Localisation of D-modules

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Notation

- K computable field of characteristic 0 contained in $\mathbb{C}$
- $R_n := K[\underline{x}] := K[x_1, .., x_n]$ the polynomial ring in n indeterminates
- $D_n := R\langle\underline{\partial}\rangle := R_n\langle\partial_1, .., \partial_n\rangle$ be the n-th Weyl algebra

Aim and Motivation

- Let $f \in R_n$ and M a holonomic (left) D_n-module
  $M \cong D_n/I$ for a left ideal I in D_n
- Compute $M[f^{-1}] := R_n[f^{-1}] \otimes_{R_n} M$
- Can be generalised to M only being holonomic on $K^n \setminus \mathcal{V}(f)$
- "localise away" non-holonomic locus
- Generalisation to last weeks with $M = R_n$

Plan

- Want to find a generator $f^a \otimes 1$ and its annihilator
- Call this generator $f^s \otimes 1 \otimes 1 \in f^s \otimes_K R_n[f^{-1}, s] \otimes_{R_n} M$

(1) Compute $J^I(f^s) := \mathrm{Ann}_{D_n[s]}(f^s \otimes 1 \otimes 1)$

(2) Compute a suitable number $a \in K$ for substituting s by a

- $\underline{x}$ operates by left multiplication on the right factor
- $\partial_i \bullet (f^s \otimes \frac{g(\underline{x},s)}{f^k} \otimes Q) =$
  $f^s \otimes \frac{sg(\underline{x},s)f_i}{f^{k+1}} \otimes Q + f^s \otimes \partial_i(\frac{g(\underline{x},s)}{f^k}) \otimes Q + f^s \otimes \frac{g(\underline{x},s)}{f^k} \otimes \partial_i Q$
- $f_i := \frac{\partial f}{\partial x_i}$.

(1) The Annihilator

Aim of this section:
Compute $J^I(f^s) := \mathrm{Ann}_{D_n[s]}(f^s \otimes 1 \otimes 1)$

Idea

- Extend $D_n[s]$ to $D_{n+1} := D_n\langle t, \partial_t \rangle$
- $t \bullet (f^s \otimes \frac{g(\underline{x},s)}{f^k} \otimes Q) := f^s \otimes \frac{g(\underline{x},s+1)f}{f^k} \otimes Q$
- $\partial_t \bullet (f^s \otimes \frac{g(\underline{x},s)}{f^k} \otimes Q) := f^s \otimes \frac{-sg(\underline{x},s-1)}{f^{k+1}} \otimes Q$
- Try to compute $J^I_{n+1}(f^s) := \mathrm{Ann}_{D_{n+1}}(f^s \otimes 1 \otimes 1)$
- "Intersect" this with $D_n[s]$
- $-\partial_t t$ acts by s, so $D_n[s] \hookrightarrow D_{n+1}$

Getting many Generators

- $\phi : D_{n+1} \xrightarrow{\sim} D_{n+1} : x_i \mapsto x_i, t \mapsto t - f, \partial_i \mapsto \partial_i + f_i \partial_t, \partial_t \mapsto \partial_t$
- Lemma: Let I be f-saturated. Then

$$J_{n+1}^{I}(f^s) =_{D_{n+1}} \langle \phi(I), t - f \rangle$$

holds.

Intersecting away

- Input: Left ideal I of D_{n+1}
- Output: $J = I \cap D_n[s] = I \cap D_n[-\partial_t t]$
- Weight vector w on $D_{n+1}[y_1, y_2]$ by $w(t) = 1, w(\partial_t) = -1, w(x_i) = w(\partial_i) = 0, w(y_1) = 1, w(y_2) = -1$
- Homogenize I by y_1 according to w
- Compute Gröbner basis $\tilde{J}$ of this ideal and $1 - y_1 y_2$ eliminating y_1 and y_2
- Take elements of $\tilde{J}$ not having y_1 or y_2 and multiply them with appropriate powers of t and ∂_t to give them a w-degree of 0
- Return these elements
- The above lemma in combination with this algorithm gives the solution to this section's problem.

(2) The Generator

Aim of this section:
Compute a suitable number $a \in K$ to get a generator and to substitute s by a in last section's result

Bernstein Polynomial

- Bernstein polynomial $b_f^I(s) \in K[s]$: the monic generator for all $b \in K[s]$, s.t. exists $Q(s) \in D_n[s]$ with:

$$b(s) \bullet (f^s \otimes 1 \otimes 1) = Q(s) \bullet (f^s \otimes f \otimes 1) = Q(s) f \bullet (f^s \otimes 1 \otimes 1)$$

- Fix $Q_f^I(s)$ as operator with above properties
- Idea: $\frac{Q_f^I(s)}{b_f^I(s)}$ is some kind of "inverse" for f

Computing the Bernstein Polynomial

- Input: $f \in R_n$ and f-saturated holonomic ideal $I \trianglelefteq D_n$
- Output: $b_f^I(s)$
- Compute $J^I(f^s)$ by means of section 1
- Compute the monic generator of ${}_{D_n[s]}\langle f, J^I(f^s)\rangle \cap K[s]$

Determine the Exponent

- Theorem: $M = D_n/I$ holonomic and $a \in K^*$, such that no element of $\{a-1, a-2, ..\}$ is root of $b_f^I(s)$, then:

$$f^a \otimes_K R_n[f^{-1}] \otimes_{R_n} M$$

$$\cong D_n \bullet (f^a \otimes 1 \otimes 1)$$

$$\cong (D_n[s]/J^I(f^s))_{|s=a}$$

- Take a as smallest negative integer root of $b_f^I(s)$. If no such number exists, then $a := -1$

Final Algorithm

- Input: $f \in R_n$, $M = D_n/I$ holonomic and f-saturated
- Output: $J \trianglelefteq D_n$ and $a \in \mathbb{Z}$ with $R_n[f^{-1}] \otimes_{R_n} M \cong D_n/J$ generated by $f^a \otimes 1$.
- Determine $J^I(f^s)$ as in section 1
- Determine $b_f^I(s)$ as in section 2
- Find the smallest integer root a of $b_f^I(s)$. If not exist, a:=-1
- Replace s by a in each generator of $J^I(f^s) \rightsquigarrow J$

Final words:

↬ Thanks For Your Attention ↫