 0}
               1 => 0
               2 => 0
               3 => 0
```

It follows that the only nonzero Lyubeznik number is $\lambda_{2,2} = 1$.

Chapter 3

Polynomial and rational solutions

Polynomial and rational solutions for linear ordinary differential equations can be obtained by algorithmic methods. For instance, the Maple package `DEtools` provides efficient functions `polysols` and `ratsols` to find polynomial and rational solutions for a given linear ordinary differential equation with rational function coefficients. On a naive level, these algorithms are based on bounding the degree of polynomial solutions and bounding the order of poles of rational solutions. Such bounds can be obtained from the indicial polynomial at infinity or the indicial polynomials at each pole.

In the several variables case, a natural analogue to the notion of a linear ordinary differential equation with rational function coefficients is the notion of a finite rank system of linear partial differential equations with rational function coefficients. These notions are analogous in the sense that they both imply a finite-dimensional vector space of holomorphic solutions at a generic point. In [12], Chyzak gave an algorithm to find rational solutions of finite rank systems by combining elimination methods in the ring of differential operators with rational function coefficients with Abramov's algorithm for rational solutions of ordinary differential equations with parameters. Chyzak's approach is analogous to solving systems of algebraic equations of zero-dimensional ideals by elimination.

In this chapter, our aim is to give new algorithms based on D-modules theory for finding polynomial and rational solutions of finite rank systems. The material here is based on our paper [35] with Oaku and Takayama. As we have seen in Section 2.1, finite rank systems can be translated to holonomic systems over the Weyl algebra D by virtue of the Weyl closure. This allows us to utilize computational methods developed for holonomic D-modules. Namely, we will replace the indicial polynomial at infinity by the notion of Gröbner deformation as introduced in [40], and we will replace the indicial polynomial at a pole by the notion of b-function with respect to a hypersurface as introduced by Kashiwara [27].

3.1 Polynomial solutions by Gröbner deformations

For the case of linear ordinary differential equations, one method to compute polynomial solutions is to compute the indicial polynomial at infinity, find its maximum non-negative integer root which provides an upper bound to the degree of all polynomial solutions, and finally determine the coefficients of polynomial solutions by linear algebra.

The analogous method works for holonomic systems by using Gröbner deformations. One difference is that there are now many directions of infinity in K^n, hence we introduce a weight vector $w \in \mathbb{R}^n$ to encode the direction. The following discussion comes entirely from [40], except for Corollary 3.1.5 and Algorithm 3.1.9.

Definition 3.1.1. *Let $w \in \mathbb{R}^n$.*

1. *The initial form with respect to w of an element $L = \sum_{\alpha,\beta\in\mathbb{N}^n} c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta \in D$ is*

$$\mathrm{in}_{(-w,w)}(L) := \sum_{-w\cdot\alpha+w\cdot\beta \ \mathit{maximal}} c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta \in D$$

2. *The Gröbner deformation with respect to w of a left ideal $I \subset D$ is the left ideal*

$$\mathrm{in}_{(-w,w)}(I) := \{\mathrm{in}_{(-w,w)}(L) : L \in I\} \subset D.$$

Theorem 3.1.2. [3] *There exist only finitely many Gröbner deformations. The equivalence classes form a fan of $\mathbb{R}^n$, and weight vectors in the open cones are called generic.*

Given a polynomial $f = \sum_{a\in\mathbb{N}^n} c_a\boldsymbol{x}^a$, we can similarly define the initial form of f with respect to a weight vector $w \in \mathbb{R}^n$ as

$$\mathrm{in}_w(f) := \sum_{\{a\in\mathbb{N}^n \ : \ c_a\neq 0, -w\cdot a \ \mathit{maximal}\}} c_a\boldsymbol{x}^a.$$

For generic w, the initial form $in_w(f)$ is a single monomial $c_a\boldsymbol{x}^a$, and we can bound the possibilities of the exponent a by the following lemmas.

Lemma 3.1.3. [40, Theorem 2.5.5] *If $f(\boldsymbol{x})$ is a polynomial solution of I, then $\mathrm{in}_w(f)(\boldsymbol{x})$ is a polynomial solution of $\mathrm{in}_{(-w,w)}(I)$.*

Proof. Let us define the w-degree of an element $L = \sum_{\alpha\beta} c_{\alpha\beta}\boldsymbol{x}^\alpha\boldsymbol{\partial}^\beta \in D$ as $\deg_w(L) = \max_{\{\alpha,\beta\in\mathbb{N}^n : c_{\alpha\beta}\neq 0\}}\{-w\cdot\alpha + w\cdot\beta\}$, and similarly the w-degree of a polynomial $f = \sum_a d_a\boldsymbol{x}^a \in K[\boldsymbol{x}]$ as $\deg_w(f) = \max_{\{a\in\mathbb{N}^n : d_a\neq 0\}}\{-w\cdot a\}$. The computation $\boldsymbol{x}^\alpha\partial^\beta \bullet \boldsymbol{x}^a = [a]_\beta x^{\alpha-\beta+a}$ shows that $\deg_w(\boldsymbol{x}^\alpha\partial^\beta \bullet \boldsymbol{x}^a) = \deg_w(\boldsymbol{x}^\alpha\partial^\beta) + \deg_w(\boldsymbol{x}^a)$ if it is nonzero. Given $L \in D$ and $f \in K[\boldsymbol{x}]$, it follows that the component of $L \bullet f$ consisting of monomials of w-degree $\deg_w(L) + \deg_w(f)$ is equal to $\mathrm{in}_{(-w,w)}(L) \bullet \mathrm{in}_w(f)$. In particular if f is annihilated by L, then $\mathrm{in}_w(f)$ is annihilated by $\mathrm{in}_{(-w,w)}(L)$ as required. □

Lemma 3.1.4. [40, Theorem 2.3.11] *Let I be finite rank and w generic. Put $\theta_i = x_i\partial_i$ and $R = K(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle$. Then the indicial ideal*

$$\mathrm{ind}_{(-w,w)}(I) := R \cdot \mathrm{in}_{(-w,w)}(I) \cap K[\theta_1, \ldots, \theta_n]$$

is an Artinian ideal in $K[\theta_1, \ldots, \theta_n]$ and the polynomial solutions of $\mathrm{in}_{(-w,w)}(I)$ equals

$$\mathrm{Span}_K\{\boldsymbol{x}^a | a \in V(\mathrm{ind}_{(-w,w)}(I))\}.$$

Proof. By Lemma 2.0.7, in order to show that the indicial ideal is Artinian, it suffices to show that I' has dimension n for any torus invariant ideal I' with the property that $I' \cap K[\boldsymbol{\theta}] = \mathrm{ind}_{(-w,w)}(I)$. Since w is generic, the Gröbner deformation $\mathrm{in}_{(-w,w)}(I)$ is torus invariant, and hence $\mathrm{Cl}(\mathrm{in}_{(-w,w)}(I))$ is torus invariant as well. By definition,

$$\mathrm{ind}_{(-w,w)}(I) = R \cdot \mathrm{in}_{(-w,w)}(I) \cap D \cap K[\boldsymbol{\theta}] = \mathrm{Cl}(\mathrm{in}_{(-w,w)}(I)) \cap K[\boldsymbol{\theta}],$$

and thus it suffices to show that $\mathrm{Cl}(\mathrm{in}_{(-w,w)}(I))$ is holonomic.

For any holonomic ideal J and any weight vector $w \in \mathbb{R}^n$, it is shown in [40, Theorem 2.2.1] that $\mathrm{in}_{(-w,w)}(J)$ is also holonomic. Since I is finite rank, we know by Proposition 2.1.7 that $\mathrm{Cl}(I)$ is holonomic, hence its Gröbner deformation $\mathrm{in}_{(-w,w)}(\mathrm{Cl}(I))$ is also holonomic. Moreover it is straightforward to see that $\mathrm{in}_{(-w,w)}(\mathrm{Cl}(I)) \subset \mathrm{Cl}(\mathrm{in}_{(-w,w)}(I))$, and hence $\mathrm{Cl}(\mathrm{in}_{(-w,w)}(I))$ is holonomic as well, as required. $\square$

The *Newton polytope* of a polynomial f is the convex hull of the exponent vectors of f. For generic w, the exponent a of $\mathrm{in}_w(f) = c\boldsymbol{x}^a$ is a vertex of the Newton polytope of f, and since Lemmas 3.1.3 and 3.1.4 also imply that a is an exponent of the indicial ideal, we thus obtain the following corollary.

Corollary 3.1.5. *The Newton polytope of a polynomial solution f to I lies in the convex hull of all non-negative integer exponents of all indicial ideals of I.*

An efficient algorithm to compute indicial ideals is given in [40]. We use the notation $[\theta]_b = \prod_{i=1}^n \prod_{j=1}^{b_i-1}(\theta_i - j)$ for $b \in \mathbb{N}^n$.

Algorithm 3.1.6. (Finding the indicial ideal [40, Theorem 2.3.4])
INPUT: a left ideal $I \subset D$ and a generic weight vector $w \in \mathbb{R}^n$.
OUTPUT: generators of the inidicial ideal $\mathrm{ind}_{(-w,w)}(I)$.

1. Compute a torus invariant generating set $\mathcal{G} = \{\boldsymbol{x}^{\alpha_1} p_1(\boldsymbol{\theta})\boldsymbol{\partial}^{\beta_1}, \cdots, \boldsymbol{x}^{\alpha_r} p_r(\boldsymbol{\theta})\boldsymbol{\partial}^{\beta_r}\}$ of $\mathrm{in}_{-w,w}(I)$.

2. Return $\{[\boldsymbol{\theta}]_{\beta_1} p(\boldsymbol{\theta} - \beta_1), \cdots, [\boldsymbol{\theta}]_{\beta_r} p(\boldsymbol{\theta} - \beta_r)\}$.

Example 3.1.7. Let us consider the ideal $I = D \cdot \{\theta_x + 3\theta_y - 7, \partial_x^3 - \partial_y\}$, which is the GKZ hypergeometric system associated to the matrix $A = [1, 3]$ and the parameter value $\beta = [7]$. Let the weight vector $w = (w_1, w_2)$. Then the initial form of $L_1 = \theta_x + 3\theta_y - 7$ with respect to $(-w, w)$ is always equal to itself. On the other hand, the initial form of $L_2 = \partial_x^3 - \partial_y$ depends on w. If $3w_1 < w_2$ then $\mathrm{in}_{(-w,w)}(L_2) = -\partial_y$, if $3w_1 > w_2$ then $\mathrm{in}_{(-w,w)}(L_2) = \partial_x^3$, and if $3w_1 = w_2$ then $\mathrm{in}_{(-w,w)}(L_2) = L_2$. In fact in this case, $\mathrm{in}_{(-w,w)}(L_1)$ and $\mathrm{in}_{(-w,w)}(L_2)$ always generate $\mathrm{in}_{(-w,w)}(I)$ which we can verify using Macaulay 2, hence we have,

w	typical $(-w, w)$	$\mathrm{in}_{(-w,w)}(I)$
$3w_1 < w_2$	$(1, 1, -1, -1)$	$D \cdot \{\theta_x + 3\theta_y - 7, -\partial_y\}$
$3w_1 > w_2$	$(-1, -1, 1, 1)$	$D \cdot \{\theta_x + 3\theta_y - 7, \partial_x^3\}$
$3w_1 = w_2$	$(1, 3, -1, -3)$	I

Thus the generic weight vectors are those with $3w_1 \neq w_2$. We now get the indicial ideals and exponents,

w	$\mathrm{ind}_{(-w,w)}(I)$	exponents
$3w_1 < w_2$	$D \cdot \{\theta_x - 7, \theta_y\}$	$(7,0)$
$3w_1 > w_2$	$D \cdot \left\{ \begin{array}{c} \theta_x + 3\theta_y - 7 \\ \theta_x(\theta_x - 1)(\theta_x - 2) \end{array} \right\}$	$(0, \frac{7}{3}), (1,2), (2, \frac{5}{3})$
$3w_1 = w_2$	$--$	$--$

The convex hull of the integer exponents $\{(7,0),(1,2)\}$ is the set $\{(7,0),(4,1),(1,2)\}$, hence the general polynomial solution has the form $f = c_0x^7 + c_1x^4y + c_2xy^2$. By applying the operators L_1 and L_2 to f, setting the coefficients to 0, and solving the resulting system of linear equations, we find that the polynomial solution space is one-dimensional and spanned by $f = x^7 + 210x^4y + 2520xy^2$.

It is not necessary to find all Gröbner deformations to obtain polynomial solutions. Let $b(s)$ be the generator of $\mathrm{in}_{(-w,w)}(I) \cap K[s]$, $s = \sum_{i=1}^n w_i\theta_i$. The polynomial $b(s)$ is called the b-function of I with respect to $(-w, w)$. When I is holonomic, the b-function is nonzero for any weight vector w. The next proposition follows from the definition of $b(s)$.

Proposition 3.1.8. *Let w be a strictly negative weight vector. Consider the b-function $b(s)$ of I with respect to $(-w, w)$ and let $-k_1$ be the smallest integer root of $b(s) = 0$. The polynomial solutions of I have the form*

$$\sum_{\{p \in \mathbb{N}^n \,:\, p_i \geq 0,\, -p\cdot w \leq k_1\}} c_p x^p. \tag{1.1}$$

Algorithm 3.1.9. (Finding the polynomial solutions by a Gröbner deformation)
INPUT: a finite rank left ideal $I \subset D$.
OUTPUT: the polynomial solutions of I.

1. Take a strictly negative weight vector w and compute the Gröbner deformation $\mathrm{in}_{(-w,w)}(I)$.

2. If $\mathrm{in}_{(-w,w)}(I)$ is holonomic, compute the b-function $b(s)$ with respect to $(-w, w)$, which is the monic generator of $\mathrm{in}_{(-w,w)}(I) \cap K[s]$, where $s = \sum_i w_i\theta_i$ (see e.g. [40, Algorithm 5.15] for this procedure). If $\mathrm{in}_{(-w,w)}(I)$ is not holonomic but w is generic, compute the indicial ideal $\mathrm{ind}_{(-w,w)}(I)$ using Algorithm 3.1.6, then set $b(s)$ to be the monic generator of $\mathrm{ind}_{(-w,w)}(I) \cap K[s]$. If $\mathrm{in}_{(-w,w)}(I)$ is not holonomic and w is not generic, pick another w and start over (alternatively one could compute the monic generator of $\mathrm{Cl}(\mathrm{in}_{-w,w}(I)) \cap K[s]$ and proceed, but this is inefficient in practice).

3. Compute the smallest non-positive integer root $-k_1$ of $b(s)$. If there are no non-positive integer roots, then there is no polynomial solution other than 0.

4. If there is a minimal integer root, then determine the coefficients c_p of (1.1) by solving linear equations for the coefficients.

Remark 3.1.10. The algorithm generalizes immediately to an algorithm for computing Laurent polynomial solutions by allowing for negative exponents as well. In this case, we need to use enough b-functions to bound the convex hulls in every direction. For example, it is enough to compute b-functions with respect to the $2n$ weight vectors $(-e_i, e_i)$ and $(e_i, -e_i)$ for $i = 1, \ldots, n$.

Example 3.1.11. The following is the system of partial differential equations for the Appell function $F_1(a, b, b', c)$ [2]:

$$\begin{aligned}&\theta_x(\theta_x + \theta_y + c - 1) - x(\theta_x + \theta_y + a)(\theta_x + b),\\ &\theta_y(\theta_x + \theta_y + c - 1) - y(\theta_x + \theta_y + a)(\theta_y + b'),\\ &(x - y)\partial_x\partial_y - b'\partial_x + b\partial_y\end{aligned}$$

where a, b, b', c are complex parameters. Let us demonstrate how Algorithm 3.1.9 works for the system of parameter values $(a, b, b', c) = (2, -3, -2, 5)$. First, we choose a strictly negative weight vector $w = (-1, -2)$ and compute the b-function $b(s)$, $s = -\theta_x - 2\theta_y$, which is the generator of the principal ideal $\mathrm{in}_{(-w,w)}(I) \cap \mathbf{Q}[-\theta_x - 2\theta_y]$. We can use the V-homogenization or the homogenized Weyl algebra to get the generator (see, e.g., [40, Section 1.2]). Second, we need to find the integer roots of the b-function $b(s) = 0$.

```
i1 : I = AppellF1({2,-3,-2,5})
                  3  2     2            2  2
o1 = ideal (- x Dx  - x y*Dx*Dy + x Dx  + x*y*Dx*Dy + 3x*y*Dy + ...

i2 : b = bFunction(I, {-1,-2})
      3     2
o2 = s  + 3s  - 28s

i3 : getIntRoots b
o3 = {0, 4, -7}
```

From Proposition 3.1.8, the highest $(-w)$-degree monomial cx^py^q in a polynomial solution gives rise to an integer solution $w_1p+w_2q = -p-2q$ of the b-function. Hence, the polynomial solutions are of the form

$$f = \sum_{p,q\geq 0, p+2q\leq 7} c_{pq}x^py^q.$$

Finally, we determine the coefficients c_{pq} by applying the differential operators to f and putting the results to 0. In Macaulay 2, we have

```
i2 : PolySols I
       3 2     3     9  2 2    12  3    72  2     36    2    63  2
o2 = {x y  - 3x y - --*x y  + --*x  + --*x y + --*x*y  - --*x  - ...
                     2          5       5       5          5
```

Thus we find that there is one polynomial solution,

$$\begin{aligned} & (-\frac{1}{21}y^2 + \frac{1}{7}y - \frac{4}{35})x^3 + (\frac{3}{14}y^2 - \frac{24}{35}y + \frac{3}{5})x^2 \\ + & (-\frac{12}{35}y^2 + \frac{6}{5}y - \frac{6}{5})x + \frac{1}{5}y^2 - \frac{4}{5}y + 1. \end{aligned}$$

3.2 Rational solutions by Gröbner deformations

For a linear ordinary differential equation $(a_n(x)\partial^n + \cdots + a_0(x)) \bullet f = 0$ with $a_i(x) \in K[x]$, a simple algorithm to find rational solutions consists of the following. Find the singular points, which are the roots λ_i of $a_n(x)$. For each λ_i, calculate the indicial polynomial p_i at λ_i, and let m_i be the minimum negative integer root of p_i. Also let N be the maximum non-negative integer root of the indicial polynomial at ∞. Then all rational solutions have the form $g(x)\prod_i(x - \lambda_i)^{m_i}$ with $g(x) \in K[x]$, where the total degree is $\leq N$.

To generalize this approach to many variables, we need to identify analogs of singular points and indicial polynomials. The analog of singular points is the singular locus, which was introduced in Chapter 2 as Definition 2.1.3. Moreover, Theorem 2.1.8 of Cauchy, Kovalevskii, and Kashiwara implies that any rational solution to I has its poles contained inside the singular locus of D/I. Thus if $f(\mathbf{x})$ defines the codimension 1 component of $\mathrm{Sing}(D/I)$, we may limit our search for rational solutions to $K[\mathbf{x}][f^{-1}]$.

We now would like a way to bound the order of the poles along $f = 0$ for rational solutions. For this purpose, the analog of the indicial polynomial is the notion of the b-function for f and a section u of a holonomic system, which was introduced by Kashiwara [27]:

Definition 3.2.1. *For a holonomic D-module M and a polynomial f, let*

$$N = \mathbb{C}[f^{-1}, s]f^s \otimes_{K[\mathbf{x}]} M, \tag{2.2}$$

which has a structure of a left $D[s]$-module via the Leibnitz rule. Let u be an element of M. Then the b-function for f and u (or for $f^s u$) at $p \in \mathbb{C}^n$ is the minimum degree monic polynomial $0 \neq b(s) \in \mathbb{C}[s]$ satisfying a functional equation of the form

$$b(s)f^s \otimes u = Lf(f^s \otimes u) \tag{2.3}$$

for some $L \in D[s][g^{-1}]$ with $g(p) \neq 0$. The global b-function for f and u (or for $f^s u$) is the minimum degree monic polynomial satisfying (2.3) for some $L \in D[s]$.

The b-function depends on the point p, and as a function of p, there is a stratification of $\mathbb{C}^n$ for which the b-function does not change on each stratum (see e.g. [32] for an algorithmic proof of this fact). The global b-function is the least common multiple of the b-functions at every point. The following theorem is due to my co-authors Oaku and Takayama.

Theorem 3.2.2. *Let u be the image of 1 in D/I, and let $b(s)$ be the b-function for f and u at a point $p \in \mathbb{C}^n$ where $f(p) = 0$. Assume that I admits an analytic solution of the form gf^r around p, where $r \in \mathbb{C}$, g is a holomorphic function on a neighborhood of p, and $g(p) \neq 0$. Then $s + r + 1$ divides $b(s)$.*

Proof. Let $\mathcal{D}^{\mathrm{an}}$ and $\mathcal{O}^{\mathrm{an}}$ be respectively the sheaf of analytic differential operators and the sheaf of holomorphic functions on $\mathbb{C}^n$. We may similarly define the analytic b-function of a $\mathcal{D}^{\mathrm{an}}$-module $\mathcal{M}$ at a point p as the minimum degree monic polynomial $b(s)$ such that $b(s)f^s \otimes u \in \mathcal{D}^{\mathrm{an}}[s]f(f^s \otimes u)$ holds in $\mathcal{N} = \mathcal{O}^{\mathrm{an}}[f^{-1}, s] \otimes_{\mathcal{O}^{\mathrm{an}}} \mathcal{M}$ at p. Since the b-function is an analytic invariant and the analytic and the algebraic b-functions coincide (see e.g. [32, Section 8]), we may work in the analytic category. We do this to consider solutions gf^r where g is holomorphic at p. If we only wish to consider solutions gf^r where g is a polynomial, then we may work in the algebraic category.

In general, given a map of left $\mathcal{D}^{\mathrm{an}}$-modules $\phi : \mathcal{M}_1^{\mathrm{an}} \to \mathcal{M}_2^{\mathrm{an}}$ and a section u of $\mathcal{M}_1^{\mathrm{an}}$, the b-function for $f^s u$ at a point p is divisible by the b-function for $f^s\phi(u)$ at p. We apply this basic fact to the following map φ. Let J^{an} be the annihilating ideal of gf^r in $\mathcal{D}^{\mathrm{an}}$. Since $J^{\mathrm{an}} \supseteq I^{\mathrm{an}} := \mathcal{D}^{\mathrm{an}} I$ and $g(p) \neq 0$, we have a left $\mathcal{D}^{\mathrm{an}}$-homomorphism

$$\varphi : \mathcal{D}^{\mathrm{an}}/I^{\mathrm{an}} \longrightarrow \mathcal{D}^{\mathrm{an}} g f^r = \mathcal{D}^{\mathrm{an}} f^r \hookrightarrow \mathcal{O}^{\mathrm{an}}[f^{-1}]f^r$$

which sends u to gf^r. This map extends to a left $\mathcal{D}^{\mathrm{an}}[s]$-homomorphism

$$\begin{aligned} 1 \otimes \varphi : \quad & \mathcal{O}^{\mathrm{an}}[f^{-1}, s]f^s \otimes_{\mathcal{O}^{\mathrm{an}}} \mathcal{D}^{\mathrm{an}}/I^{\mathrm{an}} \\ & \longrightarrow \mathcal{O}^{\mathrm{an}}[f^{-1}, s]f^s \otimes_{\mathcal{O}^{\mathrm{an}}} \mathcal{O}^{\mathrm{an}}[f^{-1}]f^r = \mathcal{O}^{\mathrm{an}}[f^{-1}, s]f^{s+r} \end{aligned}$$

which sends $f^s \otimes u$ to gf^{s+r}. By the definition of $b(s)$, there exists a germ $P(s)$ of $\mathcal{D}[s]$ at p such that

$$(P(s)f - b(s))(f^s \otimes u) = 0.$$

Since $1 \otimes \varphi$ is a left $\mathcal{D}^{\mathrm{an}}$-homomorphism, applying it to the above equation gives the equation $(P(s)f - b(s))(gf^{s+r}) = 0$, or in other words,

$$g^{-1}P(s)gf^{s+r+1} = b(s)f^{s+r}.$$

Thus, we see that the Bernstein-Sato polynomial $b_f(s)$ of f at p divides $b(s-r)$. Note that $s+1$ divides $b_f(s)$ since $f(p) = 0$ (see [26]). In conclusion, we have proved that $s+1$ divides $b(s-r)$. This completes the proof. □

By virtue of the above theorem, we can obtain upper bounds by computing the b-functions for $f_i^s u$ at a smooth point of each irreducible component $f_i = 0$ of the singular locus $f = 0$ of I where u is the image of 1 in D/I. From now on, let us take $f \in K[\mathbf{x}]$ to be a square-free polynomial defining the codimension one component of the singular locus, and let $f = f_1 \cdots f_m$ be its irreducible decomposition in $K[\mathbf{x}]$. Let us also fix u to be the image of 1 in D/I.

Theorem 3.2.3. *Let $b_i(s)$ be the b-function for $f_i^s u$ at a generic point of $f_i = 0$. Denote by r_i the maximum integer root of $b_i(s) = 0$. Then any rational solution (if any) to I can be written in the form $gf_1^{-r_1-1} \cdots f_m^{-r_m-1}$ with a polynomial $g \in \mathbb{C}[\mathbf{x}]$. If some $b_i(s)$ has no integral root, then there exist no rational solutions to I other than zero.*

Proof. An arbitrary rational solution to I can be written in the form $gf_1^{-\nu_1}\cdots f_m^{-\nu_m}$ with integers $\nu_1,\ldots,\nu_m$ and $g \in \mathbb{C}[\mathbf{x}]$. Since the space of the rational solutions with coefficients in $\mathbb{C}$ is spanned by those with coefficients in K, we may assume $g \in K[\boldsymbol{x}]$, and f and g are relatively prime in $K[\mathbf{x}]$. Let p be a generic point of $f_i = 0$. We may assume that f_i is smooth at p, $g(p) \neq 0$, and $f_j(p) \neq 0$ for $j \neq i$. It follows from Theorem 3.2.2 that $b_i(\nu_i - 1) = 0$. This implies $\nu_i \leq r_i + 1$. □

Since b-functions divide the global b-function, an upper bound can also be obtained from the global b-function.

Corollary 3.2.4. *Let $b_i(s)$ be the global b-function for $f_i^s u$, and let r_i denote the maximum integer root of $b_i(s) = 0$. Then any rational solution (if any) to I can be written in the form $gf_1^{-r_1-1}\cdots f_m^{-r_m-1}$ with a polynomial $g \in \mathbb{C}[\mathbf{x}]$.*

Example 3.2.5. Put $\theta_i = x_i\partial_i$. Then the GKZ system associated to the matrix and parameter,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

is the ideal,

$$H_A(\beta) = D \cdot \left\{ \begin{array}{l} L_1 = \theta_1 + \theta_2 + \theta_3 - \beta_1 \\ L_2 = \theta_2 + 2\theta_3 - \beta_2 \\ L_3 = \partial_1\partial_3 - \partial_2^2 \end{array} \right\}$$

The singular locus is $x_1x_3\Delta = 0$, where $\Delta = x_2^2 - 4x_1x_3$. Using an algorithm due to Oaku to compute b-functions which we will describe shortly, we find multiples of the global b-functions with respect to the various factors of the singular locus.

f_i	multiple of b-function wrt f_i
x_1	$(s+1)(s+\beta_1-\beta_2+1)$
x_3	$(s+1)(s+\beta_2-\beta_1+1)$
Δ	$(s+1)(s+\beta_1+3/2)$

For example, when $\beta_1 = 0$ and $\beta_2 = -1$, then

$$\left. \begin{array}{l} x_1 : (s+1)s \\ x_3 : (s+1)(s+2) \\ \Delta : (s+1)(s+3/2) \end{array} \right\} \Rightarrow \{\text{rational solutions of } I\} \subset x_1^{-1}\mathbb{C}[\boldsymbol{x}]$$

In fact, this GKZ system has rank 2 and its solutions are spanned by the roots $t = \zeta_1$ and $t = \zeta_2$ of the generic quadratic polynomial $x_1t^2 + x_2t + x_0 = 0$ regarded as functions $\zeta_1 = \zeta_1(x_1,x_2,x_3)$ and $\zeta_2 = \zeta_2(x_1,x_2,x_3)$ of the coefficients. By the quadratic formula these roots are $\zeta_1 = (-x_2 + \sqrt{\Delta})/2x_1$ and $\zeta_2 = (-x_2 - \sqrt{\Delta})/2x_1$, hence we see that $-\zeta_1 - \zeta_2 = x_2/x_1$ is the rational solution. Moreover, the solution $\zeta_1 - \zeta_2 = \sqrt{\Delta}/x_1$ gives rise to the root $-3/2$ of the b-function with respect Δ.

In [10], Cattani, D'Andrea, and Dickenstein show that the only rational solutions to GKZ systems coming from homogeneous matrices of size 2 by n are Laurent solutions. In other words, the discriminant Δ in our example above never appears as the pole of a rational solution, regardless of the parameter value β. We can also read this fact from the b-functions with respect to Δ. Namely, note that the ring $K[\boldsymbol{x}][\Delta^{-1}]$ is spanned as a K-vector space by elements of the form $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\Delta^j$, with $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{N}$ and $j \in \mathbb{Z}$. Next we note that $x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\Delta^j$ is an eigenvector of L_1 and L_2 with eigenvalues $(\alpha_1+\alpha_2+\alpha_3+2j-\beta_1)$ and $(\alpha_2+2\alpha_3+2j-\beta_2)$. It thus follows that rational solutions can occur only if $\beta_1, \beta_2 \in \mathbb{Z}$. If this is the case, then the only integer root of $b_\Delta(s)$ is -1, and hence Δ will not appear as a denominator. The question of which "discriminants" appear as poles of rational solutions to GKZ systems has been studied more generally by Cattani, Dickenstein, and Sturmfels in [11].

Example 3.2.6. We mention Corollary 3.2.4 since the algorithm to compute global b-functions is simpler than the algorithm to compute b-functions. However, the b-function offers finer information. For instance, the well-known example $f = x^2 + y^2 + z^2 + w^2$ has Bernstein-Sato polynomial $(s+1)(s+2)$ coming from the functional equation, $\frac{1}{4}(\partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_w^2) \cdot f^{s+1} = (s+1)(s+2)f^s$. Now consider the module $M = D \cdot f^{-1}$ and let u be the element $f^{-1} \in M$. Then M gets the presentation $M \simeq D/I$ where the generator corresponds to u and $I = \mathrm{Ann}_D(f^{-1})$. By construction, the only rational solution of I is f^{-1}. However, the global b-function for $f^s u$ is $s(s+1)$, and hence Corollary 3.2.4 implies that rational solutions of I all have the form gf^{-1} or gf^0, where g is a polynomial not divisible by f. On the other hand, the Bernstein-Sato polynomial of f at any nonsingular point p of $f=0$ (i.e. except for the origin) is $s+1$. It follows that the b-function for $f^s u$ equals s at the generic point of $f=0$ and hence Theorem 3.2.3 implies that all rational solutions of I actually have the form gf^{-1}.

An algorithm to compute the b-function and the global b-function for $f^s u$ was first given by Oaku in [32] based upon tensor product computation, which is memory intensive. Shortly thereafter, Walther refined this algorithm in [51] to give a more efficient method to compute the global b-function for $f^s u$. Both methods give the global b-function exactly, under the condition that I is f-saturated. Otherwise, we get a multiple of the global b-function. Here, an ideal I is said to be f-saturated if $(I : f^\infty) = I$. Similarly, Oaku's method of [32] gives the b-function exactly if I is f-saturated and additionally a certain primary decomposition in $\mathbb{C}[\mathbf{x}]$ is known. If primary decomposition is only available in $K[\mathbf{x}]$, we again get a multiple of the b-function.

Let us now describe an algorithm to compute the b-function for $f^s u$ at a generic point of $f=0$ by combining the method of [51] and the primary decomposition as was used in [32]. The following exposition is due to Oaku.

Algorithm 3.2.7. (Computing a multiple of the b-function at a generic point)
INPUT: a finite set G_0 of generators of a holonomic D-ideal I and an irreducible polynomial $f \in K[\mathbf{x}]$.
OUTPUT: $b'(s) \in K[s]$, which is a multiple of the b-function $b(s)$ for $f^s u$ at a generic point of $f=0$, where u is the image of 1 in D/I.

1. Introducing a new variable t, put $\vartheta_i = \partial_i + (\partial f/\partial x_i)\partial_t$. Let $\widetilde{I}$ be the left ideal of D_{n+1}, the Weyl algebra on the variables $x_1, \ldots, x_n, t$, that is generated by

$$\{P(\mathbf{x}, \vartheta_1, \ldots, \vartheta_n) \mid P(\mathbf{x}, \partial_1, \ldots, \partial_n) \in G_0\} \cup \{t - f(\mathbf{x})\}.$$

2. Let G_1 be a finite set of generators of the left ideal $\mathrm{in}_{(-1,0,\ldots,0;1,0,\ldots,0)}(\widetilde{I})$ of D_{n+1}. Here, -1 is the weight for t and 1 is the weight for ∂_t.

3. Rewrite each element P of G_1 in the form
$$P = \partial_t^{\mu} P'(t\partial_t, \mathbf{x}, \partial_1, \ldots, \partial_n) \quad \text{or} \quad P = t^{\mu} P'(t\partial_t, \mathbf{x}, \partial_1, \ldots, \partial_n)$$
with a non-negative integer μ, and define $\psi(P)$ by,
$$\psi(P) := t^{\mu}\partial_t^{\mu} P' = t\partial_t \cdots (t\partial_t - \mu + 1)P'(t\partial_t, \mathbf{x}, \partial_1, \ldots, \partial_n)$$
$$\text{or} \quad \psi(P) := \partial_t^{\mu} t^{\mu} P' = (t\partial_t + 1) \cdots (t\partial_t + \mu)P'(t\partial_t, \mathbf{x}, \partial_1, \ldots, \partial_n).$$
Put
$$G_2 := \{\psi(P)(-s-1, \mathbf{x}, \partial_1, \ldots, \partial_n) \mid P \in G_1\}.$$

4. Compute the elimination ideal $J := K[s, \mathbf{x}] \cap D[s]G_2$. (The global b-function can be obtained at this stage by computing the monic generator of the ideal $J \cap K[s]$.)

5. Compute a primary decomposition of J in $K[s, \mathbf{x}]$ as
$$J = Q_1 \cap \cdots \cap Q_{\nu}.$$

6. For each $i = 1, \ldots, \nu$, compute $Q_{ix} := Q_i \cap K[\mathbf{x}]$, which is a primary ideal of $K[\mathbf{x}]$.

7. Let $b'(s)$ be the monic generator of the ideal
$$\bigcap \{Q_i \cap K[s] \mid \sqrt{Q_{ix}} \subset K[\mathbf{x}]f\}$$
of $K[s]$. (Note that $\sqrt{Q_{ix}} \subset K[\mathbf{x}]f$ implies that $\sqrt{Q_{ix}}$ equals $K[\mathbf{x}]f$ or $\{0\}$.)

Theorem 3.2.8. *In the above algorithm, the polynomial $b'(s)$ is precisely the b-function for $f^s u$ at a generic point of $f = 0$ if I is f-saturated (i.e., $I : f^{\infty} = I$) and each $\mathbb{C}[s, \mathbf{x}]Q_i$ remains primary in $\mathbb{C}[s, \mathbf{x}]$. Otherwise, the polynomial $b'(s)$ is a multiple of the b-function for $f^s u$ at a generic point of $f = 0$.*

Proof. The proof follows by combining the proofs of [51, Lemma 4.1], [32, Theorem 6.14], and [32, Theorem 4.7]. □

Remark 3.2.9. Since I is f-saturated if and only if the torsion module $H^0_f(D/I)$ is nonzero, we may compute $H^0_f(I)$ and likewise the f-saturation $D[f^{-1}] \cdot I \cap D$ by using the methods of Chapter 2. Namely, if I is holonomic, then we may apply Oaku's torsion algorithm which was summarized in Algorithm 2.4.3. If I is not holonomic, then we apply Algorithms 2.4.3 and 2.4.5.

Remark 3.2.10. Only Steps 1 through 4 of Algorithm 3.2.7 for b-functions have been implemented in Macaulay 2. Thus, we have a script to compute multiples of global b-functions, but we do not currently have scripts to compute multiples of b-functions at a point p.

Once we have determined the integers $r_1, \ldots, r_m$ of Theorem 3.2.3, we can use Gröbner deformations to obtain the rational solutions. Put $k_i = -(r_i + 1)$. Then by virtue of Theorem 3.2.3, we have only to determine rational solutions of the form $g f_1^{k_1} \cdots f_m^{k_m}$ for some polynomial g. This amounts to computing polynomial solutions of some twisted ideal $I_{(k_1,\ldots,k_m)}$ of I.

Lemma 3.2.11. *Let $I \subset D$ be a left ideal generated by $\{P_1, \ldots, P_r\}$, let $f = f_1 \cdots f_m \in K[\boldsymbol{x}]$, and let $\{k_1, \ldots, k_m\} \subset \mathbb{C}$. For each generator P_i, let $a_i \in \mathbb{N}$ be sufficiently large so that $f^{a_i} P_i$ may be expressed as*

$$f^{a_i} P_i = p_i(x_1, \ldots, x_n, f\partial_1, \ldots, f\partial_n) \in K\langle x_1, \ldots, x_n, f\partial_1, \ldots, f\partial_n \rangle.$$

Now consider the ideal

$$I_{(k_1,\ldots,k_m)} := D\{p_i(x_1, \ldots, x_n, L_1, \ldots, L_n)\}_{i=1}^{m}$$

where

$$L_i = f\partial_i + \sum_{j=1}^{m} k_j \frac{f}{f_j} \frac{\partial f_j}{\partial x_i}. \tag{2.4}$$

Then the space V of polynomial solutions of $I_{(k_1,\ldots,k_m)}$ is isomorphic to the space W of solutions of I inside the $\mathbb{C}[\boldsymbol{x}]$-module $\mathbb{C}[\boldsymbol{x}] f_1^{k_1} \ldots f_m^{k_m}$ by the map $V \to W$ sending $g \mapsto g f_1^{k_1} \cdots f_m^{k_m}$. Moreover, $\operatorname{rank}(I) = \operatorname{rank}(I_{(k_1,\ldots,k_m)})$.

Proof. Consider how $f\partial_i$ acts on an element $g f_1^{k_1} \cdots f_m^{k_m} \in \mathbb{C}[\boldsymbol{x}] f_1^{k_1} \ldots f_m^{k_m}$:

$$f\partial_i \bullet (g f_1^{k_1} \cdots f_m^{k_m}) = \left(f \frac{\partial g}{\partial x_i} + \sum_{j=1}^{m} k_j \frac{f}{f_j} \frac{\partial f_j}{\partial x_i} g \right) f_1^{k_1} \cdots f_m^{k_m}.$$

In other words, $f\partial_i$ acts on the polynomial part g as the differential operator L_i of (2.4), and the part of the lemma on solutions follows.

Given a point p of I away from both the singular locus of I and the zero locus of f, then the map $V \to W$ also extends to a map between the holomorphic solution spaces of $I_{(k_1,\ldots,k_m)}$ and I at p (here a branch of $f_1^{k_1} \cdots f_m^{k_m}$ at p is chosen so that it may be regarded as a holomorphic function at p). Since rank of an ideal is generically equal to the dimension of the holomorphic solution space, the part of the lemma on rank follows. This can also be shown algebraically by observing that

$$\mathrm{in}_{(0,e)}(p(x_1, \ldots, x_n, f\partial_1, \ldots, f\partial_n)) = \mathrm{in}_{(0,e)}(p(x_1, \ldots, x_n, L_1, \ldots, L_n)).$$

We also remark that the definition of $I_{(k_1,\ldots,k_m)}$ is ambiguous but can be made well-defined by applying the Weyl closure operation. ☐

Algorithm 3.2.12. (Computing the rational solutions of a finite rank ideal)
INPUT: a finite rank left ideal $I \subset D$.
OUTPUT: a basis of the rational solutions $h \in K(\mathbf{x})$ of $I \bullet h = 0$.

1. Compute a polynomial f defining the codimension 1 component of $\mathrm{Sing}(I)$ using Algorithm 2.1.4.

2. Compute the irreducible decomposition $f = f_1 \cdots f_m$ in $K[\mathbf{x}]$.

3. For each $i = 1, \ldots, m$, compute the output $b'(s)$ of Algorithm 3.2.7 with I and f_i as input. Let r_i be the maximum integer root of $b'(s) = 0$ and put $k_i = -(r_i + 1)$. If $b'(s)$ has no integral root for some i, then there exists no rational solution other than zero.

4. Form the twisted ideal $I_{(k_1,\ldots,k_m)}$ described in Lemma 3.2.11.

5. Compute a basis $\{g_1, \ldots, g_k\}$ of the polynomial solutions of $I_{(k_1,\ldots,k_m)}$ using Algorithm 3.1.9.

6. Return $\{g_1 f_1^{k_1} \cdots f_m^{k_m}, \ldots, g_k f_1^{k_1} \cdots f_m^{k_m}\}$, a basis of the rational solutions of I.

Example 3.2.13. Let I be the left ideal generated by

$$L_1 = \theta_x(\theta_x + \theta_y) - x(\theta_x + \theta_y + 3)(\theta_x - 1)$$

$$L_2 = \theta_y(\theta_x + \theta_y) - y(\theta_x + \theta_y + 3)(\theta_y + 1).$$

The Appell function $F_1(3, -1, 1, 1; x, y)$ is a solution of this system. The singular locus of I is $xy(x-1)(y-1)(x-y) = 0$. We can compute the global b-functions of u along its components f_i using Macaulay 2:

```
i1 : factorBFunction globalBFunction(I,x)
               2
o1 = (s + 2) (s + 1)

i2 : factorBFunction globalBFunction(I,y)
o2 = (s + 1)(s)(s + 2)

i3 : factorBFunction globalBFunction(I,x-1)
o3 = (s + 1)(s)

i4 : factorBFunction globalBFunction(I,y-1)
o4 = (s - 2)(s + 1)

i5 : factorBFunction globalBFunction(I,x-y)
               2
o5 = (s + 2) (s + 1)(s + 3)
```

We conclude from this that any rational solution to I, if it exists, can be written in the form $g(x, y)y^{-1}(x-1)^{-1}(y-1)^{-3}$ with a polynomial g. Now we may compute the twisted ideal $I_{(0,-1,0,-1,-3)}$, where $f_1 = x, f_2 = y, f_3 = x - y, f_4 = x - 1, f_5 = y - 1$, and f is the product. Multiplying by f^2, we get the expressions,

$$\begin{aligned}
f^2 L_1 &= (x^2 - x^3)(f\partial_x)^2 + x((1-3x)f - (1-x)y\tfrac{\partial f}{\partial y} - (1-x)x\tfrac{\partial f}{\partial x})(f\partial_x) + \\
&\quad x(1-x)y(f\partial_y)(f\partial_x) + xyf(f\partial_y) + 3xf^2 \\
f^2 L_2 &= (y^2 - y^3)(f\partial_y)^2 + y((1-5y)f - (1-y)x\tfrac{\partial f}{\partial x} - (1-y)y\tfrac{\partial f}{\partial y})(f\partial_y) + \\
&\quad y(1-y)x(f\partial_x)(f\partial_y) - yxf(f\partial_x) - 3yf^2,
\end{aligned}$$

and we set T_1 and T_2 to be the operators obtained from the substitution of L_i into $f\partial_i$ as defined by (2.4). We remark that the ideal $I_{(0,-1,0,-1,-3)}$ generated by T_1 and T_2 is neither holonomic nor specializable with respect to the weight vector $(1, 1, -1, -1)$, hence we use the indicial ideal in step 2 of Algorithm 3.1.9 to get its polynomial solutions. Our Macaulay 2 script finds,

```
i6 :  RatSols(I, {x,y,x-1,y-1,x-y}, {10,1})

                                          3         2
              - x + y          -x*y   + 3*x*y   - 3*x*y + 4*x - 3*y
o6 = {--------------------, --------------------------------------}
           4     3     2                 4       3       2
        - y   + 3y   - 3y   + y        -y   + 3*y   - 3*y   + y
```

Here, the second argument to the function `RatSols` is a list of factors of the singular locus, and the third argument is a weight vector for the Gröbner deformation in Algorithm 3.1.9. After some simplification, we find that the rational function solutions are x/y and $(xy^2 - 3xy + 3x - 1)/(y-1)^3$.

Chapter 4

Homomorphisms between holonomic D-modules

Let $\mathrm{Hom}_D(M, N)$ denote the set of left D-module maps between two left D-modules M and N. Then $\mathrm{Hom}_D(M, N)$ is a K-vector space and can also be regarded as the solutions of M inside N. Namely, given a presentation $M \simeq D^{r_0}/D \cdot \{L_1, \ldots, L_{r_1}\}$, let S denote the system of vector-valued linear partial differential equations,

$$S = \{L_1 \bullet f = \cdots = L_{r_1} \bullet f = 0\},$$

and let $\mathrm{Sol}(S; N)$ denote the N-valued solutions $f \in N^{r_0}$ to S. Then the homomorphism space $\mathrm{Hom}_D(D^{r_0}/D \cdot \{L_1, \ldots, L_{r_1}\}, N)$ is isomorphic to the solution space $\mathrm{Sol}(S; N)$ where the identification is as follows. A homomorphism φ in $\mathrm{Hom}_D(D^{r_0}/D \cdot \{L_1, \ldots, L_{r_1}\}, N)$ corresponds to the solution $[\varphi(e_1), \ldots, \varphi(e_{r_0})]^T \in N^{r_0}$ of S, while a solution $f = [f_1, \ldots, f_{r_0}]^T \in N^{r_0}$ of S corresponds to the homomorphism which sends e_i to f_i.

If M and N are holonomic, then the set $\mathrm{Hom}_D(M, N)$ as well as the higher derived functors $\mathrm{Ext}^i_D(M, N)$ are finite-dimensional K-vector spaces. In this chapter, we give algorithms that compute explicit bases for $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}^i_D(M, N)$ in this situation. The material presented here is based on joint work with Oaku, Takayama, and Walther in [35], [49]. Algebraically, the problem of computing a basis of homomorphisms is easy to describe. Namely, since a map of left D-modules from M to N is uniquely determined by the images of a set of generators of M, we must simply determine which sets of elements of N constitute legal choices for the images of a homomorphism (of a fixed set of generators of M). It is perhaps surprising that this is a difficult computation. One of the reasons is that $\mathrm{Hom}_D(M, N)$ lacks any D-module structure in general and is just a K-vector space.

In recent years, one of the fundamental advances in computational D-modules has been the development of algorithms by Oaku and Takayama [32], [33] to compute the derived restriction modules $\mathrm{Tor}^D_i(D/\{x_1, \ldots, x_d\} \cdot D, M)$ and derived integration modules $\mathrm{Tor}^D_i(D/\{\partial_1, \ldots, \partial_d\} \cdot D, M)$ of a holonomic D-module M to a linear subspace $x_1 = \cdots = x_d = 0$. We give a summary of these algorithms in the Appendix. These algorithms have been the basis for the local cohomology algorithm discussed in Section 2.5 as well as for various deRham cohomology algorithms [34], [52], and they have been extended to algorithms for derived restriction and integration of complexes with holonomic cohomology by Walther [52].

Similarly, as was pointed out by Takayama, the computation of $\mathrm{Hom}_D(M, N)$ and $\mathrm{Ext}^i_D(M, N)$ can also be reduced to a restriction computation by using isomorphisms of Kashiwara [27] and Björk [7]. These isomorphisms are,

$$\mathrm{Ext}^i_D(M, N) \simeq \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), N), \tag{0.1}$$

which turns an Ext computation into a Tor computation and

$$\mathrm{Tor}^D_j(M', N) \simeq \mathrm{Tor}^{D_{2n}}_j(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}^n_{i=1} \cdot D_{2n}, \tau(M') \boxtimes N), \tag{0.2}$$

which turns a Tor computation into a twisted restriction computation in twice as many variables (an explanation of the notation used above can be found in Section 4.5).

In this chapter, we obtain an algorithm to compute an explicit basis of $\mathrm{Ext}^i_D(M, N)$ by analyzing the isomorphisms (0.1) and (0.2) and making them compatible with the restriction algorithm. In Section 4.1, we establish notation and review the construction of free resolutions and the holonomic dual. In Section 4.2, we present a proof of isomorphism (0.1) adapted from [7]. In Section 4.3, we give an algorithm for the case $N = K[x_1, \ldots, x_n]$, which is used to compute polynomial solutions of a system S. In Section 4.4, we give an algorithm for the case $N = K[x_1, \ldots, x_n][f^{-1}]$, which can be used to compute rational solutions of S. In Section 4.5, we give our main result, which is an algorithm to compute $\mathrm{Hom}_D(M, N)$ for general holonomic modules M, N. In Section 4.6, we give a companion algorithm which computes the derived functors $\mathrm{Ext}^i_D(M, N)$ and their representations as Yoneda Ext groups. In Section 4.7, we give an algorithm to determine whether M and N are isomorphic and if so to find an isomorphism. In the Appendix, we review Oaku and Takayama's restriction and integration algorithms.

4.1 Left versus right modules and the holonomic dual

Maps of left and right D-modules. Let us start by explaining our convention for writing maps of left or right D-modules. As usual, maps between finitely generated modules will be represented by matrices, but some attention has to be given to the order in which elements are multiplied due to the noncommutativity of D. Let us denote the identity matrix of size r by id_r, and similarly the identity map on the module M by id_M.

Given an $r \times s$ matrix $A = [a_{ij}]$ with entries in D, we get a map of free left D-modules,

$$D^r \xrightarrow{\cdot A} D^s \quad : \quad [\ell_1, \ldots, \ell_r] \mapsto [\ell_1, \ldots, \ell_r] \cdot A,$$

where D^r and D^s are regarded as modules of row vectors, and the map is matrix multiplication. Under this convention, the composition of maps $D^r \xrightarrow{\cdot A} D^s$ and $D^s \xrightarrow{\cdot B} D^t$ is the map $D^r \xrightarrow{\cdot AB} D^t$ where AB is usual matrix multiplication.

In general, suppose M and N are left D-modules with presentations D^r/M_0 and D^s/N_0. Then the matrix A induces a left D-module map between M and N, denoted $(D^r/M_0) \xrightarrow{\cdot A} (D^s/N_0)$, precisely when $L \cdot A \in N_0$ for all row vectors $L \in M_0$. This condition need only be checked for a generating set of M_0. Conversely, any map of left D-modules between M and N can be represented by some matrix A in the manner above.

Now let us discuss maps of right D-modules. The $r \times s$ matrix A also defines a map of right D-modules in the opposite direction,

$$(D^s)^T \xrightarrow{A\cdot} (D^r)^T \quad : \quad [\ell'_1, \ldots, \ell'_s]^T \mapsto A \cdot [\ell'_1, \ldots, \ell'_s]^T,$$

where the superscript-T means to regard the free modules $(D^s)^T$ and $(D^r)^T$ as consisting of column vectors. We will suppress the superscript-T when the context is clear. As before, the matrix A induces a right D-module map between right D-modules $N' = (D^s)^T/N'_0$ and $M' = (D^r)^T/M'_0$ whenever $A \cdot L \in M'_0$ for all column vectors $L \in N'_0$. We denote the map by $(D^s)^T/N'_0 \xrightarrow{A\cdot} (D^r)^T/M'_0$.

Free resolutions and the Ext modules. Given a left D-module M, a free resolution of M can be constructed using Gröbner bases. Under the convention described above, we write a resolution in the form,

$$X^\bullet : \cdots \to \underbrace{D^{r-a}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r-1} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \to M \to 0.$$

The Ext modules $\mathrm{Ext}^i_D(M, D)$ are the cohomology modules of the complex $\mathrm{Hom}_D(X^\bullet, D)$. We have an isomorphism $\mathrm{Hom}_D(D^r, D) \simeq (D^r)^T$ by identifying a homomorphism ϕ with the column vector $[\phi(e_1), \ldots, \phi(e_r)]^T$. Moreover, if we apply $\mathrm{Hom}_D(-, D)$ to a map $D^r \xrightarrow{\cdot A} D^s$ and use the identification $\mathrm{Hom}_D(D^r, D) \simeq (D^r)^T$, then the map $\mathrm{Hom}_D(\cdot A, D)$ is identified with $(D^s)^T \xrightarrow{A\cdot} (D^r)^T$. Thus, $\mathrm{Ext}^i_D(M, D)$ are the cohomology modules of the complex

$$\mathrm{Hom}_D(X^\bullet, D) : 0 \leftarrow \underbrace{(D^{r-a})^T}_{\text{degree } a} \xleftarrow{M_{-a+1}\cdot} \cdots \leftarrow (D^{r-1})^T \xleftarrow{M_0\cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0$$

The Ext modules of M are closely related to the dimension of M. Recall that the dimension of a D-module is the dimension of its characteristic variety. Let $j(M)$ be the smallest non-negative integer such that $\mathrm{Ext}^{j(M)}_D(M, D) \neq 0$.

Theorem 4.1.1. (see e.g. [7]) *Let M be a left D-module. Then*

1. $\dim(M) + j(M) = 2n$
2. $\dim(\mathrm{Ext}^i_D(M, D)) \leq 2n - i$. *In particular,* $\mathrm{Ext}^i_D(M, D) = 0$ *for* $i > n$.

Corollary 4.1.2. *If M is holonomic, then $\mathrm{Ext}^n_D(M, D)$ is the only nonzero Ext module and has dimension n.*

Left-right correspondence and the top differential forms. The category of left D-modules is equivalent to the category of right D-modules, and for convenience, we will sometimes prefer to work in one category rather than the other – for instance, we will phrase all algorithms in terms of left D-modules. In the Weyl algebra, the correspondence is given by the algebra involution

$$D \xrightarrow{\tau} D \quad : \quad x^\alpha \partial^\beta \mapsto (-\partial)^\beta x^\alpha.$$

The map τ is called the adjoint operator or standard transposition. Given a left D-module D^r/M_0, the corresponding right D-module is

$$\tau\left(\frac{D^r}{M_0}\right) := \frac{D^r}{\tau(M_0)}, \qquad \tau(M_0) = \{\tau(L) | L \in M_0\},$$

Similarly, given a homomorphism of left D-modules $\phi : D^r/M_0 \longrightarrow D^s/N_0$ defined by right multiplication by the $r \times s$ matrix $A = [a_{ij}]$, the corresponding homomorphism of right D-modules $\tau(\phi) : D^r/\tau(M_0) \longrightarrow D^s/\tau(N_0)$ is defined by left multiplication by the $s \times r$ matrix $\tau(A) := [\tau(a_{ij})]^T$. The map τ is used similarly to go from right to left D-modules.

On a theoretical level, the left-right correspondence is established through the module of top dimensional differential forms Ω, which is a right D-module under the action of the Lie derivative. This correspondence works more generally for modules over rings of differential operators on smooth algebraic varieties, but we only describe the case of the Weyl algebra below. Given a left D-module M, there is a corresponding right D-module $\Omega \otimes_{K[\mathbf{x}]} M$ where the structure is given by extending the actions,

$$(w \otimes m)f = wf \otimes m \qquad (w \otimes m)\xi = w\xi \otimes m - w \otimes \xi m$$

for $w \in \Omega$, $m \in M$, $f \in K[\mathbf{x}]$ and $\xi \in \mathrm{Der}(K[\mathbf{x}])$. Given a presentation D^r/M_0 for M with generators denoted $\{e_i\}_{i=1}^r$, then in $\Omega \otimes_{K[\mathbf{x}]} M$ we have

$$(1 \otimes e_i)x^\alpha \partial^\beta = (1 \otimes x^\alpha e_i)\partial^\beta = 1 \otimes (-\partial)^\beta x^\alpha e_i = 1 \otimes \tau(x^\alpha \partial^\beta) e_i.$$

It follows that $\Omega \otimes_{K[\mathbf{x}]} M$ is generated by $\{1 \otimes e_i\}_{i=1}^r$ and gets the presentation $D_n^r/\tau(M_0)$.

Conversely, given a right D-module N, there is a corresponding left D-module $\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N)$ where the structure is given by extending the actions,

$$(f\varphi)(w) = \varphi(w)f \qquad (\xi\varphi)(w) = \varphi(w\xi) - \varphi(w)\xi$$

for $\varphi \in \mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N)$, $w \in \Omega$, $f \in K[\mathbf{x}]$, and $\xi \in \mathrm{Der}(K[\mathbf{x}])$. A morphism $\varphi \in \mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N)$ can be identified with its image $\varphi(1) \in N$. Since

$$\begin{aligned} (x^\alpha \partial^\beta \varphi)(1) &= (x^\alpha(\partial^\beta \varphi))(1) = (\partial^\beta \varphi)(1)x^\alpha = \varphi(1)(-\partial)^\beta x^\alpha \\ &= \varphi(1)\tau(x^\alpha \partial^\beta), \end{aligned}$$

the morphism $x^\alpha \partial^\beta \varphi$ gets identified with $\varphi(1)\tau(x^\alpha \partial^\beta)$. In particular, given a presentation D^s/N_0 of N, then $\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N)$ is generated as a left D-module by the morphisms $\{\varphi_i\}_{i=1}^s$ such that $\varphi_i(1) = e_i$. By the computation above, a relation $\sum_i e_i g_i = 0$ in N corresponds to a relation $\sum_i \tau(g_i)\varphi_i$ in $\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N)$ because $(\sum_i \tau(g_i)\varphi_i)(1) = \sum_i e_i \tau(\tau(g_i)) = \sum_i e_i g_i$. It follows that $\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N)$ is generated by $\{\varphi_i\}_{i=1}^s$ and gets the presentation

$$\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, N) \simeq D_n^s/\tau(N_0). \tag{1.3}$$

Holonomic dual. As we saw in Corollary 4.1.2, if M is holonomic then the only nonzero Ext module is $\mathrm{Ext}^n_D(M, D)$, which is a right D-module of dimension n. By applying the left-right correspondence, we thus get a duality for the category of holonomic left D-modules.

Definition 4.1.3. *The holonomic dual of a holonomic left D-module M, denoted* $\mathbf{D}(M)$, *is the holonomic left D-module,* $\mathbf{D}(M) := \mathrm{Hom}_{K[x]}(\Omega, \mathrm{Ext}^n_D(M, D)) \simeq \tau(\mathrm{Ext}^n_D(M, D))$.

Let us discuss how the holonomic dual $\mathbf{D}(M)$ can be computed.

Algorithm 4.1.4. (Computing the holonomic dual)
INPUT: a presentation $M \simeq D^{r_0}/D \cdot \{L_1, \dots, L_{r_1}\}$, of a holonomic left D-module.
OUTPUT: the holonomic dual $\mathbf{D}(M)$.

1. Compute the first $n+1$ steps of any free resolution of M. Let the n-th part of the resolution be $D^p \xrightarrow{\cdot P} D^q \xrightarrow{\cdot Q} D^r$.

2. Dualize and apply the adjoint operator (recall if $P = [p_{ij}]$, then $\tau(P) = [\tau(p_{ij})]^T$) to get $D^p \xleftarrow{\cdot\tau(P)} D^q \xleftarrow{\cdot\tau(Q)} D^r$.

3. Return $\ker(\cdot\tau(P))/\mathrm{im}(\cdot\tau(Q))$.

Proof: Let the first $n+1$ steps of a free resolution of M be denoted,

$$F^\bullet : D^{r_{n+1}} \xrightarrow{\cdot P} D^{r_n} \xrightarrow{\cdot Q} D^{r_{n-1}} \to \cdots \to D^{r_0} \to 0.$$

Applying $\mathrm{Hom}_D(D, -)$ yields a complex of right D-modules,

$$\mathrm{Hom}_D(D, F^\bullet) : (D^{r_{n+1}})^T \xleftarrow{P\cdot} (D^{r_n})^T \xleftarrow{Q\cdot} (D^{r_{n-1}})^T \leftarrow \cdots \leftarrow (D^{r_0})^T \leftarrow 0,$$

and by definition,

$$\mathrm{Ext}^n_D(M, D) \simeq \frac{\ker(D^{r_{n+1}} \xleftarrow{P\cdot} D^{r_n})}{\mathrm{im}(D^{r_n} \xleftarrow{Q\cdot} D^{r_{n-1}})}.$$

Since $\mathbf{D}(M) = \mathrm{Hom}_{K[\mathbf{x}]}(\Omega, \mathrm{Ext}^n_D(M, D))$, it only remains to determine the effect of applying $\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, -)$. Using the equation (1.3), if $\{\vec{L}_1, \dots, \vec{L}_k\}$ are generators of $K = \ker(D^{r_{n+1}} \xleftarrow{P\cdot} D^{r_n})$, and $\sum_i \vec{L}_i g_i \in I = \mathrm{im}(D^{r_n} \xleftarrow{Q\cdot} D^{r_{n-1}})$ is a relation, then the corresponding relation $\sum_i \tau(g_i)\varphi_i$ in $\mathrm{Hom}_{K[\mathbf{x}]}(\Omega, \mathrm{Ext}^n_D(M, D))$ can be realized as the relation $\sum_i \tau(\vec{L}_i g_i) = \tau(g_i)\tau(\vec{L}_i) \in \tau(I)$. It follows that

$$\mathbf{D}(M) \simeq \frac{\ker(D^{r_{n+1}} \xleftarrow{\cdot\tau(P)} D^{r_n})}{\mathrm{im}(D^{r_n} \xleftarrow{\cdot\tau(Q)} D^{r_{n-1}})},$$

which is the output of step 3. $\square$

Example 4.1.5. The Appell system $F_1(2,-3,-2,5)$ of Example 3.1.11 has the resolution $0 \to D^1 \overset{\cdot Q_1}{\to} D^2 \overset{\cdot Q_0}{\to} D^1 \to 0$, where

$$Q_0 = \begin{bmatrix} (\theta_x - 3)\partial_y - (\theta_y - 2)\partial_x \\ (y^2 - y)(\partial_x\partial_y + \partial_y^2) - 2(y+x)\partial_x + 4y\partial_y + 2\partial_x - 8\partial_y - 4 \end{bmatrix}^T$$

$$Q_1 = \begin{bmatrix} (y^2 - y)(\partial_x\partial_y + \partial_y^2) - 2x\partial_x + 6y\partial_y + \partial_x - 9\partial_y \\ -(\theta_x - 3)\partial_y + (\theta_y - 1)\partial_x \end{bmatrix}$$

The holonomic dual $\mathbf{D}(F_1(2,-3,-2,5))$ is the cokernel of $\tau(Q_1)$ and is the Appell system $F_1(-1,4,2,-3)$. We verify this in Macaulay 2 by,

```
i1 : (I = AppellF1({2,-3,-2,5});
      W = ring I;
      Idual = Ddual I)
o1 = cokernel {0} | xDxDy-1yDxDy-2Dx-4/-1Dy y2DxDy-1/-1y2Dy^2-1yDxDy+...

i2 : ideal relations Idual == AppellF1({-1,4,2,-3})
o2 = true
```

4.2 Homological isomorphism

The following identification, which we take from Björk [7], is our main theoretical tool to explicitly compute homomorphisms of holonomic D-modules.

Theorem 4.2.1. [7] *Let M and N be holonomic left D-modules. Then*

$$\mathrm{Ext}^i_D(M,N) \cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M,D),N). \tag{2.4}$$

Proof. Since it will be useful to us later, we give the main steps of the proof here. The interesting bit of the construction is the transformation of a Hom into a tensor product. The presentation is adapted from [7]. Let $X^\bullet$ be a free resolution of M,

$$X^\bullet : 0 \to D^{r_{-a}} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r_{-1}} \xrightarrow{\cdot M_0} D^{r_0} \to M \to 0$$

We may assume it is of finite length by virtue of Hilbert's syzygy theorem – namely, Schreyer's proof and method carries over to D (see e.g. [16]). The dual of $X^\bullet$ is the complex of right D-modules,

$$\mathrm{Hom}_D(X^\bullet, D) : 0 \leftarrow \underbrace{(D^{r_{-a}})^T}_{\text{degree } a} \xleftarrow{M_{-a+1}\cdot} \cdots \leftarrow (D^{r_{-1}})^T \xleftarrow{M_0\cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0$$

Since $\mathrm{Hom}_D(D^r, D) \otimes_D N \simeq \mathrm{Hom}_D(D^r, N)$, we see that $\mathrm{Hom}_D(X^\bullet, D) \otimes_D N \simeq \mathrm{Hom}_D(X^\bullet, N)$, whose cohomology groups are by definition $\mathrm{Ext}^i_D(M,N)$. Now as is customary, replace N by a free resolution $Y^\bullet$, which we may also take to be of finite length,

$$Y^\bullet : 0 \to D^{s_{-b}} \xrightarrow{\cdot N_{-b+1}} \cdots \to D^{s_{-1}} \xrightarrow{\cdot N_0} D^{s_0} \to N \to 0 \tag{2.5}$$

We get the double complex $\mathrm{Hom}_D(X^\bullet, D)\otimes_D Y^\bullet$,

$$\begin{tikzcd}
 & 0 & & & 0 & 0 & \\
0 & (D^{r_{-a}})^T\otimes_D D^{s_0} \arrow[l] \arrow[u] & \cdots \arrow[l, "(M_{-a+1}\cdot)\otimes\mathrm{id}_{s_0}"'] & (D^{r_{-1}})^T\otimes_D D^{s_0} \arrow[l] & & & \\
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & 0 & & 0 & 0 & \\
0 & (D^{r_{-a}})^T\otimes_D D^{s_0} \arrow[l] \arrow[u] & \cdots \arrow[l, "(M_{-a+1}\cdot)\otimes\mathrm{id}_{s_0}"'] & (D^{r_{-1}})^T\otimes_D D^{s_0} \arrow[l] \arrow[u] & (D^{r_0})^T\otimes_D D^{s_0} \arrow[l, "(M_0\cdot)\otimes\mathrm{id}_{s_0}"'] \arrow[u] & 0 \arrow[l] \\
0 & (D^{r_{-a}})^T\otimes_D D^{s_{-1}} \arrow[l] \arrow[u, "(-\mathrm{id}_{r_{-a}})^a\otimes(\cdot N_0)"] & \cdots \arrow[l, "(M_{-a+1}\cdot)\otimes\mathrm{id}_{s_{-1}}"'] & (D^{r_{-1}})^T\otimes_D D^{s_{-1}} \arrow[l] \arrow[u, "-\mathrm{id}_{r_{-1}}\otimes(\cdot N_0)"'] & (D^{r_0})^T\otimes_D D^{s_{-1}} \arrow[l, "(M_0\cdot)\otimes\mathrm{id}_{s_{-1}}"'] \arrow[u, "\mathrm{id}_{r_0}\otimes(\cdot N_0)"'] & 0 \arrow[l] \\
 & \vdots \arrow[u] & & \vdots \arrow[u] & \vdots \arrow[u] & \\
0 & (D^{r_{-a}})^T\otimes_D D^{s_{-b}} \arrow[l] \arrow[u, "(-\mathrm{id}_{r_{-a}})^a\otimes(\cdot N_{-b+1})"] & \cdots \arrow[l, "(M_{-a+1}\cdot)\otimes\mathrm{id}_{s_{-b}}"'] & (D^{r_{-1}})^T\otimes_D D^{s_{-b}} \arrow[l] \arrow[u, "-\mathrm{id}_{r_{-1}}\otimes(\cdot N_{-b+1})"'] & (D^{r_0})^T\otimes_D D^{s_{-b}} \arrow[l, "(M_0\cdot)\otimes\mathrm{id}_{s_{-b}}"'] \arrow[u, "\mathrm{id}_{r_0}\otimes(\cdot N_{-b+1})"'] & 0 \arrow[l] \\
 & 0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{2.6}$$

Since the columns of the double complex are exact except at positions in the top row, it follows that the cohomology of the total complex equals the cohomology of the complex induced on the table of E_1 terms (vertical cohomologies),

$$0 \leftarrow \underbrace{\mathrm{Hom}_D(D^{r_{-a}}, N)}_{\text{degree } a} \xleftarrow{\mathrm{Hom}_D((M_{-a+1}\cdot),N)} \cdots \xleftarrow{\mathrm{Hom}_D((M_0\cdot),N)} \underbrace{\mathrm{Hom}_D(D^{r_0}, N)}_{\text{degree } 0} \leftarrow 0 \tag{2.7}$$

As stated earlier, these cohomology groups are $\mathrm{Ext}^i_D(M,N)$.

On the other hand, since M is holonomic, the complex $\mathrm{Hom}_D(X^\bullet, D)$ is exact except in degree n, where its cohomology is by definition $\mathrm{Ext}^n_D(M,D)$. Hence the rows of the double complex are also exact except at positions in the n-th column, i.e. the column containing terms $(D^{r_{-n}}\otimes_D(-))$. It follows that the cohomology of the total complex also equals the cohomology of the complex induced on the other table of E_1 terms (horizontal cohomologies), which in this case is

$$0\to \mathrm{Ext}^n_D(M,D)\otimes_D D^{s_{-b}} \to \cdots \xrightarrow{\mathrm{id}_{\mathrm{Ext}^n_D(M,D)}\otimes(\cdot N_0)} \mathrm{Ext}^n_D(M,D)\otimes_D D^{s_0}\to 0 \tag{2.8}$$

By definition, the above complex has cohomology groups $\mathrm{Tor}^D_j(\mathrm{Ext}^n_D(M,D),N)$, which establishes the identification. $\square$

In the next few sections, our goal will be to compute an explicit basis of cohomology classes of the complex (2.7). In particular, the cohomology in degree 0 corresponds explicitly to $\mathrm{Hom}_D(M, N)$ because any map $\psi \in \mathrm{Hom}_D(D^{r_0}, N)$ which is in the degree 0 kernel, i.e. in

$$H^0(\mathrm{Hom}_D(D^{r_{-1}}, N) \xleftarrow{\mathrm{Hom}_D((M_0\cdot),N)} \underbrace{\mathrm{Hom}_D(D^{r_0}, N)}_{\text{degree } 0} \leftarrow 0), \tag{2.9}$$

factors through $M \simeq D^{r_0}/M_0$, hence defines a homomorphism $\overline{\psi} : M \to N$. The reason why it is hard to compute these cohomology classes is that the modules $\mathrm{Hom}_D(D^{r_i}, N)$ in the complex (2.7) are left D-modules while the maps $\mathrm{Hom}_D((M_i\cdot), N)$ are not maps of left D-modules. In the next few sections, we will explain how the ingredients of the proof of Theorem 4.2.1 can be combined with the restriction algorithm to compute the desired cohomology classes.

4.3 Polynomial solutions by duality

In this section, we give an algorithm to compute $\mathrm{Hom}_D(M, K[\mathbf{x}])$ for holonomic M. This vector space is more efficiently computed by Gröbner deformations as described in Chapter 3, but we wish to discuss this special case in order to introduce the general methodology.

For $N = K[\boldsymbol{x}]$, the isomorphism (2.4) of Theorem 4.2.1 specializes to

$$\mathrm{Ext}^i_D(M, K[\boldsymbol{x}]) \simeq \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), K[\boldsymbol{x}]). \tag{3.10}$$

If we are only interested in the dimensions of these vector spaces, then by applying the adjoint operator to the right-hand side, we get $\mathrm{Ext}^i_D(M, K[\boldsymbol{x}]) \simeq \mathrm{Tor}^D_{n-i}(\Omega, \mathbf{D}(M))$, which are the derived integrations of $\mathbf{D}(M)$ to the origin.

Algorithm 4.3.1. (Evaluating dimensions of polynomial solution spaces)
INPUT: a holonomic left D-module M.
OUTPUT: dimensions of $\mathrm{Ext}^i_D(M, K[\mathbf{x}])$.

1. Compute the dual $\mathbf{D}(M)$ using Algorithm 4.1.4
2. Compute the derived integrations of $\mathbf{D}(M)$ to the origin using Algorithm 4.8.7. They are finite dimensional vector spaces.
3. Return the dimensions.

Example 4.3.2. Let us use the duality method to evaluate the dimension of polynomial solutions of the Appell system $F_1(2, -3, -2, 5)$. We saw earlier that the holonomic dual is the Appell system $F_1(-1, 4, 2, -3)$, and hence we would like to know the dimensions of its derived integration to the origin. Using Macaulay 2, we find,

```
i1 : PolyExt AppellF1({2,-3,-2,5})
                        1
o1 = HashTable{0 => QQ }
                        1
               2 => QQ
                        2
               1 => QQ
```

Here, the output `i => QQ^j` means that $\dim \mathrm{Ext}^i_D(M, K[\mathbf{x}]) = j$. In other words, there is one polynomial solution, as we computed earlier.

To obtain an explicit basis of polynomial solutions, the proof of Theorem 4.2.1 also leads directly to an alternative algorithm. As a D-module, the polynomial ring has the presentation $K[\boldsymbol{x}] \simeq D/D \cdot \{\partial_1, \ldots, \partial_n\}$ and can be resolved by the Koszul complex,

$$K^\bullet : 0 \to \underbrace{D}_{\text{degree } n} \xrightarrow{\cdot[(-1)^{n-1}\partial_n, \cdots, \partial_1]} D^n \to \cdots \to D^n \xrightarrow{\cdot\begin{bmatrix}\partial_1 \\ \vdots \\ \partial_n\end{bmatrix}} \underbrace{D}_{\text{degree } 0} \to 0.$$

The complex (2.8) whose cohomology computes $\mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), K[\boldsymbol{x}])$ then specializes to $\mathrm{Ext}^n_D(M, D) \otimes_D K^\bullet$ and is equivalently the derived integration complex of $\mathrm{Ext}^n_D(M, D)$. in the category of right D-modules. Oaku and Takayama's integration algorithm can now be applied to obtain a basis of explicit cohomology classes in $H^n(\mathrm{Ext}^n_D(M, D) \otimes_D K^\bullet) \simeq \mathrm{Tor}^D_n(\mathrm{Ext}^n_D(M, D), K[\boldsymbol{x}])$. These classes can then be transferred via the double complex (2.6) to cohomology classes in the complex (2.9), where they represent homomorphisms in $\mathrm{Hom}_D(M, K[\boldsymbol{x}])$. The method and details are best illustrated through an example.

Example 4.3.3. Consider the GKZ hypergeometric system $M_A(\beta)$ associated to the matrix $A = [1, 2]$ and parameter vector $\beta = [5]$, i.e. the D-module associated to the equations,

$$u = \theta_1 + 2\theta_2 - 5 \qquad v = \partial_1^2 - \partial_2$$

A resolution for $M_A(\beta)$ is

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot[-v \;\; u+2]} D^2 \xrightarrow{\cdot\left[\begin{smallmatrix}u \\ v\end{smallmatrix}\right]} D^1 \to 0$$

while a resolution for $K[x_1, x_2]$ is the Koszul complex,

$$K^\bullet : 0 \to D \xrightarrow{\cdot[\partial_1, \partial_2]} D^2 \xrightarrow{\cdot\left[\begin{smallmatrix}\partial_2 \\ -\partial_1\end{smallmatrix}\right]} D \to 0$$

The augmented double complex $\mathrm{Hom}_D(X^\bullet, D) \otimes_D K^\bullet$ is

$$\begin{tikzcd}
 & K[x_1,x_2] & K[x_1,x_2]^2 \arrow[l, "{[-v\ \ u+2]\cdot}"'] & K[x_1,x_2] \arrow[l, "{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\cdot}"'] \\
\mathrm{Ext}^2_D(M_A(\beta), D) & D^1 \arrow[l] \arrow[u] & D^2 \arrow[l, "{[-v\ \ u+2]\cdot}"'] \arrow[u] & D^1 \arrow[l, "{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\cdot}"'] \arrow[u] \\
\mathrm{Ext}^2_D(M_A(\beta), D)^2 \arrow[u, "{\cdot\left[\begin{smallmatrix} \partial_2 \\ -\partial_1 \end{smallmatrix}\right]}"] & D^2 \arrow[l] \arrow[u, "{\cdot\left[\begin{smallmatrix} \partial_2 \\ -\partial_1 \end{smallmatrix}\right]}"] & D^4 \arrow[l, "{\left[\begin{smallmatrix} -v & u+2 & 0 & 0 \\ 0 & 0 & -v & u+2 \end{smallmatrix}\right]\cdot}"'] \arrow[u, "{\cdot\left[\begin{smallmatrix} \partial_2 & 0 \\ 0 & \partial_2 \\ -\partial_1 & 0 \\ 0 & -\partial_1 \end{smallmatrix}\right]}"] & D^2 \arrow[l, "{\left[\begin{smallmatrix} u & 0 \\ v & 0 \\ 0 & u \\ 0 & v \end{smallmatrix}\right]\cdot}"'] \arrow[u, "{\cdot\left[\begin{smallmatrix} \partial_2 \\ -\partial_1 \end{smallmatrix}\right]}"'] \\
\mathrm{Ext}^2_D(M_A(\beta), D) \arrow[u, "{\cdot[\partial_1\ \ \partial_2]}"] & \boxed{D^1} \arrow[l] \arrow[u, "{\cdot[\partial_1\ \ \partial_2]}"] & D^2 \arrow[l, "{[-v\ \ u+2]\cdot}"'] \arrow[u, "{\cdot\left[\begin{smallmatrix} \partial_1 & 0 & \partial_2 & 0 \\ 0 & \partial_1 & 0 & \partial_2 \end{smallmatrix}\right]}"] & D^1 \arrow[l, "{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\cdot}"'] \arrow[u, "{\cdot[\partial_1\ \ \partial_2]}"']
\end{tikzcd}$$

Here, we interpret an element of a module in the above diagram as a column vector for purposes of the horizontal maps and as a row vector for purposes of the vertical maps. The induced complex at the left-hand wall is the derived integration to the origin of $\mathrm{Ext}^2_D(M_A(\beta), D)$ in the category of right D-modules. Applying the integration algorithm, we find that the horizontal cohomology at the module $\boxed{D^1}$ in the bottom left-hand corner is 1-dimensional and spanned by the residue class of

$$L_{1,0} = -(2x_1^5x_2 - 40x_1^3x_2^2 + 120x_1x_2^3)\partial_1 - (x_1^6 - 30x_1^4x_2 + 180x_1^2x_2^2 - 120x_2^3).$$

We lift this class to a cohomology class of the complex induced at the top row via a "transfer" sequence in the total complex given schematically by

$$\begin{tikzcd}
 & D^2 & D^1 \ni L_{1,2} \arrow[l, "{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\cdot}"'] \\
D^2 & D^4 \ni L_{1,1} \arrow[l, "{\left[\begin{smallmatrix} -v & u+2 & 0 & 0 \\ 0 & 0 & -v & u+2 \end{smallmatrix}\right]\cdot}"'] \arrow[u, "{\cdot\left[\begin{smallmatrix} \partial_2 & 0 \\ 0 & \partial_2 \\ -\partial_1 & 0 \\ 0 & -\partial_1 \end{smallmatrix}\right]}"'] & \\
D^1 \ni L_{1,0} \arrow[u, "{\cdot[\partial_1\ \ \partial_2]}"] & &
\end{tikzcd}$$

In other words, $L_{1,1}$ is obtained by taking the image of $L_{1,0}$ under the vertical map and then a pre-image under the horizontal map, and similarly for $L_{1,2}$. We find that,

$$
\begin{aligned}
L_{1,1} &= \begin{bmatrix} 2x_1^5x_2 - 40x_1^3x_2^2 + 120x_1x_2^3 \\ -(x_1^5 - 20x_1^3x_2 + 60x_1x_2^2) \\ -(x_1^6 - 20x_1^4x_2 + 60x_1^2x_2^2) \\ (x_1^5 - 20x_1^3x_2 + 60x_1x_2^2)\partial_1 + (10x_1^4 - 120x_1^2x_2 + 120x_2^2) \end{bmatrix}, \\
L_{1,2} &= \begin{bmatrix} x_1^5 - 20x_1^3x_2 + 60x_1x_2^2 \end{bmatrix}.
\end{aligned}
$$

The space of polynomial solutions is spanned by the residue class of $L_{1,2}$ in $K[x_1, x_2]$, which is $x_1^5 - 20x_1^3x_2 + 60x_1x_2^2$.

Remark 4.3.4. The transfer sequence above is used to show that Tor is a balanced functor in Weibel [54, Section 2.7]. A generalization of the transfer sequence is also used to compute the cup product structure for deRham cohomology of the complement of an affine variety in [53].

From a practical standpoint, the method outlined above is not quite the final story. The detail we have left out is how Oaku and Takayama's integration algorithm actually computes the cohomology classes of a Koszul complex such as $\operatorname{Ext}^n_D(M, D) \otimes_D K^\bullet$. Their algorithm does not compute these classes directly. Rather, their method (phrased in terms of right D-modules) is to first compute a $\widetilde{V}$-adapted resolution $Z^\bullet$ of $\operatorname{Ext}^n_D(M, D)$. Then they give a technique to compute explicitly the cohomology of $Z^\bullet \otimes_D K[\boldsymbol{x}]$. This complex is quasi-isomorphic to $\operatorname{Ext}^n_D(M, D) \otimes_D K^\bullet$, and cohomology classes are transferred by setting up another double complex $Z^\bullet \otimes_D K^\bullet$. Thus, our method as described to compute polynomial solutions requires two transfers via two double complexes.

Given the true nature of the integration algorithm, the two transfers can be collapsed into a single step. Namely, we start with $\operatorname{Hom}_D(X^\bullet, D)$,

$$
\operatorname{Hom}_D(X^\bullet, D) : 0 \leftarrow \cdots \xleftarrow{M_{-n}\cdot} \underbrace{(D^{r_{-n}})^T}_{\text{degree } n} \xleftarrow{M_{-n+1}\cdot} \cdots \xleftarrow{M_0\cdot} \underbrace{(D^{r_0})^T}_{\text{degree } 0} \leftarrow 0
$$

which is exact except in cohomological degree n because M is holonomic. We are interested in explicit cohomology classes for $H^0(\operatorname{Hom}_D(X^\bullet, D) \otimes_D K[\boldsymbol{x}])$. To obtain them, we replace $\operatorname{Hom}_D(X^\bullet, D)$ with a quasi-isomorphic $\widetilde{V}$-adapted resolution $E^\bullet$ along with an explicit quasi-isomorphism $\pi_\bullet$ from $E^\bullet$ to $\operatorname{Hom}_D(X^\bullet, D)$. That is, we make a map π_n from a free module $(D^{s_{-n}})^T$ onto some choice of generators of $\ker(M_{-n}\cdot)$, take the pre-image P of $\operatorname{im}(M_{-n+1}\cdot)$ under π_n, and compute a $\widetilde{V}$-adapted resolution $E^\bullet$ of $D^{s_{-n}}/P$. Schematically,

$$
\begin{tikzcd}
0 & \frac{(D^{s_{-n}})^T}{P} \arrow[l] & (D^{s_{-n}})^T \arrow[l] \arrow[d, "\pi_n"'] & (D^{s_{-n+1}})^T \cdots \arrow[l, "N_{-n+1}\cdot"'] \arrow[d, dashed] & (D^{s_0})^T \arrow[l, "N_0\cdot"'] \arrow[d, dashed] & (D^{s_1})^T \arrow[l] & \cdots \arrow[l] \\
0 \leftarrow \cdots \leftarrow (D^{r_{-n-1}})^T & \underbrace{(D^{r_{-n}})^T}_{\text{degree } n} \arrow[l, "M_{-n}\cdot"'] & & (D^{r_{-n+1}})^T \cdots \arrow[ll, "M_{-n+1}\cdot"'] & \underbrace{(D^{r_0})^T}_{\text{degree } 0} \arrow[l, "M_0\cdot"'] & 0 \arrow[l] &
\end{tikzcd}
$$

Using the integration algorithm, the cohomology classes of $E^\bullet \otimes_D K[\boldsymbol{x}]$ can now be computed. In order to transfer them to $\mathrm{Hom}_D(X^\bullet, D) \otimes_D K[\boldsymbol{x}]$, a chain map lifting π_n is computed and utilized as suggested by the dashed arrows. We summarize the algorithm as follows. To keep computations in terms of left D-modules, we make use of the transposition τ at various places. Applying τ to the polynomial ring $K[\boldsymbol{x}]$ gives the top differential forms $\Omega = D/\{\partial_1, \ldots, \partial_n\} \cdot D = \tau(K[\boldsymbol{x}])$.

Algorithm 4.3.5. (Polynomial solutions by duality)
INPUT: $\{L_1, \ldots, L_{r_1}\} \subset D^{r_0}$ such that $M = D^{r_0}/D \cdot \{L_1, \ldots, L_{r_1}\}$ is holonomic.
OUTPUT: the polynomial solutions $R \in K[\mathbf{x}]^{r_0}$ of the system of differential equations given by $\{L_i \bullet R = 0 \ : \ i = 1, \ldots, r_1\}$.

1. Compute a free resolution $X^\bullet$ of M of length $n+1$. Let its part in cohomological degree $-n$ be denoted:

$$\cdots \to D^{r_{-n-1}} \xrightarrow{\cdot M_{-n}} D^{r_{-n}} \xrightarrow{\cdot M_{-n+1}} D^{r_{-n+1}} \to \cdots$$

2. Form the complex $\tau(\mathrm{Hom}_D(X^\bullet, D))$ obtained by dualizing $X^\bullet$ and then applying the standard transposition. Its part in cohomological degree n now looks like:

$$\cdots \leftarrow D^{r_{-n-1}} \xleftarrow{\cdot\tau(M_{-n})} D^{r_n} \xleftarrow{\cdot\tau(M_{-n+1})} D^{r_{-n+1}} \leftarrow \cdots$$

3. Compute a surjection $\pi_n : D^{s_{-n}} \twoheadrightarrow \ker(\cdot\tau(M_{-n}))$, and find the pre-image $\tau(P) := {\pi_n}^{-1}(\mathrm{im}\,(\cdot\tau(M_{-n+1})))$. This yields the presentation $D^{s_{-n}}/\tau(P) \simeq \tau(\mathrm{Ext}^n_D(M, D))$.

4. Compute the derived integration module $H^0((\Omega \otimes^L_D (D^{s_{-n}}/\tau(P))[n])$ using Algorithm 4.8.7. In particular, this algorithm produces

   (i.) A $\widetilde{V}$-adapted free resolution $E^\bullet$ of $D^{s_{-n}}/\tau(P)$ of length $n+1$,

$$E^\bullet : 0 \leftarrow \underbrace{D^{s_{-n}}}_{\text{degree } n} \leftarrow D^{s_{-n+1}} \leftarrow \cdots \leftarrow D^{s_{-1}} \leftarrow \underbrace{D^{s_0}}_{\text{degree } 0} \leftarrow D^{s_1}.$$

   (ii.) Elements $\{g_1, \ldots, g_k\} \subset D^{s_0}$ whose images modulo $\mathrm{im}(\Omega \otimes D^{s_1})$ form a basis for

$$H^0\left(\left(\Omega \otimes^L_D \left(\frac{D^{s_{-n}}}{\tau(P)}\right)\right)[n]\right) \simeq H^0(\Omega \otimes_D E^\bullet) \simeq \frac{\ker(\Omega \otimes_D D^{s_{-1}} \leftarrow \Omega \otimes_D D^{s_0})}{\mathrm{im}(\Omega \otimes_D D^{s_0} \leftarrow \Omega \otimes_D D^{s_1})}$$

5. Lift the map π_n to a chain map $\pi_\bullet : E^\bullet \to \tau(\mathrm{Hom}_D(X^\bullet, D))$. Denote these maps $\pi_i : D^{s_{-i}} \to D^{r_{-i}}$.

6. Evaluate $\{\tau(\pi_0(g_1)), \ldots, \tau(\pi_0(g_k))\}$ and let $\{R_1(\mathbf{x}), \ldots, R_k(\mathbf{x})\}$ be their images in $(D/D \cdot \{\partial_1, \ldots, \partial_n\})^{r_0} \simeq K[\mathbf{x}]^{r_0}$.

7. Return $\{R_1(\mathbf{x}), \ldots, R_k(\mathbf{x})\}$, a basis for the polynomial solutions to M.

Example 4.3.6. Let us return to the GKZ example and apply the revised algorithm. For Step 1, we have already computed a resolution $X^\bullet$. Its length equals the global homological dimension. Thus for Step 2, we get a complex which is a resolution for the holonomic dual,

$$\tau(\mathrm{Hom}(X^\bullet, D)) : 0 \leftarrow D^1 \xleftarrow{\cdot[v' \ u']} D^2 \xleftarrow{\cdot\left[\begin{smallmatrix} u'-2 \\ -v' \end{smallmatrix}\right]} D^1 \leftarrow 0,$$

where $u' = -(\theta_1 + 2\theta_2 + 6)$ and $v' = -(\partial_1^2 + \partial_2)$. For Step 3, it follows that we get the presentation,

$$\tau(\mathrm{Ext}_D(M, D)) \simeq \frac{D^1}{D \cdot \{u', v'\}} = \frac{D^1}{D \cdot \{\theta_1 + 2\theta_2 + 6, \partial_1^2 + \partial_2\}}$$

For Step 4, we compute the derived integration of this module. It turns out that the complex $\tau(\mathrm{Hom}(X^\bullet, D))$ is already a $\widetilde{V}$-adapted resolution when taken with the shifts $0 \leftarrow D^1[0] \leftarrow D^2[1,0] \leftarrow D^1[1] \leftarrow 0$. The integration b-function is $s-4$, hence according to the integration algorithm, $\Omega \otimes_D \tau(\mathrm{Hom}(X^\bullet, D))$ is quasi-isomorphic to the finite-dimensional subcomplex,

$$0 \leftarrow \widetilde{F}^4(\Omega^1[0]) \xleftarrow{\cdot[-v \ -u-1]} \widetilde{F}^4(\Omega^2[1,0]) \xleftarrow{\cdot\left[\begin{smallmatrix} -u-3 \\ v \end{smallmatrix}\right]} \widetilde{F}^4(\Omega^1[1]) \leftarrow 0.$$

Here, $\widetilde{F}^4(\Omega^1[1])$ is spanned by the 21 monomials of degree ≤ 5,

$$\{1, x_1, x_2, \ldots, x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4, x_2^5\},$$

while $\widetilde{F}^4(\Omega^2[0,1])$ is spanned by the 36 monomials

$$\begin{aligned} &\{1, x_1, x_2, \ldots, x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^4\} \cdot e_1 \quad \cup \\ &\quad \{1, x_1, x_2, \ldots, x_1^5, x_1^4x_2, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^4, x_2^5\} \cdot e_2. \end{aligned}$$

The matrix $\left[\begin{smallmatrix} u'-2 \\ v' \end{smallmatrix}\right]$ induces a map of K-vector spaces between them whose kernel is spanned by the degree 5 polynomial $R_1 = (x_1^5 - 20x_1^3x_2 + 60x_1x_2^2)$. The Macaulay 2 session is,

```
i1 : I = gkz(matrix{{1,2}}, {5})
                   2
o1 = ideal (D  - D , x D  + 2x D  - 5)
             1    2   1 1     2 2
i2 : PolySols (I, Alg => Duality)
         5      3          2
o2 = {2x  + 40x x  + 120x x }
        1      1 2        1 2
```

We use the option `Alg => Duality` in `PolySols` above to indicate that we would like to use Algorithm 4.3.5 based on duality as opposed to Algorithm 3.1.9 based on Gröbner deformations.

4.4 Rational solutions by duality

In this section, we give a duality algorithm to compute an explicit basis of the space $\mathrm{Hom}_D(M, K[\mathbf{x}][f^{-1}])$ for holonomic M. The method is essentially the same as the algorithm for polynomial solutions. Since a rational solution has its poles inside the singular locus of M, we obtain an algorithm to compute the rational solutions of M. Here, as otherwise, we shall use $N[f^{-1}]$ to denote $N \otimes_{K[\mathbf{x}]} K[\mathbf{x}][f^{-1}]$.

If we are only interested in dimensions of the vector spaces $\mathrm{Ext}^i_D(M, K[\boldsymbol{x}][f^{-1}])$, then the isomorphism (2.4) of Theorem 4.2.1 specializes to (see e.g the proof of Algorithm 4.4.3),

$$\begin{aligned}
\mathrm{Ext}^i_D(M, K[\boldsymbol{x}][f^{-1}]) &\simeq \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D), K[\boldsymbol{x}][f^{-1}]) \\
&\simeq \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M, D)[f^{-1}], K[\boldsymbol{x}]) \\
&\simeq \mathrm{Tor}^D_{n-i}(\Omega, \mathbf{D}(M)[f^{-1}])
\end{aligned}$$

On the vector space level, it thus suffices to compute the derived integrations of $\mathbf{D}(M)[f^{-1}]$ to the origin.

Algorithm 4.4.1. (Evaluating dimensions of rational solution spaces)
INPUT: a holonomic left D-module M.
OUTPUT: dimensions of $\mathrm{Ext}^i_D(M, K[\mathbf{x}][f^{-1}])$.

1. Compute a polynomial f vanishing on the codimension 1 component of the singular locus of M using Algorithm 2.1.4.

2. Compute the dual $\mathbf{D}(M)$ using Algorithm 4.1.4.

3. Compute the localization $\mathbf{D}(M)[f^{-1}]$ using Algorithm 2.2.3.

4. Compute the derived integrations of $\mathbf{D}(M)[f^{-1}]$ to the origin using Algorithm 4.8.7. They are finite dimensional vector spaces.

5. Return the dimensions.

Example 4.4.2. Let us now evaluate the dimension of rational solutions to the Appell system $M = F_1(2, -3, -2, 5)$. As we computed earlier, $\mathbf{D}(M) \simeq F_1(-1, 4, 2, -3)$. We first compute the singular locus of M,

```
i1 : singLocus (M = AppellF1({2,-3,-2,5}))
                3 2    2 3    3        3    2       2
o1 = ideal(x y  - x y  - x y + x*y  + x y - x*y )
```

It factors as $f = xy(x - y)(x - 1)(y - 1)$. It will be difficult to compute the localization of $\mathbf{D}(M)$ at the full polynomial f. Let us instead start by evaluating the dimension of solutions in $K[x, y][x^{-1}]$. Going through the algorithm, we find,

```
i2 : RatlExt (M, x)
                         2
o2 = HashTable{0 => QQ }
                         3
               2 => QQ
                         5
               1 => QQ
```

Since we already computed a polynomial solution, this means there is one rational solution with pole along x. Similarly, we get the exact same dimensions for $\mathrm{Ext}^i_D(M, K[x,y][y^{-1}])$, which means that there is also one rational solution with pole along y. We will list these rational solutions in Example 4.4.5. We now compute the rank,

```
i3 : Drank M
o3 = 3
```

which is 3, and therefore we have accounted for a full basis of solutions. We could nevertheless also compute,

```
i4 : RatlExt (M, f)
                         1
o4 = HashTable{0 => QQ }
                         2
               2 => QQ
                         3
               1 => QQ
```

where f is any of the polynomials $x-y$, $x-1$, or $y-1$. As expected, there are no rational solutions with poles along $x-y$, $x-1$, or $y-1$, but in all cases there are new Ext^1 and Ext^2.

An explicit algorithm for computing rational solutions can be made in a similar way as for the polynomial case.

Algorithm 4.4.3. (Rational solutions by duality)
INPUT: $\{L_1, \ldots, L_{r_1}\} \subset D^{r_0}$ such that $M = (D^{r_0}/D \cdot \{L_1, \ldots, L_{r_1}\})$ is holonomic.
OUTPUT: the rational solutions $R \in K(\mathbf{x})^{r_0}$ of the system of differential equations given by $\{L_i \bullet R = 0 : i = 1, \ldots, r_1\}$.

1. Compute a polynomial f vanishing on the codimension 1 component of the singular locus of M using Algorithm 2.1.4.

2. Compute a free resolution $X^\bullet$ of M up to length $n+1$. Let its part in cohomological degree $-n$ be denoted:

$$\cdots \to D^{r_{-n-1}} \xrightarrow{\cdot M_{-n}} D^{r_{-n}} \xrightarrow{\cdot M_{-n+1}} D^{r_{-n+1}} \to \cdots$$

3. Form the complex $\tau(\mathrm{Hom}_D(X^\bullet, D))$. Its part in cohomological degree n now looks like:

$$\cdots \leftarrow D^{r-n-1} \xleftarrow{\cdot\tau(M_{-n})} D^{r-n} \xleftarrow{\cdot\tau(M_{-n+1})} D^{r-n+1} \leftarrow \cdots$$

4. Compute a surjection

$$\varpi_n : D^{s-n} \twoheadrightarrow \ker(\cdot\tau(M_{-n})),$$

and find the preimage $\tau(P) := {\varpi_n}^{-1}(\mathrm{im}(\cdot\tau(M_{-n+1})))$. Denote by $\varpi_{f,n}$ the induced map on the localizations, $D^{s-n}[f^{-1}] \twoheadrightarrow \ker(\cdot\tau(M_{-n}))[f^{-1}]$.

5. Compute the localization of $D^{s-n}/\tau(P)$ at f using Algorithm 2.2.3. This produces a presentation,

$$\bar{\varphi} : \frac{D^{s-n}}{\tau(Q)} \xrightarrow{\simeq} \left(\frac{D^{s-n}[f^{-1}]}{\tau(P)[f^{-1}]}\right)$$

$$e_i \mathrm{mod} \tau(Q) \longrightarrow (e_i \mathrm{mod} \tau(P)) \otimes f^{-a_i}$$

6. Compute the derived integration $H^0((\Omega \otimes^L_D D^{s-n}/\tau(Q))[n])$ using Algorithm 4.8.7. In particular, this algorithm produces

   (i.) A $\widetilde{V}$-adapted free resolution $E^\bullet$ of $D^{s-n}/\tau(Q)$ of length $n+1$,

$$E^\bullet : 0 \leftarrow \underbrace{D^{s-n}}_{\text{degree } n} \leftarrow D^{s-n+1} \leftarrow \cdots \leftarrow D^{s-1} \leftarrow \underbrace{D^{s_0}}_{\text{degree } 0} \leftarrow D^{s_1}.$$

   (ii.) Elements $\{g_1, \ldots, g_k\} \subset D^{s_0}$ whose images form a basis for

$$H^0\left(\left(\Omega \otimes^L_D \frac{D^{s-n}}{\tau(Q)}\right)[n]\right) \simeq H^0(\Omega \otimes_D E^\bullet) \simeq \frac{\ker(\Omega\otimes_D D^{s-1} \leftarrow \Omega\otimes_D D^{s_0})}{\mathrm{im}(\Omega\otimes_D D^{s_0} \leftarrow \Omega\otimes_D D^{s_1})}$$

7. Let π_n be the composition

$$\varpi_{f,n} \circ \varphi : D^{s-n} \to D^{s-n}[f^{-1}] \to \ker(\cdot\tau(M_{-n}))[f^{-1}],$$

where $\varphi : D^{s-n} \longrightarrow D^{s-n}[f^{-1}]$ is the map defined by $e_i \mapsto e_i \otimes f^{-a_i}$. Lift π_n to a chain map $\pi_\bullet : E^\bullet \to \mathrm{Hom}_D(X^\bullet, D)[f^{-1}]$.

8. Evaluate $\{\tau(\pi_0(g_1)), \ldots, \tau(\pi_0(g_k))\} \subset D^{r_0}[f^{-1}]$ and let $\{R_1(\mathbf{x}), \ldots, R_k(\mathbf{x})\}$ be their residues in $(D[f^{-1}]/D[f^{-1}] \cdot \{\partial_1, \ldots, \partial_n\})^{r_0} \simeq K[\mathbf{x}][f^{-1}]^{r_0}$.

9. Return $\{R_1(\mathbf{x}), \ldots, R_k(\mathbf{x})\}$, a basis for the rational solutions to M.

Proof. Any rational solution of M has its poles contained inside the singular locus of M. The proof is now essentially the same as for the polynomial case. The space of rational solutions can be identified with the 0-th cohomology of the complex (2.7), which specializes to

$$\begin{aligned} \mathrm{Hom}_D(X^\bullet, D) \otimes_D K[\mathbf{x}][f^{-1}] &\simeq \mathrm{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_{D[f^{-1}]} K[\boldsymbol{x}][f^{-1}] \\ &\simeq \mathrm{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_{D[f^{-1}]} D[f^{-1}] \otimes_D K[\boldsymbol{x}] \\ &\simeq \mathrm{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_D K[\boldsymbol{x}] \end{aligned}$$

Since the complex $\mathrm{Hom}_D(X^\bullet, D)$ is exact except in cohomological degree n where its cohomology is $\mathrm{Ext}^n_D(M, D)$, and since localization is also exact, the localized complex $\mathrm{Hom}_D(X^\bullet, D)[f^{-1}]$ remains exact except in cohomological degree n where its cohomology becomes $\mathrm{Ext}^n_D(M, D)[f^{-1}]$. Hence $\mathrm{Hom}_D(X^\bullet, D)[f^{-1}] \otimes_D K[\boldsymbol{x}]$ computes the derived integration modules of $\mathrm{Ext}^n_D(M, D)[f^{-1}]$ in the category of right D-modules. The above algorithm computes cohomology classes for the derived integration modules and transfers them back to cohomology classes of $\mathrm{Hom}_D(X^\bullet, D) \otimes_D K[\mathbf{x}][f^{-1}]$. $\square$

Remark 4.4.4. Let us explain how the lifting of π_n to a chain map in Step 7 may be accomplished algorithmically. We wish to do computations in terms of D and not $D[f^{-1}]$. The idea is that localization is exact, hence any boundary in $\mathrm{Hom}_D(X^\bullet, D)[f^{-1}]$ is the localization of a boundary in $\mathrm{Hom}_D(X^\bullet, D)$. Suppose we have computed $\pi_j : D^{s_{-j}} \to D^{r_{-j}}[f^{-1}]$. Then to compute π_{j-1}, we first compute the images ℓ_i of $e_i \in D^{s_{j+1}}$ under $D^{s_{-j+1}} \to D^{s_{-j}} \xrightarrow{\pi_j} D^{r_{-j}}[f^{-1}]$. Because the existing $\pi_n, \ldots, \pi_j$ are the beginning of a chain map, the ℓ_i are in the image of $D^{r_{-j+1}}[f^{-1}] \to D^{r_{-j}}[f^{-1}]$. Now we use the fact that localization is exact, which means for sufficiently large m_i, $f^{m_i}\ell_i$ is in the image of $D^{r_{-j+1}} \to D^{r_{-j}}$. To find valid m_i, we can multiply ℓ_i by successively higher powers of f and test for membership at each step via Gröbner basis over D. Now compute any preimage P_i of $f^{m_i}\ell_i$ in $D^{r_{-j}}$. The map $\pi_{j-1} : D^{s_{-j+1}} \to D^{r_{-j+1}}[f^{-1}]$ may be defined by sending $e_i \mapsto f^{-m_i}P_i$.

Example 4.4.5. Let us continue Example 4.4.2 and obtain the solution of the Appell system $F_1(2, -3, -2, 5)$ with pole along x explicitly. We already computed the resolution,

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot M_{-1}} D^2 \xrightarrow{\cdot M_0} D^1 \to 0,$$

hence $\tau(\mathrm{Hom}_D(X^\bullet, D))$ is a resolution for $\tau(\mathrm{Ext}^n_D(M, D)) = F_1(-1, 4, 2, -3)$,

$$\tau(\mathrm{Hom}_D(X^\bullet, D)) : 0 \leftarrow D^1 \xleftarrow{\cdot \tau(M_{-1})} D^2 \xleftarrow{\cdot \tau(M_0)} D^1 \leftarrow 0.$$

where,

$$\begin{aligned}
\tau(M_{-1}) &= \begin{bmatrix} (\theta_x + 4)\partial_y - (\theta_y + 2)\partial_x \\ (y^2 - y)(\partial_x\partial_y + \partial_y^2) + 2(x+y)\partial_x - 2y\partial_y - 2\partial_x + 7\partial_y - 2 \end{bmatrix} \\
\tau(M_0) &= \begin{bmatrix} (y^2 - y)(\partial_x\partial_y + \partial_y^2) + 2(x+2y)\partial_x - 3\partial_x + 6\partial_y - 4 \\ -(\theta_x + 4)\partial_y + (\theta_y + 3)\partial_x \end{bmatrix}^T.
\end{aligned}$$

Using Algorithm 2.2.3, we compute that the localization has presentation

$$\bar{\varphi} : \frac{D}{\tau(Q)} \xrightarrow{\simeq} \left(\frac{D[x^{-1}]}{\mathrm{im}(\cdot\tau(M_{-1}))[x^{-1}]} \right) \simeq \tau(\mathrm{Ext}^n_D(M, D))[x^{-1}]$$

$$1 \bmod \tau(Q) \longrightarrow (1 \bmod \mathrm{im}(\cdot\tau(M_{-1}))) \otimes x^{-7}$$

where

$$\tau(Q) = D \cdot \left\{ \begin{array}{c} (\theta_x\theta_y + \theta_y^2 + 8\theta_y + 2\theta_x + 12) - (\theta_x + \theta_y + 4)\partial_y \\ (\theta_x\theta_y + 2\theta_x + 7\theta_y + 14) - (\theta_x + 10)x\partial_y \end{array} \right\}.$$

We then compute that $D/\tau(Q)$ has a $\widetilde{V}$-adapted resolution,

$$E^\bullet : 0 \longleftarrow D^1[0] \xleftarrow{\cdot\left[\begin{smallmatrix}u_1\\u_2\end{smallmatrix}\right]} D^2[0,-1] \xleftarrow{\cdot[v_1,v_2]} D^1[-1] \longleftarrow 0,$$

where

$$\begin{aligned}
u_1 &= x^2\partial_x\partial_y - xy\partial_x\partial_y - 2x\partial_x + 11x\partial_y - 7y\partial_y - 14\\
u_2 &= x^3\partial_x^2 + x^3\partial_x\partial_y - x^2\partial_x^2 - x^2\partial_x\partial_y + 16x^2\partial_x + 11x^2\partial_y +\\
&\quad 4xy\partial_y - 9x\partial_x - 11x\partial_y + 52x - 7\\
v_1 &= x^3\partial_x^2 + x^3\partial_x\partial_y - x^2\partial_x^2 - x^2\partial_x\partial_y + 16x^2\partial_x + 12x^2\partial_y +\\
&\quad 4xy\partial_y - 8x\partial_x - 11x\partial_y + 52x - 6\\
v_2 &= -x^2\partial_x\partial_y + xy\partial_x\partial_y + 2x\partial_x - 11x\partial_y + 6y\partial_y + 12.
\end{aligned}$$

We would like to construct a chain map $\pi_\bullet : E^\bullet \to \tau(\mathrm{Hom}_D(X^\bullet, D))$ which lifts the map $\pi_2 : D^1[0] \to D[x^{-1}]^1$ defined by $1 \mapsto x^{-7}$. To compute the next map $\pi_1 : D^2[0,-1] \to D[x^{-1}]^2$, we need to find preimages of the elements $\pi_2 \circ (\cdot[u_1,u_2]^T)(e_1) = \pi_2(u_1)$ and $\pi_2 \circ (\cdot[u_1,u_2]^T)(e_2) = \pi_2(u_2)$ under the map $(\cdot\tau(M_{-1})) : D^1[x^{-1}] \longleftarrow D^2[x^{-1}]$. Note that

$$\begin{aligned}
\pi_2 \circ (\cdot[u_1,u_2]^T)(e_1) &= u_1 \cdot x^{-7}\\
&= x^{-6}((\theta_x + 4)\partial_y - (\theta_y + 2)\partial_x)
\end{aligned}$$

It follows that $\pi_2 \circ (\cdot[u_1,u_2]^T)(x^6 e_1) = (\cdot\tau(M_{-1}))(e_1')$ so that we may set $\pi_1(e_1) = x^{-6}e_1'$. In a similar manner, we obtain the chain map,

$$\begin{tikzcd}
E^\bullet : 0 & D^1[0] \arrow[l] \arrow[d, "\pi_2"', "{\cdot[x^{-7}]}"] & D^2[0,-1] \arrow[l, "\cdot\left[\begin{smallmatrix}u_1\\u_2\end{smallmatrix}\right]"'] \arrow[d, "\pi_1"', "\cdot\left[\begin{smallmatrix}x^{-6} & 0\\ x^{-6}a & x^{-6}b\end{smallmatrix}\right]"] & D^1[-1] \arrow[l, "{\cdot[v_1,v_2]}"'] \arrow[d, "\pi_0"', "{\cdot[x^{-5}c]}"] & 0 \arrow[l] \\
\tau(\mathrm{Hom}_D(X^\bullet, D)) : 0 & D^1[x^{-1}] \arrow[l] & D^2[x^{-1}] \arrow[l, "\cdot\tau(M_{-1})"'] & D^1[x^{-1}] \arrow[l, "\cdot\tau(M_0)"'] & 0, \arrow[l]
\end{tikzcd}$$

where

$$\begin{aligned}
a &= -\tfrac{1}{2}(y^2 - y)(\partial_x + \partial_y) + x + 2y - \tfrac{9}{2}\\
b &= \tfrac{1}{2}(x - y)\partial_x + 2\\
c &= \tfrac{1}{2}(x - y)\partial_x + \tfrac{5}{2}
\end{aligned}$$

The integration b-function is $(s-11)(s-4)(s-1)$, hence according to the integration algorithm, $\Omega \otimes_D E^\bullet$ is quasi-isomorphic to its subcomplex $\widetilde{F}^{11}(\Omega \otimes_D E^\bullet)$. Using Macaulay 2, we find that $\ker(\Omega \xleftarrow{\cdot[v_1,v_2]} \Omega^2)$ is 2-dimensional and spanned by,

$$\begin{aligned}
g_1 &= x^9y - \tfrac{3}{2}x^9 - 6x^8y + \tfrac{48}{5}x^8 + \tfrac{72}{5}x^7y\\
&\quad -\tfrac{126}{5}x^7 - \tfrac{84}{5}x^6y + \tfrac{168}{5}x^6 + \tfrac{42}{5}x^5y - 21x^5\\
g_2 &= -x^3 + \tfrac{2}{7}x^2y + \tfrac{4}{7}x^2 - \tfrac{3}{14}xy - \tfrac{3}{28}x + \tfrac{1}{21}y
\end{aligned}$$

The residue class of $\tau(\pi_0(g_1))$ yields the polynomial solution,

$$\begin{aligned}
&(2y^2 - 6y + \frac{24}{5})x^3 + (-9y^2 + \frac{144}{5}y - \frac{165}{5})x^2\\
+\;&(\frac{72}{5}y^2 - \frac{252}{5}y + \frac{252}{5})x + (-\frac{42}{5}y^2 + \frac{168}{5}y - 42),
\end{aligned}$$

while the residue class of $\tau(\pi_0(g_2))$ yields the rational solution,

$$\left(-6x^4+4x^3y-\frac{6}{7}x^2y^2+4x^3-\frac{24}{7}x^2y+\frac{6}{7}xy^2-\frac{6}{7}x^2+\frac{6}{7}xy-\frac{5}{21}y^2\right)x^{-6}.$$

By similar methods, we obtain the rational solutions with pole along y,

$$\left(\begin{array}{c} x^3y^2-\frac{21}{4}x^2y^3+\frac{21}{2}xy^4-\frac{35}{4}y^5-\frac{5}{4}x^3y+6x^2y^2 \\ -\frac{21}{2}xy^3+7y^4+\frac{5}{12}x^3-\frac{15}{8}x^2y+3xy^2-\frac{7}{4}y^3 \end{array}\right)y^{-7}.$$

Together with the polynomial solution found in Example 3.1.11, these solutions span the holomorphic solution space in a neighborhood of any point away from $xy=0$.

Remark 4.4.6. In the next section, we give an algorithm to compute $\mathrm{Hom}_D(M,N)$ for arbitrary holonomic M and N. Using it with $N=K[\mathbf{x}][f^{-1}]$, we get a similar but computationally different duality method to compute rational solutions. The basic difference is that the algorithm of this section uses computations over D and in principle over $D[f^{-1}]$, while the algorithm of the next section uses computations over D_{2n}, the Weyl algebra in twice as many variables. From the computational perspective, we believe the algorithm of this section is more efficient.

4.5 Holonomic solutions

In this section, we give an algorithm to compute a basis of $\mathrm{Hom}_D(M,N)$ for holonomic left D-modules M and N. We will use the following notation. As before, D will denote the ring of differential operators in the variables $x_1,\ldots,x_n$ with derivations $\partial_1,\ldots,\partial_n$. Occasionally we will also write D_n or D_x for D. In a similar fashion, D_y will stand for the ring of differential operators in the variables $y_1,\ldots,y_n$ with derivations $\delta_1,\ldots,\delta_n$.

If X is a D_x-module and Y a D_y-module then we denote by $X\boxtimes Y$ the external product of X and Y. It equals the tensor product of X and Y over the field K, equipped with its natural structure as a module over $D_{2n}=D_x\boxtimes D_y$, the ring of differential operators in $x_1,\ldots,x_n,y_1,\ldots,y_n$ with derivations $\{\partial_i,\delta_j\}_{1\le i,j\le n}$. In addition, let η denote the algebra isomorphism,

$$\eta: D_{2n}\longrightarrow D_{2n}\qquad \left\{\begin{array}{ll} x_i\mapsto \frac{1}{2}x_i-\delta_i, & \partial_i\mapsto \frac{1}{2}y_i+\partial_i, \\ y_i\mapsto -\frac{1}{2}x_i-\delta_i, & \delta_i\mapsto \frac{1}{2}y_i-\partial_i \end{array}\right\}_{i=1}^{n},$$

and let Δ and Λ denote the right D_{2n}-modules,

$$\Delta := \frac{D_{2n}}{\{x_i-y_i,\partial_i+\delta_i : 1\le i\le n\}\cdot D_{2n}}\qquad \Lambda := \frac{D_{2n}}{\mathbf{x}D_{2n}+\mathbf{y}D_{2n}}=\eta(\Delta).$$

As mentioned in the introduction to this chapter, an algorithm to compute the dimensions of $\mathrm{Ext}^i_D(M,N)$ will be based upon the isomorphisms (0.1) and (0.2):

$$\begin{array}{l} \mathrm{Ext}^i_D(M,N)\cong \mathrm{Tor}^D_{n-i}(\mathrm{Ext}^n_D(M,D),N) \\ \mathrm{Tor}^D_j(M',N)\simeq \mathrm{Tor}^{D_{2n}}_j(D_{2n}/\{x_i-y_i,\partial_i+\delta_i\}_{i=1}^n\cdot D_{2n},\tau(M')\boxtimes N). \end{array}$$

Combining these isomorphisms where $M' = \mathrm{Ext}^n_D(M, D)$ produces

$$\mathrm{Ext}^i_D(M,N) \simeq \mathrm{Tor}^{D_{2n}}_j(D_{2n}/\{x_i - y_i, \partial_i + \delta_i\}_{i=1}^n \cdot D_{2n}, \tau(\mathrm{Ext}^n_D(M,D)) \boxtimes N) \tag{5.11}$$

In order to compute $\mathrm{Hom}_D(M,N)$ explicitly, we will trace the isomorphism (5.11). We explain how to do this step by step in the following algorithm. The motivation behind the algorithm is discussed in the proof.

Algorithm 4.5.1. (Holonomic solutions by duality)
INPUT: presentations $M = D^{r_0}/M_0$ and $N = D^{s_0}/N_0$ of holonomic left D-modules.
OUTPUT: a basis for $\mathrm{Hom}_D(M,N)$.

1. Compute finite free resolutions $X^\bullet$ and $Y^\bullet$ of M and N,

$$X^\bullet : 0 \to \underbrace{D^{r_{-a}}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r_{-1}} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \to M \to 0$$

$$Y^\bullet : 0 \to \underbrace{D^{s_{-b}}}_{\text{degree } -b} \xrightarrow{\cdot N_{-b+1}} \cdots \to D^{s_{-1}} \xrightarrow{\cdot N_0} \underbrace{D^{s_0}}_{\text{degree } 0} \to N \to 0$$

Also, dualize $X^\bullet$ and apply the standard transposition to obtain,

$$\tau(\mathrm{Hom}_D(X^\bullet, D)) : 0 \leftarrow \underbrace{D^{r_{-a}}}_{\text{degree } a} \xleftarrow{\cdot \tau(M_{-a+1})} \cdots \leftarrow D^{r_{-1}} \xleftarrow{\cdot \tau(M_0)} \underbrace{D^{r_0}}_{\text{degree } 0} \leftarrow 0.$$

2. Form the double complex $\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ of left D_{2n}-modules and its total complex

$$Z^\bullet : 0 \leftarrow \underbrace{{D_{2n}}^{t_a}}_{\text{degree } a} \leftarrow \cdots \leftarrow \underbrace{{D_{2n}}^{t_0}}_{\text{degree } 0} \leftarrow \cdots \leftarrow {D_{2n}}^{t_{-b}} \leftarrow 0$$

where

$${D_{2n}}^{t_k} = \bigoplus_{i-j=k} D^{r_{-i}} \boxtimes D^{s_{-j}}.$$

Let the part of $Z^\bullet$ in cohomological degree n be denoted,

$${D_{2n}}^{t_{n+1}} \xleftarrow{\cdot T_n} {D_{2n}}^{t_n} \xleftarrow{\cdot T_{n-1}} {D_{2n}}^{t_{n-1}}$$

3. Compute a surjection $\pi_n : {D_{2n}}^{u_n} \twoheadrightarrow \ker(\cdot \eta(T_n))$, and find generators for the preimage $P := \pi_n^{-1}(\mathrm{im}(\cdot \eta(T_{n-1})))$.

4. Compute the derived restriction $H^0((\Lambda \otimes^L_{D_{2n}} {D_{2n}}^{u_n}/P)[n])$ using Algorithm 4.8.6. In particular, this algorithm produces,

   (i.) A V-adapted free resolution $E^\bullet$ of D^{s_n}/P of length $n+1$,

$$E^\bullet : 0 \leftarrow \underbrace{{D_{2n}}^{u_n}}_{\text{degree } n} \leftarrow {D_{2n}}^{u_{n-1}} \leftarrow \cdots \leftarrow {D_{2n}}^{u_1} \leftarrow \underbrace{{D_{2n}}^{u_0}}_{\text{degree } 0} \leftarrow {D_{2n}}^{u_{-1}}.$$

(ii.) Elements $\{g_1, \ldots, g_k\} \subset {D_{2n}}^{u_0}$ whose residues in $\Lambda \otimes_{D_{2n}} E^\bullet$ form a basis for

$$H^0\left(\left(\Lambda \otimes^L_{D_{2n}} \frac{{D_{2n}}^{u_n}}{P}\right)[n]\right) \simeq H^0(\Lambda \otimes_{D_{2n}} E^\bullet) \simeq \frac{\ker\left(\Lambda \otimes_{D_{2n}} {D_{2n}}^{u_1} \leftarrow \Lambda \otimes_{D_{2n}} {D_{2n}}^{u_0}\right)}{\mathrm{im}\left(\Lambda \otimes_{D_{2n}} {D_{2n}}^{u_0} \leftarrow \Lambda \otimes_{D_{2n}} {D_{2n}}^{u_{-1}}\right)}$$

5. Lift the map π_n to a chain map $\pi_\bullet : E^\bullet \to \eta(Z^\bullet)$. Denote the maps $\pi_i : D^{u_i} \to D^{r_i}$.

6. Compute the image of each g_i under the composition of chain maps,

$$\begin{tikzcd}
E^\bullet \arrow[d, "\pi_0"'] & \Delta \otimes_{D_{2n}} Z^\bullet \arrow[r, "\simeq"] & \mathrm{Tot}^\bullet(\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet) \arrow[d, "p_1"'] \\
\eta(Z^\bullet) \arrow[r, "\eta^{-1}"] & Z^\bullet \arrow[u] & \mathrm{Hom}_D(X^\bullet, N)
\end{tikzcd}$$

Here p_1 is the projection onto $\mathrm{Hom}_D(X^\bullet, D) \otimes Y^0$ followed by factorization through N_0.

These are all chain maps of complexes of vector spaces. Step by step, we do the following. Evaluate $\{L_1 = \eta^{-1}(\pi_0(g_1)), \ldots, L_k = \eta^{-1}(\pi_0(g_k))\}$, and write each L_i in terms of the decomposition,

$$L_i = \oplus_j L_{i,j} \in \bigoplus_j D_x^{r-j} \boxtimes D_y^{s-j} \qquad \left(= {D_{2n}}^{t_0}\right).$$

Now re-express $L_{i,0}$ modulo $\{x_i - y_i, \partial_i + \delta_i : 1 \leq i \leq n\} \cdot D_{2n} \otimes_{D_{2n}} (D_x^{r_0} \boxtimes D_y^{s_0})$ so that x_i and ∂_j do not appear in any component. Using the identification $D_x^{r_0} \boxtimes D_y^{s_0} \simeq D_{2n}^{s_0} e_1 \oplus \cdots \oplus D_{2n}^{s_0} e_r$, where $\{e_i\}$ forms the canonical D-basis for $D_x^{r_0}$, we then get an expression

$$L_{i,0} = \ell_{i,1} e_1 + \cdots + \ell_{i,r_0} e_{r_0} \in (D_y)^{s_0} e_1 \oplus \cdots \oplus (D_y)^{s_0} e_{r_0}.$$

Let $\{\overline{\ell}_{i,1}, \ldots, \overline{\ell}_{i,r_0}\}$ be the residues in $(D^{s_0}/N_0) \simeq N$. Finally, set $\phi_i \in \mathrm{Hom}_D(M, N)$ to be the map induced by

$$\{e_1 \mapsto \bar{\ell}_{i,1}, e_2 \mapsto \bar{\ell}_{i,2}, \ldots, e_{r_0} \mapsto \bar{\ell}_{i,r_0}\}.$$

7. Return $\{\phi_1, \ldots, \phi_k\}$, a basis for $\mathrm{Hom}_D(M, N)$.

Proof. The main idea behind the algorithm is to adapt the proof of Theorem 4.2.1. In that proof, we saw that $\mathrm{Tot}^\bullet(\mathrm{Hom}(X^\bullet, D) \otimes_D Y^\bullet) \xrightarrow{p_1} \mathrm{Hom}_D(X^\bullet, N)$ is a quasi-isomorphism. Thus it suffices to compute explicit generating classes for

$$H^0(\mathrm{Tot}^\bullet(\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet)) \xrightarrow{\simeq} H^0(\mathrm{Hom}_D(X^\bullet, N)) \simeq \mathrm{Hom}_D(M, N).$$

Here, the double complex $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ is in some sense easier to digest because it consists entirely of free D-modules. However, it too only carries the structure of a complex of infinite-dimensional vector spaces, making its cohomology no easier to compute than the cohomology of $\mathrm{Hom}_D(X^\bullet, N)$.

Thus, we instead are led to consider the double complex $\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet$ of Step 2, whose total complex $T^\bullet$ does carry the structure of a complex of left D_{2n}-modules. Moreover, we can get back to the original double complex by "restricting back to the diagonal". In other words, we claim that as a double complex of vector spaces, $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ can be naturally identified with the double complex,

$$\Delta \otimes_D (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet).$$

To make the identification, first note that the natural map

$$D_y \longrightarrow \frac{D_{2n}}{\{x_i - y_i, \partial_i + \delta_i : 1 \le i \le n\} \cdot D_{2n}} = \Delta$$

is an isomorphism of left D_y-modules. Let $\{e_1, \ldots, e_r\}$ denote the canonical basis of a free module D^r. Then an arbitrary element of $\Delta \otimes_{D_{2n}} ({D_x}^r \boxtimes {D_y}^s)$ can be expressed uniquely as $\sum_k e_k \boxtimes m_k$, where $m_k \in {D_y}^s$. Similarly, an element of $D^r \otimes_D D^s$ can be expressed uniquely as $\sum_k e_k \otimes m_k$ where $m_k \in D^s$. Hence we get an isomorphic identification as D_n-modules of $\Delta \otimes_{D_{2n}} ({D_x}^r \boxtimes {D_y}^s)$ and $D^r \otimes_D D^s$. In particular, this shows that the modules appearing in the double complexes are the same.

It remains to show that the maps in the double complexes can also be identified. An arbitrary vertical map of $\Delta \otimes_{D_{2n}} (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ acts on an arbitrary element $\sum_k 1 \otimes e_k \boxtimes m_k$ according to,

$$\begin{array}{ccc}
\Delta \otimes_{D_{2n}} ({D_x}^{r_i} \boxtimes {D_y}^{s_j}) & & \sum_i (-1)^i e_k \boxtimes (\cdot N_j)(m_k) \\
{\scriptstyle \mathrm{id}_\Delta \otimes (-\mathrm{id}_{r_i})^i \boxtimes (\cdot N_j)} \Big\uparrow & & \Big\uparrow \\
\Delta \otimes_{D_{2n}} ({D_x}^{r_i} \boxtimes {D_y}^{s_{j+1}}) & & \sum_k 1 \otimes e_k \boxtimes m_k
\end{array}$$

This is exactly the way the corresponding vertical map in $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ works on the corresponding element:

$$\begin{array}{ccc}
{D_x}^{r_i} \otimes_D {D_y}^{s_j} & & \sum_k (-1)^i e_k \otimes (\cdot N_j)(m_k) \\
{\scriptstyle (-\mathrm{id}_{r_i})^i \otimes (\cdot N_j)} \Big\uparrow & & \Big\uparrow \\
{D_x}^{r_i} \otimes_D {D_y}^{s_{j+1}} & & \sum_k e_k \otimes m_k
\end{array}$$

Likewise, an arbitrary horizontal map of $\Delta \otimes_{D_{2n}} (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ acts on an arbitrary element according to,

$$\begin{array}{ccc}
\Delta \otimes_{D_{2n}} ({D_x}^{r_{i+1}} \boxtimes {D_y}^{s_j}) & \xrightarrow{\mathrm{id}_\Delta \otimes (\cdot \tau(M_i)) \boxtimes 1} & \Delta \otimes_{D_{2n}} ({D_x}^{r_i} \boxtimes {D_y}^{s_j}) \\
\sum_k 1 \otimes e_k \boxtimes m_k & \longrightarrow & \sum_k 1 \otimes (\cdot \tau(M_i))(e_k) \boxtimes m_k.
\end{array}$$

Here, we would like to re-express the image $\sum_k 1 \otimes (\cdot\tau(M_i))(e_k) \boxtimes m_k$ in the form $\sum_k 1 \otimes e_k \boxtimes n_k$. To help us, note the following computation in $\Delta \otimes_{D_{2n}} ({D_x}^r \boxtimes {D_y}^s)$:

$$(1 \otimes x^\alpha \partial^\beta e_i \boxtimes m) = 1 \otimes \partial^\beta e_i \boxtimes y^\alpha m = 1 \otimes e_i \boxtimes (-\delta)^\beta y^\alpha m = 1 \otimes e_i \boxtimes \tau(y^\alpha \delta^\beta) m.$$

Using it, we get that

$$\begin{aligned}
\sum_k 1 \otimes (\cdot\tau(M_i))(e_k) \boxtimes m_k &= \sum_k \sum_j 1 \otimes \tau(M_i)_{jk} e_j \boxtimes m_k \\
&= \sum_k \sum_j 1 \otimes e_j \boxtimes \tau(\tau(M_i)_{jk}) m_k \\
&= \sum_k \sum_j 1 \otimes e_j \boxtimes (M_i)_{jk} m_k
\end{aligned}$$

This is exactly the way the corresponding horizontal map in $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$ works on an arbitrary element:

$$\begin{array}{ccc}
D^{r_{i+1}} \otimes_D D^{s_j} & \xrightarrow{(M_i\cdot)\otimes \mathrm{id}_{s_j}} & D^{r_i} \otimes_D D^{s_j} \\
& & \\
\sum_k e_k \otimes m_k & \longrightarrow & \sum_k \sum_j e_j \otimes (M_i)_{jk} m_k
\end{array}$$

Thus, we have explicitly identified $\Delta \otimes_D (\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$ with $\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet$.

The task now becomes to compute explicit cohomology classes which are a basis for $H^0(\Delta \otimes_{D_{2n}} Z^\bullet)$. To do this, we note that $Z^\bullet$ is exact except in cohomological degree n, where its cohomology is $\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N$. This follows because $\tau(\mathrm{Hom}_D(X^\bullet, D))$ is exact by holonomicity except in degree n, where its cohomology is $\tau(\mathrm{Ext}^n_D(M, D))$, and $Y^\bullet$ is exact except in degree 0, where its cohomology is N. In other words, the complex $\Delta \otimes_{D_{2n}} Z^\bullet$ is in some sense a restriction complex. Namely, after applying the algebra isomorphism η, we get an honest restriction complex $\Lambda \otimes \eta(Z^\bullet)$ for the restriction of $\eta(\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N)$ to the origin (the restriction complex of a left D_{2n}-module M' is by definition $\Lambda \otimes^L_{D_{2n}} M'$).

We can thus compute the cohomology groups of $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$ by applying the restriction algorithm. However, since we are after explicit representatives for the cohomology classes, we need to use a presentation of $\eta(\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N)$ which is compatible with $\eta(Z^\bullet)$. This is the content of Step 3. Once equipped with a compatible presentation, we apply the restriction algorithm to it, which is the content of Step 4. This step produces explicit cohomology classes of $\Lambda \otimes_{D_{2n}} E^\bullet$, where $E^\bullet$ is a V-adapted resolution of $\eta(\tau(\mathrm{Ext}^n_D(M, D)) \boxtimes N)$. To then get explicit cohomology classes of $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$, we construct a chain map between $E^\bullet$ and $\eta(Z^\bullet)$, which is the content of Step 5. The cohomology classes can now be transported to $\Lambda \otimes_{D_{2n}} \eta(Z^\bullet)$ using the chain map, then to $\Delta \otimes_{D_{2n}} Z^\bullet$ using η^{-1}, then to $\mathrm{Tot}^\bullet(\mathrm{Hom}_D(X^\bullet, D) \otimes_D Y^\bullet)$ using the natural identification described earlier, and finally to the complex $\mathrm{Hom}_D(X^\bullet, N)$ using the natural augmentation map. These steps are all grouped together in Step 6. This completes the proof of the algorithm. □

Example 4.5.2. Let $M = D/D \cdot (\partial - 1)$ and $N = D/D \cdot (\partial - 1)^2$, where D is the first Weyl algebra. Then for Step 1, we have the resolutions,

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot(\partial-1)} D^1 \to 0 \qquad Y^\bullet : 0 \to D^1 \xrightarrow{\cdot(\partial-1)^2} D^1 \to 0$$

For Step 2, we form the complex $Z^\bullet = \mathrm{Tot}(\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$,

$$Z^\bullet : 0 \leftarrow \underbrace{D_2{}^1}_{\text{degree } 1} \xleftarrow{\cdot\begin{bmatrix}(\partial_x+1)\\(\partial_y-1)^2\end{bmatrix}} \underbrace{D_2{}^2}_{\text{degree } -1} \xleftarrow{\cdot[(\partial_y-1)^2, -(\partial_x+1)]} \underbrace{D_2{}^1}_{\text{degree } 0} \leftarrow 0$$

For Steps 3-5, we get the output,

$$\begin{tikzcd}
\eta(Z^\bullet) : 0 \leftarrow D_2{}^1 & & D_2{}^2 \arrow[ll, "\cdot\begin{bmatrix}\frac{1}{2}y+\partial_x+1\\(\frac{1}{2}y-\partial_x-1)^2\end{bmatrix}"'] & & D_2{}^1 \leftarrow 0 \arrow[ll, "\cdot[(\frac{1}{2}y-\partial_x-1)^2, -\frac{1}{2}y-\partial_x-1]"'] \\
E^\bullet : 0 \leftarrow D_2{}^1[0] \arrow[u, "\pi_1 = \cdot[1]"'] & & D_2{}^2[-1,2] \arrow[ll, "\cdot\begin{bmatrix}\frac{1}{2}y+\partial_x+1\\y^2\end{bmatrix}"'] \arrow[u, "\pi_0 = \cdot\begin{bmatrix}1 & 0\\ \frac{3}{2}y-\partial_x-1 & 1\end{bmatrix}"'] & & D_2{}^1[1] \leftarrow 0 \arrow[ll, "\cdot[y^2, -\frac{1}{2}y-\partial_x-1]"']
\end{tikzcd}$$

The complex $E^\bullet$ is a V-adapted resolution of the cohomology of $\eta(Z^\bullet)$ at degree 1, and the restriction b-function is $b(s) = (s+1)(s+2)$. Hence $\Lambda \otimes_D E^\bullet$ is quasi-isomorphic to its sub-complex $F^{-1}(\Lambda \otimes_D E^\bullet)$

$$0 \leftarrow 0 \xleftarrow{\cdot\begin{bmatrix}\frac{1}{2}y+\partial_x+1\\y^2\end{bmatrix}} \mathrm{Span}_K \left\{ \begin{array}{c} 0 \oplus \overline{1} \\ 0 \oplus \overline{\partial_x} \\ 0 \oplus \overline{\partial_y} \end{array} \right\} \xleftarrow{\cdot[y^2, -\frac{1}{2}y-\partial_x-1]} \mathrm{Span}_K\{\overline{1}\} \leftarrow 0$$

Hence the cohomology $H^0(\Lambda \otimes_D E^\bullet)$ is spanned by $\{0 \oplus \overline{1}, 0 \oplus \overline{\partial_y}\}$. Applying π_0, we see that the vector space $H^0(\Lambda \otimes_D \eta(Z^\bullet))$ is spanned by the residue classes of $\{(\frac{3}{2}y - \partial_x - 1) \oplus 1, \partial_y(\frac{3}{2}y - \partial_x - 1) \oplus \partial_y\}$. Next applying η^{-1}, $H^0(\Delta \otimes_D Z^\bullet)$ is spanned by the residue classes of $\{L_1 = (\partial_x + 2\partial_y - 1) \oplus 1, L_2 = -\frac{1}{2}(x\partial_x + 2y\partial_y + y\partial_x + 2x\partial_y - x - y) \oplus -\frac{1}{2}(x+y)\}$. Modulo the right ideal generated by $\{x - y, \partial_x + \partial_y\}$, we can re-express these cohomology classes by $\{(\partial_y - 1) \oplus 1, (y\partial_y - y - 1) \oplus -y\}$. Applying p_1 we get $\{L_{1,0} = \partial_y - 1, L_{2,0} = y\partial_y - y - 1\}$, which corresponds to a basis of $\mathrm{Hom}_D(M, N)$ given by,

$$\phi_1 : \frac{D}{D \cdot (\partial - 1)} \xrightarrow{\cdot[\partial - 1]} \frac{D}{D \cdot (\partial - 1)^2}$$
$$\phi_2 : \frac{D}{D \cdot (\partial - 1)} \xrightarrow{\cdot[x\partial - x - 1]} \frac{D}{D \cdot (\partial - 1)^2}.$$

The Macaulay 2 session is,

```
i1 : (W = QQ[x,Dx,WeylAlgebra => {x=>Dx}];
      M = cokernel matrix{{Dx-1}};
      N = cokernel matrix{{(Dx-1)^2}};
      DHom(M,N))

o1 = {{0} | -1xDx+x+1 |, {0} | 1/-1Dx-1/-1 |}
```

Remark 4.5.3. When M is a holonomic D-module and N is an arbitrary finitely generated D-module, then $\mathrm{Hom}_D(M,N) = \mathrm{Hom}_D(M, \mathrm{Hol}(N))$, where $\mathrm{Hol}(N)$ denotes the maximal holonomic submodule of N. This submodule can be computed by a dualizing complex, as suggested in Remark 2.5.7. When M is not holonomic, then $\mathrm{Hom}_D(M,N)$ is in general not finite-dimensional, and as far as we know there is not a good theory for dealing with this situation.

4.6 Extensions of holonomic D-modules

In this section we explain how one can modify our algorithm for the computation of $\mathrm{Hom}_D(M,N)$ in order to compute explicitly the higher derived functors $\mathrm{Ext}^i_D(M,N)$ for holonomic D-modules M and N.

A useful way to represent $\mathrm{Ext}^i_D(M,N)$ is as the i-th Yoneda Ext group (see e.g. [54, Section 3.4]), which consists of equivalence classes of exact sequences,

$$\xi : 0 \to \ N \longrightarrow Q \longrightarrow X^{-i+2} \longrightarrow \cdots \longrightarrow X^0 \longrightarrow M \ \longrightarrow 0,$$

for any list of (not necessarily free) D-modules $Q, X^{-i+2}, \ldots, X^0$. Two exact sequences ξ and ξ' are considered equivalent when there is a chain map of the form,

$$\begin{tikzcd}
\xi : 0 \arrow[r] & N \arrow[r] \arrow[d, "\mathrm{id}_N"'] & Q \arrow[r] \arrow[d] & X^{-i+2} \arrow[r] \arrow[d] & \cdots \arrow[r] & X^0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "\mathrm{id}_M"] & 0 \\
\xi' : 0 \arrow[r] & N \arrow[r] & Q' \arrow[r] & X'^{-i+2} \arrow[r] & \cdots \arrow[r] & X'^{0} \arrow[r] & M \arrow[r] & 0.
\end{tikzcd}$$

In our modified algorithm we follow the same steps as in Algorithm 4.5.1, except that in Step 4 we compute $H^{-n+i}(\Lambda \otimes^L_{D_{2n}} (D_{2n}{}^{u_n}/P))$ instead of $H^{-n}(\Lambda \otimes^L_{D_{2n}} (D_{2n}{}^{u_n}/P))$. The output is a basis $\{\varphi_1, \ldots, \varphi_k\}$ of the finite-dimensional vector space $H^i(\mathrm{Hom}_D(X^\bullet, N))$, where $X^\bullet$ is a free resolution of M,

$$X^\bullet : 0 \to \underbrace{D^{r_{-a}}}_{\text{degree } -a} \xrightarrow{\cdot M_{-a+1}} \cdots \to D^{r_{-1}} \xrightarrow{\cdot M_0} \underbrace{D^{r_0}}_{\text{degree } 0} \to M \to 0.$$

To obtain the i-th Yoneda Ext group from our output for $\mathrm{Ext}^i_D(M,N)$, we follow the [54, Section 3.4] and associate to a cohomology class $\varphi \in H^i(\mathrm{Hom}_D(X^\bullet, N))$ the exact sequence,

$$\xi(\varphi) : 0 \longrightarrow N \longrightarrow Q \longrightarrow D^{r_{-i+2}} \longrightarrow \cdots \longrightarrow D^{r_0} \to M \to 0.$$

Here, Q is the cokernel of $(\cdot M_{-i+1}, \varphi) : D^{r_{-i}} \longrightarrow D^{r_{-i+1}} \oplus N$, and the maps are all the natural ones. Notice that the only difference between any $\xi(\varphi)$ and $\xi(\varphi')$ are their corresponding Q's and the maps to and from them. In terms of the basis $\{\varphi_1, \ldots, \varphi_k\}$ of $H^i(\mathrm{Hom}_D(X^\bullet, N))$, the set of possible Q's which appear can thus be packaged as the set,

$$V_i = \left\{ \frac{D^{r_{-i}} \oplus D^{s_0}}{(0 \oplus N_0) + D \cdot \{(\cdot M_{-i} + \sum_{h=1}^k \kappa_h \varphi_h)(e_j)\}_{j=1}^{r_{-i-1}}} \;\middle|\; (\kappa_1, \ldots, \kappa_k) \in K^k \right\}.$$

When $i = 1$, the 1-st Yoneda Ext group consists of equivalence classes of extensions of M by N,

$$\xi : 0 \to N \longrightarrow Q \longrightarrow M = (D^{r_0}/D^{r_{-1}} \cdot M_0) \to 0$$

where $Q = X^{-1} \oplus N$ modulo $X^{-2} \cdot M_{-1}$. Thus, once we have computed a basis $\{\varphi_1, \ldots, \varphi_k\}$ of $H^1(\mathrm{Hom}_D(X^\bullet, N))$ via the modified Algorithm 4.5.1, the possible extensions Q of M by N are,

$$V_1 = \left\{ \frac{D^{r_{-1}} \oplus D^{s_0}}{(0 \oplus N_0) + D \cdot \{(\cdot M_{-1} + \sum_{h=1}^{k} \kappa_h \varphi_h)(e_j)\}_{j=1}^{r-2}} \;\middle|\; (\kappa_1, \ldots, \kappa_k) \in K^k \right\}.$$

Example 4.6.1. Let $D = K\langle x, \partial\rangle$ be the first Weyl algebra and $M = D/D\cdot\partial$, $N = D/D\cdot x$. Then for Step 1 of Algorithm 4.5.1, we have the resolutions,

$$X^\bullet : 0 \to D^1 \xrightarrow{\cdot\partial} D^1 \to 0, \qquad Y^\bullet : 0 \to D^1 \xrightarrow{\cdot x} D^1 \to 0$$

For Step 2, we form the complex $Z^\bullet = \mathrm{Tot}(\tau(\mathrm{Hom}_D(X^\bullet, D)) \boxtimes Y^\bullet)$,

$$Z^\bullet : 0 \leftarrow \underbrace{{D_2}^1}_{\text{degree 1}} \xleftarrow{\cdot\left[\begin{smallmatrix}\partial_x \\ y\end{smallmatrix}\right]} {D_2}^2 \xleftarrow{\cdot[y, -\partial_x]} {D_2}^1 \leftarrow 0$$

For Steps 3-6, we find that $H^1(\Delta \otimes_{D_2} Z^\bullet)$ is spanned by $\{1\}$, and projecting by p_1, $\mathrm{Ext}^1_D(M, N) \simeq H^1(\mathrm{Hom}_D(X^\bullet, D/D\cdot x)$ is spanned by the natural projection $\varphi : D \to (D/D \cdot x)$. For $\kappa \in K$, the cohomology classes $\kappa\varphi$ correspond to the extensions on the bottom row of the following diagram,

$$\begin{tikzcd} 0 \arrow[r] & D \arrow[r, "\cdot\partial"] \arrow[d, "\cdot\kappa"'] & D \arrow[r] \arrow[d, "\cdot{[0,1]}"'] & \frac{D}{D\cdot\partial} \arrow[r] \arrow[d, "\mathrm{id}_{D/D\cdot\partial}"'] & 0 \\ 0 \arrow[r] & \frac{D}{D\cdot x} \arrow[r, "\cdot{[1,0]}"] & \frac{D\cdot e_1 \oplus D\cdot e_2}{D\cdot xe_1 + D\cdot(\kappa e_1 + \partial e_2)} \arrow[r, "\cdot\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]"] & \frac{D}{D\cdot\partial} \arrow[r] & 0 \end{tikzcd}$$

When $\kappa \neq 0$, the module $Q(\kappa) = (D \cdot e_1 \oplus D \cdot e_2)/(D \cdot xe_1 + D \cdot (\kappa e_1 + \partial e_2))$ is generated by e_2 and is always isomorphic to $D/D \cdot x\partial$. When $\kappa = 0$, the module is no longer generated by e_2 and is isomorphic to $D/D \cdot \partial \oplus D/D \cdot x$ but not to $D/D \cdot x\partial$.

In fact, the module $(D \cdot e_1 \oplus D \cdot e_2)/(D \cdot xe_1 + D \cdot (\kappa e_1 + \partial e_2))$ is always generated by the residue class of $e_1 + e_2$ and has the cyclic presentation $D/D \cdot \{\partial^2 x + \kappa x\partial, x^2\partial\}$ with respect to this generator. Using this presentation, the extensions take the form,

$$0 \to \frac{D}{D\cdot x} \xrightarrow{\cdot[-x\partial]} \frac{D}{D\cdot\{\partial^2 x + \kappa x\partial, x^2\partial\}} \xrightarrow{\cdot[x\partial+1]} \frac{D}{D\cdot\partial} \to 0.$$

Remark 4.6.2. The algorithm for computing Yoneda Ext explicitly is not completely implemented in Macaulay 2. Currently there is an implementation which returns the dimensions of the Ext groups. For instance, let $M(a, b)$ be the GKZ system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix} \quad , \quad \partial_1\partial_3 - \partial_2^2,$$

and let $N = D/D \cdot \{x_1, x_2, x_3\}$ be the module of the delta functions with support at the origin $(0,0,0)$. We wish to compute dimensions of the Ext groups between $M(a,b)$ and N. For $M(0,0)$, we find,

```
i1 : (A = matrix{{1,1,1},{0,1,2}};
      M = cokernel gens gkz(A, {0,0});
      N = cokernel matrix{{x_1,x_2,x_3}};
      DExt(M,N))

o1 = HashTable{0 => 0}
               1 => 0
               2 => 0
               3 => 0
```

On the other hand, for $\mathbf{D}(M(0,0)) \simeq M(-1,-1)$, we find

```
i3 : (A = matrix{{1,1,1},{0,1,2}};
      M = cokernel gens gkz(A, {-1,-1});
      N = cokernel matrix{{x_1,x_2,x_3}};
      DExt(M,N))

o3 = HashTable{0 => 0  }
                     1
               1 => QQ
                     1
               3 => QQ
                     2
               2 => QQ
```

4.7 Isomorphisms of holonomic D-modules

In this section, we give an algorithm to determine if two holonomic D-modules M and N are isomorphic and if so to produce an explicit isomorphism. We then make some well-known remarks on using the endomorphism ring $\mathrm{End}_D(M)$ of D-linear maps from M to itself to obtain a decomposition of M into a direct sum of indecomposables.

If holonomic M and N are isomorphic, then $\mathrm{Hom}_D(M,N) \simeq \mathrm{End}_D(M)$ is a finite-dimensional K-algebra. In the theory of finite dimensional K-algebras, the Jacobson radical J is the intersection of all maximal left ideals of E, and it has the property that the quotient E/J is a semi-simple K-algebra. By the Wedderburn-Artin theorem, a semi-simple algebra is isomorphic to a direct product of matrix rings over division algebras, and hence by taking the algebraic closure, we find that $E/J \otimes_K \overline{K}$ is isomorphic to the direct product of matrix rings over the field $\overline{K}$. One consequence of this decomposition is that the non-units of $E/J \otimes_K \overline{K}$ form a determinantal hypersurface. In particular, the units of $E/J \otimes_K \overline{K}$ form a Zariski open set, and hence the units of E/J also form a Zariski open set. Moreover, units and non-units respect the Jacobson radical in the sense that if j is in the Jacobson radical of E and if u is a unit of E then $u+j$ is also a unit, and similarly, if n is not a unit of E then $n+j$ is not a unit. We can thus conclude the following lemma.

Lemma 4.7.1. *Let M be a holonomic D-module. Then the space of D-linear isomorphisms $\mathrm{Iso}_D(M)$ from M to itself is open in $\mathrm{End}_D(M)$ under the Zariski topology.*

The lemma says that if holonomic M and N are isomorphic then most maps from M to N are isomorphisms. We now give an algorithm to determine if M and N are isomorphic based on Algorithm 4.5.1 and Lemma 4.7.1.

Algorithm 4.7.2. *(Is M isomorphic to N?)*
INPUT: presentations $M \simeq D^{m_M}/D \cdot \{P_1, \ldots, P_a\}$ and $N \simeq D^{m_N}/D \cdot \{Q_1, \ldots, Q_b\}$ of left holonomic D-modules.
OUTPUT: "No" if $M \not\simeq N$; and "Yes" together with an isomorphism $\iota : M \to N$ if $M \simeq N$.

1. Compute bases $\{s_1, \ldots, s_\sigma\}$ and $\{t_1, \ldots, t_\tau\}$ for the vector spaces $V = \mathrm{Hom}_D(M, N)$ and $W = \mathrm{Hom}_D(N, M)$ using Algorithm 4.5.1, where s_i and t_j are respectively $m_M \times m_M$ and $m_N \times m_N$ matrices with entries in D representing homomorphisms by right multiplication. Recall that we veiw D^{m_M} and D^{m_N} as consisting of row vectors. If $\sigma \neq \tau$, return "No" and exit.

2. Introduce new indeterminates $\{\mu_i\}_1^\tau$ and $\{\nu\}_1^\tau$, and form the "generic homomorphisms" $\sum_i \mu_i s_i \in \mathrm{Hom}_D(M, N)$ and $\sum_j \nu_j t_j \in \mathrm{Hom}_D(N, M)$. Then the compositions $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j : M \to N \to M$ and $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i : N \to M \to N$ are respectively $m_M \times m_M$ and $m_N \times m_N$-matrices with entries in $D[\mu_1, \ldots, \mu_{m_M}, \nu_1, \ldots, \nu_{m_N}]$.

3. Reduce the rows of the matrix $\sum_{i,j} \mu_i \nu_j s_i \cdot t_j - \mathrm{id}_{m_M}$ modulo a Gröbner basis for $D \cdot \{P_1, \ldots, P_a\} \subset D^{m_M}$. Force this reduction to be zero by setting the coefficients (which are inhomogeneous bilinear polynomials in μ_i, ν_j) of every standard monomial in every entry to be zero. Collect these relations in the ideal $I_M \subset K[\mu_1, \ldots, \mu_{m_M}, \nu_1, \ldots, \nu_{m_N}]$

4. Similarly, reduce the rows of the matrix $\sum_{i,j} \mu_i \nu_j t_j \cdot s_i - \mathrm{id}_{m_N}$ modulo a Gröbner basis for $D \cdot \{Q_1, \ldots, Q_b\} \subset D^{m_N}$. Force this reduction to be zero by setting the coefficients of every standard monomial in every entry to be zero, and collect these relations in the ideal $I_N \subset K[\boldsymbol{\mu}, \boldsymbol{\nu}]$.

5. Put $I(V, W) = I_M + I_N \subset K[\boldsymbol{\mu}, \boldsymbol{\nu}]$. If $I(V, W)$ contains a unit, return "No" and exit.

6. Otherwise compute an isomorphism $\sum_{i=1}^\tau k_i s_i$ in $\mathrm{Hom}_D(M, N)$ by finding the first τ coordinates of any point in the zero locus of $I(V, W)$. For instance, we can do this by inductively finding $k_i \in K$ for each i from 1 to τ such that $I(V, W) + (\mu_1 - k_1, \ldots, \mu_i - k_i)$ is a proper ideal. At each step i, this can be accomplished by trying different numbers for k_i until a suitable choice is found.

7. Return "Yes" and the isomorphism $(\sum_{i=1}^\tau k_i s_i) : M \to N$.

Remark 4.7.3. Algorithm 4.7.2 can also be modified to detect whether M is a direct summand of N. Namely M is a direct summand of N if and only if the ideal I_M of step 3 is not the unit ideal. Similarly N is a direct summand of M if and only if the ideal I_N of step 4 is not the unit ideal

Proof. Reduction of the generic matrix $\sum_{i,j} \mu_i\nu_j s_i \cdot t_j - \mathrm{id}_{m_M}$ modulo $D \cdot \{P_1, \ldots, P_a\}$ in step 3 leads to a generic remainder which depends on the parameters μ_i, ν_j. Moreover, since a Gröbner basis of $D \cdot \{P_1, \ldots, P_a\}$ is parameter-free, this generic remainder has the property that its specialization to a fixed choice of parameters $\mu_i = a_i, \nu_j = b_j$ gives the remainder of $\sum_{i,j} a_i b_j s_i \cdot t_j - \mathrm{id}_{m_M}$ modulo $D \cdot \{P_1, \ldots, P_a\}$. Thus setting the remainder to zero in step 3 corresponds to deriving conditions on the parameters μ_i, ν_j which makes the endomorphism given by $\sum_{i,j} \mu_i\nu_j s_i \cdot t_j$ equal to the identity on M. This is possible if and only if M is a direct summand of N. The analogous statement holds for reduction of $\sum_{i,j} \mu_i\nu_j t_j \cdot s_i - \mathrm{id}_{m_N}$ modulo $D \cdot \{Q_1, \ldots, Q_b\}$ and setting its resulting remainder to zero. Here, setting a remainder to zero is equivalent to the vanishing of the coefficients of its standard monomials, and we collect these vanishing conditions in the ideal $I(V, W)$ of $K[\boldsymbol{\mu}, \boldsymbol{\nu}]$.

Now a linear combination $\sum_i a_i s_i : M \to N$ is an isomorphism with inverse $\sum b_j t_j : N \to M$ if and only if the composition $\sum_{i,j} a_i b_j s_i \cdot t_j$ is congruent to id_{m_M} modulo $D \cdot \{P_1, \ldots, P_a\}$ and the opposite composition $\sum_{i,j} a_i b_j t_j \cdot s_i$ is congruent to id_{m_N} modulo $D \cdot \{Q_1, \ldots, Q_b\}$. Thus the common zeroes $(a_1, \ldots, a_\tau, b_1, \ldots, b_\tau)$ of $I(V, W)$ correspond to isomorphisms $\sum_i a_i s_i$ and their inverses $\sum_j b_j t_j$. In particular, if $I(V, W)$ is the entire ring, which we detect by searching for 1 in a Gröbner basis of $I(V, W)$, then there are no isomorphisms.

On the other hand if $I(V, W)$ is proper, then M and N are isomorphic and we obtain an explicit isomorphism from finding any common solution of $I(V, W)$. By Lemma 4.7.1, the invertible homomorphisms from M to N are Zariski dense in the vector space $\mathrm{Hom}_D(M, N)$. Hence, a common solution can be explicitly found by by intersecting the zero locus of $I(V, W)$ with a suitable number of generic hyperplanes $\{\mu = k_i\}$. Because of denseness, each of these hyperplanes can be found in a finite number of steps. In other words, if $I(V, W) + \langle \mu_1 - k_1, \ldots, \mu - k_{i-1}\rangle$ is proper, then there are only finitely many k_i for which the sum $I(V, W) + \langle \mu_1 - k_1, \ldots, \mu - k_i\rangle$ is the unit ideal. □

Remark 4.7.4. Once we have specialized the μ_i in a common solution of $I(V, W)$, then the ν_j are determined because of the bilinear nature of the relations (which gives linear relations for the ν_j once all μ_i are chosen). This also means that if there is any solution, then the μ_i are rational functions in the ν_j and vice versa. In particular, if $\phi \in \mathrm{Hom}_D(M, N)$ is defined over the field K then ϕ^{-1} is defined over K as well and no field extensions are required. We now give two simple examples, one where M and N are isomorphic, and one where they are not.

Example 4.7.5. Let $n = 1$ and $M = N = D/D \cdot \partial^2$. One checks that $V = W = \mathrm{Hom}_D(M, N)$ is generated by the 4 morphisms $s_1 = \cdot(\partial)$, $s_2 = \cdot(x\partial)$, $s_3 = \cdot(1)$, and

$s_4 = \cdot(x^2\partial - x)$. We obtain the generic morphism

$$\begin{aligned}\sum_{i=1}^{4}\sum_{j=1}^{4}\mu_i\nu_j t_j \cdot s_i - 1 &= (\mu_3\nu_3 - \mu_1\nu_4 - 1)\\ &\quad + (-\mu_4\nu_3 - \mu_2\nu_4 - \mu_3\nu_4)x\\ &\quad + (\mu_3\nu_1 + \mu_1\nu_2 + \mu_1\nu_3)\partial\\ &\quad + (-\mu_4\nu_1 + \mu_2\nu_2 + \mu_3\nu_2 + \mu_2\nu_3 + \mu_1\nu_4)x\partial\\ &\quad + (\mu_4\nu_3 + \mu_2\nu_4 + \mu_3\nu_4)x^2\partial\end{aligned}$$

plus 9 other terms which are in $D \cdot \partial^2$ independently of the parameters.

Hence in order for $\sum_{i=1}^4 \mu_i s_i$ to be an isomorphism, the μ_i need to be part of a solution to the ideal

$$\begin{aligned}I(V, W) &= (\mu_3\nu_3 - \mu_1\nu_4 - 1,\\ &\quad -\mu_4\nu_3 - \mu_2\nu_4 - \mu_3\nu_4,\\ &\quad \mu_3\nu_1 + \mu_1\nu_2 + \mu_1\nu_3,\\ &\quad -\mu_4\nu_1 + \mu_2\nu_2 + \mu_3\nu_2 + \mu_2\nu_3 + \mu_1\nu_4,\\ &\quad \mu_4\nu_3 + \mu_2\nu_4 + \mu_3\nu_4).\end{aligned}$$

This ideal is not the unit ideal and has degree 8. Hence there are isomorphisms between M and N. Pick "at random" $\mu_1 = 1$, $\mu_2 = 2$, and $\mu_3 = 0$. Then the ideal $I(V, W) + (\mu_1 - 1, \mu_2 - 2, \mu_3 - 0)$ equals the ideal $(\mu_1 - 1, \mu_2 - 2, \mu_3, \nu_4 + 1, \nu_2 + \nu_3, \nu_1 + \frac{1}{2}\nu_3, \mu_4\nu_3 - 2)$. We see that we have to avoid $\mu_4 = 0$ but otherwise have complete choice.

Example 4.7.6. Let $n = 1$, $M = D/D \cdot \partial^2$, and $N = D/D \cdot \partial$. One checks that $V = \mathrm{Hom}_D(N, M)$ is generated by $t_1 = \cdot(\partial)$ and $t_2 = \cdot(x\partial - 1)$ while $W = \mathrm{Hom}_D(M, N)$ is generated by $s_1 = \cdot(1)$ and $s_2 = \cdot(x)$. The sum $\sum \mu_i\nu_j s_i \cdot t_j$ takes the form

$$\mu_2\nu_2 x^2\partial + (\mu_1\nu_2 + \mu_2\nu_1)x\partial + \mu_1\nu_1\partial - (\mu_1\nu_2 + \mu_2\nu_2).$$

Modulo $D \cdot \partial$ we want this to be 1, so we get the relation

$$\mu_2\nu_1 - \mu_1\nu_2 = 1.$$

We note that this equation has plenty of solutions, which means that M can be realized as a summand of N. On the other hand, the sum $\sum \mu_i\nu_j t_j \cdot s_i$ takes the form

$$\mu_1\nu_1\partial + (\mu_1\nu_2 + \mu_2\nu_1)x\partial - \mu_1\nu_2 - \mu_2\nu_2 x + \mu_2\nu_2 x^2\partial.$$

Modulo $D \cdot \partial^2$ we want this to be 1, so we get the relations

$$\begin{aligned}-\mu_1\nu_2 &= 1,\\ \mu_1\nu_1 &= 0,\\ \mu_1\nu_2 + \mu_2\nu_1 &= 0,\\ \mu_2\nu_2 &= 0.\end{aligned}$$

Putting all the equations together, we obtain the unit ideal, and hence M and N are not isomorphic.

Let us end by mentioning the general relationship coming from ring theory between $E = \mathrm{End}_D(M)$ and M. An element e of E is said to be an *idempotent* if $e^2 = e$, and two idempotents e and f are said to be *orthogonal* if $ef = fe = 0$. The relationship between E and M can now be stated as follows: There is a bijective correspondence between (1) the decompositions of M into a direct sum of submodules and (2) the decompositions of the identity element $1 = e_1 + \cdots + e_s$ of E into pairwise orthogonal idempotents [17, Theorem 1.7.2]. The correspondence is gotten by taking a set of orthogonal idempotents $\{e_1, \ldots, e_s\}$ and producing the decomposition $M = e_1 \cdot M \oplus \cdots \oplus e_s \cdot M$. Moreover, the Krull-Schmidt-Azumaya Theorem states that M has a decomposition into a direct sum of indecomposable submodules, and this decomposition is unique up to re-ordering and isomorphism (see e.g. [24, Theorem 19.21]). Here, a D-module is said to be *indecomposable* if it cannot be written as a direct sum of two nonzero submodules. Thus, it would be useful to have an algorithm which produces a full set of orthogonal idempotents for the K-algebra $\mathrm{End}_D(M)$ since combined with Algorithm 4.5.1, this would give a method to decompose holonomic D-modules into indecomposables.

There is an active area of research concerning computations in finite-dimensional K-algebras E. For instance, when K is a number field, early work of Friedl and Ronyai provides polynomial-time algorithms to find the radical of E and the decomposition of E into simple algebras if E is semi-simple [20]. These algorithms take as inputs a set of matrices which generate E as a subalgebra of a matrix ring (for instance, consider E embedded as its regular representation) and output matrices which generate the radical of E or which correspond to the identity elements of the simple subalgebras of the semi-simple E. Thus the latter algorithm is already enough to decompose M into a direct sum of submodules, where each submodule corresponds to the sum of all indecomposable submodules of a given type.

We should also mention that the radical of E is independent of field extension of K while the decomposition into simple algebras depends upon the field K. Thus, the analogous problem of finding a full set of orthogonal idempotents of a simple algebra depends on the field, and has been shown to be computationally difficult [39]. However, for the case of $K = \mathbb{C}$, Eberly has given Las Vegas polynomial time algorithms to find the decomposition of a simple algebra as a full matrix ring [18]. Thus if we are willing to use $K = \mathbb{C}$, then the work of Friedl and Ronyai combined with the work of Eberly will produce the radical J of E and a complete set of orthogonal idempotents of E/J. The only step which is left is to lift these idempotents modulo J to orthogonal idempotents of E (see e.g. [17, Lemma 3.2.1.] for the theory of lifting individual idempotents). Eberly has preliminary ideas on how this may be accomplished algorithmically [19].

4.8 Homomorphisms of finite rank systems

Recall the ring $R = K(x_1, \ldots, x_n)\langle\partial_1, \ldots, \partial_n\rangle$ of linear differential operators with rational function coefficients. Given a left D-module M, we will denote by M_R the extension of M to an R-module $M_R := K(\boldsymbol{x}) \otimes_{K[\boldsymbol{x}]} M$. In this section, we will study the relation between $\mathrm{Hom}_D(M, N)$ and $\mathrm{Hom}_R(M_R, N_R)$ for holonomic D-modules M and N. The material here is based on discussions with Frederic Chyzak and Michael Singer and is still in somewhat preliminary stages.

The following discussion is valid for both D and R, hence we will use $\mathcal{D}$ to stand for either D or R and similarly $\mathcal{M}$ and $\mathcal{N}$ to stand for either D-modules or R-modules. Also, we will use A to denote the corresponding $K[\boldsymbol{x}]$ or $K(\boldsymbol{x})$. It was pointed out to us by Singer that $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ has the structure of a left $\mathcal{D}$-module: A function $f \in A$ acts on $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ in the obvious way, that is for $\phi \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ and $m \in \mathcal{M}$, then $(f \bullet \phi)(m) = \phi(f \cdot m) = f \cdot \phi(m)$ Similarly a derivation θ acts on $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ by sending ϕ to $(\theta \bullet \phi)(m) = \theta \cdot \phi(m) - \phi(\theta \cdot m)$. One checks that this action satisfies the conditions,

$$\begin{array}{rcl} (f\theta) \bullet \phi & = & f \cdot (\theta \bullet \phi) \\ \theta \bullet f \bullet \phi & = & f \bullet \theta \bullet \phi + \theta(f) \bullet \phi \\ [\theta,\vartheta] \bullet \phi & = & \theta \bullet \vartheta \bullet \phi - \vartheta \bullet \theta \bullet \phi \end{array}$$

where $[\theta, \vartheta]$ is the Lie derivative, and hence the action extends to a structure of left $\mathcal{D}$-module (see e.g. [27]). This is also equivalent to saying that the standard connection

$$\mathrm{Hom}_A(\mathcal{M},\mathcal{N}) \xrightarrow{\begin{bmatrix} \partial_1 \\ \vdots \\ \partial_n \end{bmatrix} \bullet} \Omega^1 \otimes_A \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$$

is integrable. The flat sections of this connection are the elements $\varphi \in \mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ satisfying

$$\partial_i \bullet \varphi(m) = \partial_i \cdot \varphi(m) - \varphi(\partial_i \cdot m) = 0$$

for all $m \in \mathcal{M}$ and i from 1 to n. In other words, the flat sections are exactly the $\mathcal{D}$-linear maps $\mathrm{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{N})$.

When $\mathcal{M} = M_R$ and $\mathcal{N} = N_R$ are finite rank, then there is a natural algorithmic way to compute the flat sections, observed by Singer and discussed for the case $n = 1$ in [41]. For simplicity, let us assume that $M_R = R/I$ and $N_R = R/J$ have cyclic presentations. The case where we are given a non-cyclic presentation is similar. To proceed, fix a term order on R, compute a Gröbner basis of J, and let $B_J = \{\boldsymbol{\partial}^{\alpha_1}, \ldots, \boldsymbol{\partial}^{\alpha_r}\}$ be the standard monomials of $N_R = R/J$. Then B_J is a $K(\boldsymbol{x})$-basis of N_R, where $r = \mathrm{rank}(N_R)$. An element $\varphi \in \mathrm{Hom}_R(M_R, N_R)$ is determined by the image of a cyclic generator, which for us is the class of 1. The generic image can be written

$$\varphi(\overline{1}) = \sum_{j=1}^{r} a_j(\boldsymbol{x})\boldsymbol{\partial}^{\alpha_j},$$

where $a_j(\boldsymbol{x}) \in K(\boldsymbol{x})$. Thus to find the flat sections, we need to solve for the possibilities on $a_j(\boldsymbol{x})$. Let a generating set of I be $\{L_1, \ldots, L_k\}$. For each generator L_i, we evaluate $L_i \cdot (\sum_{j=1}^{r} a_j(\boldsymbol{x})\boldsymbol{\partial}^{\alpha_j})$, and reduce it modulo the Gröbner basis of J to obtain a new expression,

$$L_i \cdot \varphi(\overline{1}) = \sum_j L_{ij} \cdot [a_0, \ldots, a_r]^t \boldsymbol{\partial}^{\alpha_j}$$

where $L_{ij} \in R^r$ are row vectors of differential operators. In particular, the element $\sum_{j=1}^{r} a_j(\boldsymbol{x})\boldsymbol{\partial}^{\alpha_j}$ defines a flat section if and only if $[a_0, \ldots, a_r]^t$ is a rational function solution of the vector-valued system $S = \{L_{ij} \bullet a = 0\}_{i,j=1}^{k,r}$.

We would now like to understand the system S in terms of our earlier algorithms. To this end, we note that the flat sections are equivalently,

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{N}) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\cdot\{\partial_1,\ldots,\partial_n\}, \mathrm{Hom}_A(\mathcal{M},\mathcal{N})).$$

Let us use the homological isomorphism of Section 4.2 and the fact that $\mathrm{Ext}^n_{\mathcal{D}}(A,\mathcal{D}) \simeq \Omega$. Here, Ω now stands for the top dimensional differential forms, and has the presentation $\mathcal{D}/\{\partial_1,\ldots,\partial_n\}\cdot\mathcal{D}$, while A has the presentation $\mathcal{D}/\mathcal{D}\cdot\{\partial_1,\ldots,\partial_n\}$. The following isomorphisms are now valid for $\mathcal{D}=R$ because $\mathrm{Hom}_{K(\boldsymbol{x})}(M_R,N_R)$ remains finite rank if M_R and N_R are finite rank. Similarly, they are valid for the case $\mathcal{D}=D$ whenever $\mathrm{Hom}_{K[\boldsymbol{x}]}(M,N)$ is holonomic, which we believe is the case if M and N are both holonomic. The isomorphisms are,

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(\mathcal{M},\mathcal{N}) &\simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}/\mathcal{D}\cdot\{\partial_1,\ldots,\partial_n\}, \mathrm{Hom}_A(\mathcal{M},\mathcal{N}))\\
&\simeq \mathrm{Hom}_{\mathcal{D}}(A, \mathrm{Hom}_A(\mathcal{M},\mathcal{N}))\\
&\simeq \mathrm{Tor}^n_{\mathcal{D}}(\Omega, \mathrm{Hom}_A(\mathcal{M},\mathcal{N}))\\
&\simeq \mathrm{Tor}^n_{\mathcal{D}}(\tau(\mathrm{Hom}_A(\mathcal{M},\mathcal{N})), A)\\
&\simeq \mathrm{Hom}_{\mathcal{D}}(\mathrm{Ext}^n_{\mathcal{D}}(\tau(\mathrm{Hom}_S(\mathcal{M},\mathcal{N})),\mathcal{D}), A)\\
&\simeq \mathrm{Hom}_{\mathcal{D}}(\mathbf{D}(\mathrm{Hom}_A(\mathcal{M},\mathcal{N})), A).
\end{aligned}$$

It thus follows that the system S is the dual system of $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ if either $\mathcal{D}=R$ or $\mathrm{Hom}_{K[\boldsymbol{x}]}(M,N)$ is holonomic. One algorithmic question is how to compute and represent $\mathrm{Hom}_A(\mathcal{M},\mathcal{N})$ for $\mathcal{D}=D$ and $\mathcal{D}=R$.

Let us now discuss the relation between $\mathrm{Hom}_D(M,N)$ and $\mathrm{Hom}_R(M_R,N_R)$. The basic idea comes from the computation of rational solutions. In other words, when $N=K[\boldsymbol{x}]$, then $\mathrm{Hom}_R(M_R,N_R)=\mathrm{Hom}_R(M_R,K(\boldsymbol{x}))$ corresponds to the rational solutions of M, and moreover, by Theorem 2.1.8 of Cauchy-Kovalevskiia-Kashiwara, we also know that $\mathrm{Hom}_R(M_R,K(\boldsymbol{x}))=\mathrm{Hom}_R(M,K[\boldsymbol{x}][f^{-1}])$ for any polynomial f vanishing on the codimension one component of the singular locus. The general statement is as follows.

Proposition 4.8.1. *Let $M=D^r/M_0$ and $N=D^s/N_0$ be holonomic D-modules, and let f be a polynomial defining the codimension one component of the singular locus of M. Then*

$$\mathrm{Hom}_R(M_R,N_R) \simeq \mathrm{Hom}_D((D^r/\operatorname{Cl}(M_0))[f^{-1}], (D^s/\operatorname{Cl}(N_0))[f^{-1}])$$

Proof. Let us denote $M'=(D^r/\operatorname{Cl}(M_0))[f^{-1}]$ and $N'=(D^s/\operatorname{Cl}(N_0))[f^{-1}]$. It is clear that any homomorphism in $\mathrm{Hom}_D(M',N')$ defines a unique homomorphism in $\mathrm{Hom}_R(M_R,N_R)$ by extension. We need to show that any homomorphism in $\mathrm{Hom}_R(M_R,N_R)$ also comes from a homomorphism in $\mathrm{Hom}_D(M',N')$ in this manner.

Since f defines the codimension 1 component of the singular locus, the key fact is that M' is generated as a $D[f^{-1}]$-module by $\{h_1\cdot e_1,\ldots,h_r\cdot e_r\}$ for any set of polynomials $\{h_1,\ldots,h_r\}\subset K[\boldsymbol{x}]$ where $\{e_1,\ldots,e_r\}$ is the canonical basis of $D[f^{-1}]^r$. More precisely, the submodules $D[f^{-1}]\cdot e_i$ and $D[f^{-1}]h_i\cdot e_i$ of M' are equal for all i. This property follows by an argument similar to the one used to prove that the Weyl closure of M_0 equals $D[f^{-1}]\cdot M_0\cap D^r$.

A homomorphism $\psi\in\mathrm{Hom}_R(M_R,N_R)$ is defined by its images $\{\psi(e_1),\ldots,\psi(e_r)\}$ inside N_R. Let $\{T_1,\ldots,T_r\}$ be lifts of $\{\psi(e_1),\ldots,\psi(e_r)\}$ to elements of R^r. Then we have $\{T_1,\ldots,T_r\}\subset D[(fg)^{-1}]^r$ for some polynomial $g\in K[\boldsymbol{x}]$.

For $k \gg 0$, $\{g^k T_1, \ldots, g^k T_r\} \subset D[f^{-1}]^r$. Since $D[f^{-1}] \cdot e_i$ and $D[f^{-1}]g^k \cdot e_i$ are equal as submodules of M', there exist operators $L_i \in D[\frac{1}{f}]$ such that $L_i g^k \cdot e_i = e_i \in M'$. Then $\{L_1 g^k T_1, \ldots, L_r g^k T_r\}$ can also be chosen as the lifts of $\{\psi(e_1), \ldots, \psi(e_r)\}$. Therefore ψ is the extension of the homomorphism $\psi' \in \mathrm{Hom}_D(M', N')$ which is defined by mapping e_i to the residue class of $L_i g^k T_i$ in N'. $\square$

Example 4.8.2. Suppose $M = N = D/D \cdot \{x^2\partial, \partial^2 x\}$, which is a cyclic presentation of the holonomic module $(D/D\partial) \oplus (D/Dx)$. Then $\mathrm{Hom}_D(M, N)$ is 2-dimensional and spanned by the maps $\cdot(1)$ and $\cdot(x\partial)$. On the other hand, $M_R = N_R = R/R\partial = K(\boldsymbol{x})$ so that $\mathrm{Hom}_R(M_R, N_R)$ is 1-dimensional and spanned by the identity. Moreover, the singular locus of M is $x = 0$, the Weyl closure of $D \cdot \{x^2\partial, \partial^2 x\}$ is $D \cdot \partial$, the localization $(D/D \cdot \partial)[x^{-1}]$ is isomorphic to $K[x, x^{-1}]$, and $\mathrm{Hom}_D(K[x, x^{-1}], K[x, x^{-1}])$ is similarly 1-dimensional and spanned by the identity.

Appendix on the restriction algorithm

In recent years, one of the major breakthroughs in computational algebraic analysis has been the development of algorithms to compute derived restriction and derived integration. In this thesis, these algorithms play a major role, and in this appendix, we provide a summary for the reader's convenience. An algorithm was first given by Oaku to compute the derived restriction modules of a holonomic D-module M to a hyperplane $x_1 = 0$ in [32]. This algorithm was then extended by Oaku and Takayama to compute the derived restriction of a holonomic D-module M to a linear subspace $x_1 = \cdots x_d = 0$ in [33]. Finally, the algorithm was generalized by Walther to compute the derived restriction of bounded complexes with holonomic cohomology in [52]. Moreover, algorithms for derived integration are an immediate consequence of these algorithms due to the algebraic similarities between restriction and integration. As we have seen in Chapter 2, the restriction algorithm was used to compute local cohomology in [33]. Similarly, it has been used to compute the deRham cohomology of complements of affine varieties of $\mathbb{C}^n$ in [34], [52]. In this thesis, the restriction algorithm is the main ingredient in Chapter 4 for computing homomorphisms and Ext between holonomic D-modules.

Let us recall briefly the general definition of derived restriction. Given a $\mathcal{D}_X$-module $\mathcal{M}$ on an algebraic variety X and a map of algebraic varieties $f : Y \to X$, then the inverse image of $\mathcal{M}$ in the sense of algebraic geometry is the $\mathcal{O}_Y$-module $f^*(\mathcal{M}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{M}$. Moreover, we can extend the $\mathcal{O}_Y$-module structure to a $\mathcal{D}_Y$-module structure since derivations of $\mathcal{O}_Y$ act naturally by pushing forward a tangent vector on Y to a tangent vector on X via f. Thus, the inverse image functor makes sense in the category of $\mathcal{D}$-modules, and the derived restriction modules of $\mathcal{M}$ under f are defined to be the left derived functors of f^* in the category of $\mathcal{D}$-modules.

For the case of the Weyl algebra, an arbitrary map of affine spaces $f : K^{m_1} \to K^{m_2}$ can be factored as $f = \pi \circ \phi \circ \iota : K^{m_1} \to K^{m_1} \times K^{m_2} \to K^{m_1} \times K^{m_2} \to K^{m_2}$ where $\iota(\boldsymbol{x}) = (\boldsymbol{x}, 0)$ is an inclusion, $\phi(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y} + f(\boldsymbol{x}))$ is an isomorphism, and $\pi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$ is a projection. Moreover, both ϕ^* and π^* are exact, hence the only interesting map for restriction is ι^*. Thus, we will only consider inclusions of affine space, i.e. let $X = K^{n+d}$ with coordinates $(x_1, \ldots, x_n, t_1, \ldots, t_d)$, $Y = \{t_1 = \cdots = t_d = 0\} \simeq K^n$ with coordinates $(x_1, \ldots, x_n)$, and $f = \iota : Y \hookrightarrow X$ be the inclusion $\iota(\boldsymbol{x}) = (\boldsymbol{x}, 0)$. In this case, the derived restriction modules of a left D_X-module M, which we shall denote $H^i(M^\bullet_{Y \to X})$, are equal to the Tor groups,

$$H^i(M^\bullet_{Y\to X}) := H^i(Lf^*(M)) \simeq \mathrm{Tor}_i^{D_X}(\Lambda_Y, M),$$

where Λ_Y is the right D_X-module $D_X/\{t_1,\ldots,t_d\}\cdot D_X$. In principle, these modules can thus be computed as the homology of the Koszul complex $K^\bullet(M;t_1,\ldots,t_d)$, or as the homology of the complex $\Lambda_Y \otimes_{D_X} P^\bullet$ where $P^\bullet$ is a projective resolution of M. The problem is that the maps in $K^\bullet(M;t_1,\ldots,t_d)$ are not maps of left D_X-modules while the modules in $\Lambda_Y \otimes_{D_X} P^\bullet$ are no longer left D_X-modules. Both complexes are indeed complexes of left D_Y-modules but the modules are no longer finitely generated as D_Y-modules.

The challenge then is to replace either of these complexes by a quasi-isomorphic complex which does consist of finitely generated D_Y-modules. The main ideas are to replace $P^\bullet$ by a so-called V_Y-adapted free resolution, and then to use an appropriate b-function to identify a quasi-isomorphic subcomplex of $\Lambda_Y \otimes_{D_X} P^\bullet$ which is also finitely generated over D_Y. Let us now explain the details of these ideas. The following is a brief summary of [33] although we modify the shift convention. As above, let $X = K^{n+d}$, $D_X = K\langle \boldsymbol{x}, \mathbf{t}, \boldsymbol{\partial_x}, \boldsymbol{\partial_t}\rangle$, $Y = \{t_1 = \cdots = t_d = 0\} \simeq K^n$, and $D_Y = K\langle \boldsymbol{x}, \boldsymbol{\partial_x}\rangle$.

Definition 4.8.3. *The V-filtration F_Y of a shifted free module $D^r[\vec{m}]$ with respect to Y is defined by*

$$F_Y^i(D_X^r[\vec{m}]) = \mathrm{Span}_K\{\boldsymbol{x}^\mu \boldsymbol{\partial}_{\boldsymbol{x}}^\nu \mathbf{t}^\alpha \boldsymbol{\partial}_{\mathbf{t}}^\beta e_l : \mu,\nu,\alpha,\beta \in \mathbb{N}^n, |\beta| - |\alpha| \le i + m_l\}.$$

We remark that the V-filtration induces filtrations on submodules and quotients in the usual manner.

Definition 4.8.4. *A free resolution $P^\bullet$ of M of the form,*

$$P^\bullet : \cdots \longrightarrow D_X^{r_{j+1}}[\vec{m}_{j+1}] \xrightarrow{\psi_{j+1}} D_X^{r_j}[\vec{m}_j] \longrightarrow \cdots$$

is said to be V_Y-adapted if

$$\psi_{j+1}(F_Y^i(D_X^{r_{j+1}}[\vec{m}_{j+1}])) \subset F_Y^i(D_X^{r_j}[\vec{m}_j]))$$

for all i and all j, and if a resolution is also induced on the level of associated graded modules,

$$\mathrm{gr}(P^\bullet) : \cdots \longrightarrow \mathrm{gr}(D_X^{r_{j+1}}[\vec{m}_{j+1}]) \xrightarrow{\mathrm{gr}(\psi_{j+1})} \mathrm{gr}(D_X^{r_j}[\vec{m}_j]) \longrightarrow \cdots.$$

Definition 4.8.5. *The b-function of M for restriction to Y is the monic polynomial $b(\theta) \in K[\theta]$ of least degree, if any, which satisfies $b(\theta)\mathrm{gr}^0(M) = 0$ with respect to the V_Y-filtration, where $\theta = t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}$.*

When M is holonomic and $P^\bullet$ is a V_Y-adapted resolution of M, then the b-function is nonzero and its maximum positive integer root gives a point of truncation for finding a quasi-isomorphic and D_Y-finitely-generated subcomplex of $\Lambda_Y \otimes_{D_X} P^\bullet$. We remark that Gröbner basis methods to compute V_Y-adapted resolutions and b-functions are given by Oaku and Takayama in [33]. A good exposition of these algorithms can also be found in [40, Chapter 5]. We now have the ingredients to summarize the restriction algorithm.

Algorithm 4.8.6. (Derived restriction modules of holonomic M to a linear subspace)
INPUT: a presentation $M = D_X^{r_0}/D_X \cdot \{L_1, \dots, L_s\}$ of a left holonomic D_X-module, and a coordinate subspace $Y = \{t_1 = \cdots = t_d = 0\}$.
OUTPUT: the derived restriction modules $H^i(M^\bullet_{Y \to X})$ for $i = 0, \dots, d$.

1. Compute the b-function $b(\theta)$ of M for restriction to Y.

2. Let k_1 be the maximum integer root of $b(\theta)$. If there are no integer roots, then return $H^i(M^\bullet_{Y \to X}) = 0$ for $i = 0, \dots, d$ and quit.

3. Compute a V_Y-adapted free resolution of M,
$$P^\bullet : \cdots \longrightarrow D_X^{r_{j+1}}[\vec{m}_{j+1}] \stackrel{\psi_{j+1}}{\longrightarrow} D_X^{r_j}[\vec{m}_j] \longrightarrow \cdots$$

4. Compute the truncated induced complex,
$$F_Y^{k_1}(\Lambda_Y \otimes_{D_X} P^\bullet) : \cdots \longrightarrow F^{k_1}(\Lambda_Y^{r_{j+1}}[\vec{m}_{j+1}]) \stackrel{\overline{\psi}_{j+1}}{\longrightarrow} F^k(\Lambda_Y^{r_j}[\vec{m}_j]) \longrightarrow \cdots$$
as a complex of finitely generated left D_Y-modules.

5. Compute the $-i$-th cohomology group $\ker \overline{\psi}_i / \operatorname{im} \overline{\psi}_{i+1}$ of the above complex in the form $D_Y^{s_i}/P_i$ for $i = 0, \dots, d$.

6. Return $H^i(M^\bullet_{Y \to X}) = D_Y^{s_i}/P_i$ for $i = 0, \dots, d$.

We would similarly like to define the direct image f_+ and its derived functors, which are called the derived integrations, for a $\mathcal{D}_X$-module $\mathcal{M}$ with respect to a map $f : X \to Y$ in the category of $\mathcal{D}$-modules. In general, the direct image functor is considerably more complicated than the inverse image and exists in the derived category, but for the case where X and Y are affine coordinate spaces it can be described in elementary terms. Again for a map $f : K^{m_1} \to K^{m_2}$ with factorization $f = \iota \circ \phi \circ \pi$, both ι_+ and ϕ_+ are exact, hence the only interesting map for integration is π_+. Thus we only consider the case where $X = K^{n+d}$ with coordinates $(x_1, \dots, x_n, t_1, \dots, t_d)$, $Y = K^n$ with coordinates $(x_1, \dots, x_n)$, and $f = \pi : X \to Y$ is the projection $\pi(x_1, \dots, x_n, t_1, \dots, t_d) = (x_1, \dots, x_n)$. Then the derived integrations of a left D_X-module M, which we shall denote $H^i(M^\bullet_{Y \leftarrow X})$, are also equal to the Tor groups,
$$H^i(M^\bullet_{Y \leftarrow X}) := H^i(L\pi_+(M)) \simeq \operatorname{Tor}_i^{D_X}(\Omega_Y, M),$$
where Ω_Y is the right D_X-module $D_X/\{\partial_{t_1}, \dots, \partial_{t_d}\} \cdot D_X$. There is an algebra automorphism called the Fourier transform
$$\mathcal{F}_X : D_X \to D_X \qquad \boldsymbol{x}^\mu \mathbf{t}^\alpha \boldsymbol{\partial}_{\boldsymbol{x}}^\nu \boldsymbol{\partial}_{\mathbf{t}}^\beta \mapsto (-\boldsymbol{\partial}_{\boldsymbol{x}})^\mu (-\boldsymbol{\partial}_{\mathbf{t}})^\alpha \boldsymbol{x}^\nu \mathbf{t}^\beta$$
such that $\mathcal{F}_X(\Omega_Y) = \Lambda_Y$. We can therefore use the restriction algorithm to compute integration as well.

Algorithm 4.8.7. (Derived integration modules of holonomic M to a linear subspace)
INPUT: a presentation $M = D_X^{r_0}/D_X \cdot \{L_1, \ldots, L_s\}$ of a left holonomic D_X-module, and a projection $\pi : X \to Y$ where $\pi(x_1, \ldots, x_n, t_1, \ldots, t_d) = (x_1, \ldots, x_n)$.
OUTPUT: the derived integration modules $H^i(M^\bullet_{Y \leftarrow X})$ for $i = 0, \ldots, d$.

1. Compute the derived restriction modules $H^i(\mathcal{F}_X(M)^\bullet_{Y \to X})$ of the Fourier transform of M to the linear subspace $Y = \{t_1 = \cdots = t_d = 0\}$ using Algorithm 4.8.6.

2. Return the left D_Y modules $\mathcal{F}_Y^{-1}(H^i(\mathcal{F}_X(M)^\bullet_{Y \to X}))$ for $i = 0, \ldots, d$ where $\mathcal{F}_Y^{-1}$ denotes the inverse Fourier transform of D_Y.

From the theoretical point of view, we also make the following definitions for integration.

Definition 4.8.8. *The $\widetilde{V}$-filtration $\widetilde{F}_Y$ of a shifted free module $D^r[\vec{m}]$ with respect to Y is defined by*

$$\widetilde{F}_Y^i(D^r[\vec{m}]) = \mathrm{Span}_K\{\boldsymbol{x}^\mu \boldsymbol{\partial}_{\boldsymbol{x}}^\nu \mathbf{t}^\alpha \boldsymbol{\partial}_{\mathbf{t}}^\beta \vec{e}_l : \mu, \nu, \alpha, \beta \in \mathbb{N}^n, |\alpha| - |\beta| \le i + m_l\}.$$

Definition 4.8.9. *The b-function of M for integration to Y is the monic polynomial $\widetilde{b}(\theta) \in K[\theta]$ of least degree, if any, which satisfies $\widetilde{b}(-\theta - d)\mathrm{gr}^0(M) = 0$ with respect to the $\widetilde{V}_Y$-filtration, where $\theta = t_1\partial_{t_1} + \cdots + t_d\partial_{t_d}$.*

Note that $\mathcal{F}_X(\widetilde{F}_Y^i(D^r[\vec{m}])) = F_Y^i(D^r[\vec{m}]))$, hence a $\widetilde{V}_Y$-adapted resolution M can be obtained as $\mathcal{F}_X^{-1}(P^\bullet)$ where $P^\bullet$ is a V_Y-adapted resolution of $\mathcal{F}_X(M)$. Similarly, the b-function $\widetilde{b}(\theta)$ of M for integration to Y is equal to the b-function $b(\theta)$ of $\mathcal{F}(M)$ for restriction to Y.

Bibliography

[1] Adolphson, A., Sperber, S. (1998): A remark on local cohomology. *Journal of Algebra* **206**, 555-567.

[2] Appell, P., Kampé de Fériet, J. (1926): *Fonctions Hypergéometrique et Hypersphériques — Polynomes d'Hermite.* Gauthier-Villars, Paris.

[3] Assi, A., Castro-Jiménez, F.J., Granger, M. (1998): The standard fan of a D-module. To appear in *Journal of Pure and Applied algebra.*

[4] Bernstein, J. (1972): The analytic continuation of generalized functions with respect to a parameter. *Funct. Analy. Appl.* **6**, 273-285.

[5] Bernstein, J. Mimeographed lecture notes on D-modules.

[6] Borel, A. (1987): *Algebraic D-modules.* Academic Press, Boston.

[7] Björk, J. (1979): *Rings of Differential Operators.* North Holland Publishing Company, Amsterdam.

[8] Briançon, J., Maisonobe, Ph. (1984): Ideaux de germes d'operateurs differentiels a une variable. *l'Enseignement Mathematique* t **30**, 7-38.

[9] Cannings, R., Holland, M. (1994): Right ideals of rings of differential operators. *Journal of Algebra* **167**, 116-141.

[10] Cattani, E., D'Andrea, C., Dickenstein, A. (1999): Rational solutions of the A-hypergeometric system associated with a monomial curve. *Duke Mathematical Journal* **99**, 179-207.

[11] Cattani, E., Dickenstein, A., Sturmfels, B. (1999): Rational hypergeometric functions. `http://www.arXiv.org/abs/math.AG/9911030`.

[12] Chyzak, F. (1997): An extension of Zeilberger's fast algorithm to general holonomic functions, *Formal Power Series and Algebraic Combinatorics*, 9th Conference, Universität Wien, pp. 172-183.

[13] Chyzak, F. (1998): Gröbner bases, symbolic summation and symbolic integration, in *Gröbner Bases and Applications (Linz, 1998)*, 32–60. London Math. Soc. Lecture Note Series 251, Cambridge University Press, Cambridge.

[14] Chyzak, F., Salvy, B. (1998): Non-commutative elimination in Ore algebras proves multivariate holonomic identities. *Journal of Symbolic Computation* **26**, 187-227.

[15] Coutinho, S. (1995): *A Primer of Algebraic D-modules.* London Mathematical Society Student Texts, Vol. 33. Cambridge University Press, Cambridge.

[16] Cox, D., Little, J., O'Shea, D. (1998): *Using algebraic geometry.* Springer Verlag.

[17] Drozd, Y., Kirichenko, V. (1994): *Finite Dimensional Algebras.* Springer Verlag, Berlin.

[18] Eberly, W. (1991): Decompositions of algebras over $\mathbb{R}$ and $\mathbb{C}$. *Computational Complexity* **1**, 211-234.

[19] Eberly, W. (2000): Personal communication.

[20] Friedl, K., Ronyai, L. (1985): Polynomial time solutions for some problems in computational algebra, in *Proc. 17th Ann. Symp. Theory of Computing*, 153-162.

[21] Galligo, A. (1985): Some algorithmic questions on ideals of differential operators. *Lecture Notes in Computer Science* **204**, 413-421.

[22] Granger, M., Maisonobe, P. (1993): A basic course on differential modules. In *D-modules Coherents et Holonomes*, eds. Maisonobe, P., Sabbah, C. Hermann editeurs des sciences et des arts, Paris.

[23] Grayson, D., Stillman, M. (1999): Macaulay 2: a computer algebra system for algebraic geometry, Version 0.8.56, `http://www.math.uiuc.edu/Macaulay2`.

[24] Lam, T.Y. (1991): *A First Course in Noncommutative Ring Theory.* Springer Verlag, New York.

[25] Le Bruyn, L. (1995): Moduli spaces for right ideals of the Weyl algebra. *Journal of Algebra* **172**, 32-48.

[26] Kashiwara, M. (1976): B-functions and holonomic systems. *Inventiones Mathematicae* **38**, 33-53.

[27] Kashiwara, M. (1978): On the holonomic systems of linear differential equations, II. *Inventiones Mathematicae* **49**, 121-135.

[28] Maisonobe, P., Sabbah, C. (1993): *D-modules Coherents et Holonomes.* Hermann editeurs des sciences et des arts, Paris.

[29] Musţată, M. (2000): Local cohomology at monomial ideals. To appear in *Journal of Symbolic Computation.*

[30] Musţată, M. (2000): Personal communication.

[31] Oaku, T. (1997): An algorithm of computing b-functions. *Duke Mathematical Journal* **87**, 115-132.

[32] Oaku, T. (1997): Algorithms for b-functions, restrictions, and algebraic local cohomology groups of D-modules. *Advances in Applied Mathematics* **19**, 61-105.

[33] Oaku, T., Takayama, N. (1998): Algorithms for D-modules — restrictions, tensor product, localization and algebraic local cohomology groups. To appear in *Journal of Pure and Applied Algebra.*

[34] Oaku, T., Takayama, N. (1999): An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation. *Journal of Pure and Applied Algebra* **139**, 201-233.

[35] Oaku, T., Takayama, N., Tsai, H. (2000): Polynomial and rational solutions to holonomic systems. `http://www.arXiv.org/abs/math.AG/0001064`.

[36] Oaku, T., Takayama, N., Walther, U. (1999): A localization algorithm for D-modules. To appear in *Journal of Symbolic Computation.*

[37] Pemantle, R., Wilson, M. (1999): Asymptotics of multivariate sequences, part I: smooth points of the singular variety, preprint.

[38] Petkovsek, M., Wilf, H., Zeilberger, D. (1996): $A = B$. A K Peters, Ltd., Wellesley, MA.

[39] Ronyai, L. (1987): Simple algebras are difficult, in *Proc. 17th Ann. Symp. Theory of Computing*, 398-408.

[40] Saito, M., Sturmfels, B., Takayama, N. (1999): *Gröbner Deformations of Hypergeometric Differential Equations.* Algorithms and Computation in Mathematics, Vol. 6. Springer Verlag, Berlin.

[41] Singer, M. (1996): Testing reducibility of linear differential operators: a group theoretic perspective. *Applicable Algebra in Engineering, Communication and Computing* **7**, 77-104.

[42] Smith, G. (1999): Irreducible components of characteristic varieties. `http://www.arXiv.org/abs/math.AG/9912066`.

[43] Stanley, R. (1999): *Enumerative Combinatorics 2.* Cambridge studies in advanced mathematics, Vol. 62. Cambrige University Press.

[44] Strömbeck, P. (1978): On left ideals in A_1 and their associated graded ideals. *Journal of Algebra* **55**, 116-144.

[45] Takayama, N. (1989): Gröbner basis and the problem of contiguous relation. *Japan Journal of Applied Mathematics* **6**, 147-160.

[46] Takayama, N. (1990): An algorithm of constructing the integral of a module – an infinite dimensional analog of Gröbner bases. In: Watanabe, S. and Nagata, N., Eds., *Symbolic and Algebraic Computation (ISSAC '90)*, Kyoto, Japan. ACM and Addison Wesley, 206-211.

[47] Takayama, N. (1992): An approach to the zero recognition problem by Buchberger's algorithm. *Journal of Symbolic Computation* **14**, 265-282.

[48] Tsai, H. (2000): Weyl closure of a linear differential operator. To appear in *Journal of Symbolic Computation.*

[49] Tsai, H., Walther, U. (2000): Computing homomorphism between holonomic D-modules, preprint.

[50] van Hoeij, M. (1997): Factorization of differential operators with rational function coefficients. *Journal of Symbolic Computation* **24**, 537-561.

[51] Walther, U. (1999): Algorithmic computation of local cohomology modules and the local cohomological dimension of algebraic varieties. *Journal of Pure and Applied Algebra* **139**, 303-321.

[52] Walther, U. (1999): Algorithmic computation of deRham cohomology of complements of complex affine varieties. To appear in *Journal of Symbolic Computation.*

[53] Walther, U. (1999): The cup product structure for complements of complex affine varieties, preprint.

[54] Weibel, C. (1994): *An Introduction to Homological Algebra.* Cambridge studies in advanced mathematics, Vol. 38. Cambrige University Press.