(A very personal view on) non-commutative Gröbner bases for Weyl, shift and their homogenized algebras

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Plan of Attack

Roadmap

- monomial orderings on $\mathbb{K}[\mathbf{x}]$ and $\mathbb{N}^n$
- Gröbner bases in $\mathbb{K}[\mathbf{x}]$
- Weyl, shift and homogenized algebras
- generalized framework: G-algebras
- left Gröbner bases in G-algebras
- different notations concerning GB
- application: GK dimension

Preliminaries: Monomials and Monoideals

Let $\mathbb{K}$ be a field and R be a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$.
R is infinite dimensional over $\mathbb{K}$, the $\mathbb{K}$–basis of R consists of $\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$. We call such elements **monomials** of R.
There is 1–1 correspondence

$$\mathrm{Mon}(R) \ni x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n.$$

$\mathbb{N}^n$ is a monoid with the neutral element $\overline{0} = (0, \ldots, 0)$ and the only operation $+$. A subset $S \subseteq \mathbb{N}^n$ is called a (additive) **monoid ideal (monoideal)**, if $\forall\, \alpha \in S$, $\forall\, \beta \in \mathbb{N}^n$ we have $\alpha + \beta \in S$.

Lemma (Dixon, 1913)

Every monoideal in $\mathbb{N}^n$ is finitely generated. That is, for any $S \subseteq \mathbb{N}^n$ there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{N}^n$, such that $S = {}_{\mathbb{N}^n}\langle \alpha_1, \ldots, \alpha_m \rangle$.

Orderings

Definition

1. a total ordering $\prec$ on $\mathbb{N}^n$ is called a **well–ordering**, if
   - $\forall F \subseteq \mathbb{N}^n$ there exists a minimal element of F, in particular $\forall\, a \in \mathbb{N}^n,\ 0 \prec a$
2. an ordering $\prec$ is called a **monomial ordering on** R, if
   - $\forall \alpha, \beta \in \mathbb{N}^n\ \alpha \prec \beta \ \Rightarrow\ x^\alpha \prec x^\beta$
   - $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$ such that $x^\alpha \prec x^\beta$ we have $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.
3. Any $f \in R \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with $c \in \mathbb{K}^*$ and $x^{\alpha'} \prec x^\alpha$ for any non–zero term $c'x^{\alpha'}$ of f'. We define

   $\mathrm{lm}(f) = x^\alpha$, the **leading monomial** of f

   $\mathrm{lc}(f) = c$, the **leading coefficient** of f

   $\mathrm{lex}(f) = \alpha$, the **leading exponent** of f.

Gröbner Basis: Preparations

From now on, we assume that a given ordering is a **well-ordering**.

Definition

We say that monomial x^α **divides** monomial x^β, if $\alpha_i \leq \beta_i \; \forall i = 1 \ldots n$. We use the notation $x^\alpha \mid x^\beta$.

It means that x^β is **reducible** by x^α, that is $\beta \subset {}_{\mathbb{N}^n}\langle \alpha \rangle$. Equivalently, there exists $\gamma \in \mathbb{N}^n$, such that $\beta = \alpha + \gamma$. It also means that $x^\beta = x^\alpha x^\gamma$.

Definition

Let $\prec$ be a monomial ordering on R, $I \subset R$ be an ideal and $G \subset I$ be a finite subset. G is called a **Gröbner basis** of I,
if $\forall \; f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\mathrm{lm}(g) \mid \mathrm{lm}(f)$.

Characterizations of Gröbner Bases

Definition

Let S be any subset of R.

- We define a **monoideal of leading exponents** $\mathcal{L}(S) \subseteq \mathbb{N}^n$ to be a $\mathbb{N}^n$–monoideal $\mathcal{L}(S) = {}_{\mathbb{N}^n}\langle \alpha \mid \exists s \in S,\ \text{lex}(s) = \alpha \rangle$, generated by the leading exponents of elements of S.
- $L(S)$, the **span of leading monomials of** S, is defined to be the $\mathbb{K}$–vector space, spanned by the set $\{x^\alpha \mid \alpha \in \mathcal{L}(S)\} \subseteq R$.

Equivalences

- G is a Gröbner basis of $I \Leftrightarrow \forall\, f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\text{lm}(g) \mid \text{lm}(f)$,
- G is a Gröbner basis of $I \Leftrightarrow L(G) = L(I)$ as $\mathbb{K}$–vector spaces,
- G is a Gröbner basis of $I \Leftrightarrow \mathcal{L}(G) = \mathcal{L}(I)$ as $\mathbb{N}^n$–monoideals.

Weyl and shift algebras

Let $\mathbb{K}$ be a field and R be a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$.

$$\textbf{Weyl } D = D(R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_j x_i = x_i \partial_j + \delta_{ij}\}\rangle.$$

The $\mathbb{K}$–basis of D is

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_n^{\beta_n} \mid \alpha_i \geq 0, \beta_j \geq 0\}$$

$$\textbf{Shift } S = S(R) = \mathbb{K}\langle y_1, \ldots, y_n, s_1, \ldots, s_n \mid \{s_j y_i = y_i s_j + \delta_{ij} \cdot s_j\}\rangle.$$

The $\mathbb{K}$–basis of S is

$$\{y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} s_1^{\beta_1} s_2^{\beta_2} \ldots s_n^{\beta_n} \mid \alpha_i \geq 0, \beta_j \geq 0\}$$

Weyl and shift algebras under homogenization

Let w be the weight vector $(u_1, \ldots, u_n, v_1, \ldots, v_n)$, $u_i + v_i \geq 0$. Assigning weights u_i to x_i and v_i to ∂_i, we introduce a new commutative variable h and homogenize the relation into $\partial_j x_j = x_j \partial_j + h^{u_j+v_j}$.

$$D_w^{(h)}(R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, \mathbf{h} \mid \{\partial_j x_i = x_i \partial_j + \delta_{ij} \mathbf{h}^{u_i+v_j}\}\rangle.$$

The $\mathbb{K}$–basis of D is

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_n^{\beta_n} \mathbf{h}^{\gamma} \mid \alpha_i \geq 0, \beta_j \geq 0, \gamma \geq 0\}$$

Assigning weights u_i to y_i and v_i to s_i, we introduce a new commutative variable h and homogenize the relation into $s_j y_i = y_i s_j + \delta_{ij} \cdot s_j h^{u_j}$.

$$S_w^{(h)}(R) = \mathbb{K}\langle y_1, \ldots, y_n, s_1, \ldots, s_n, \mathbf{h} \mid \{s_j y_i = y_i s_j + \delta_{ij} \cdot s_j \mathbf{h}^{u_j}\}\rangle.$$

The $\mathbb{K}$–basis of S is

$$\{y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} s_1^{\beta_1} s_2^{\beta_2} \ldots s_n^{\beta_n} \mathbf{h}^{\gamma} \mid \alpha_i \geq 0, \beta_j \geq 0, \gamma \geq 0\}$$

Yet another homogenization

Let w be the weight vector $(u_1, \ldots, u_n, v_1, \ldots, v_n)$, such that $u_i + v_i = 0$, in other words $u_i = -w_i$, $v_i = w_i$.

Since we need nonnegative weights for Gröbner basis, we do the following. We introduce a new commutative variable h and homogenize the relation into $\partial_j(x_j h^{w_j}) = (x_j h^{w_j})\partial_j + h^{w_j}$. In what follows, we denote $x_j h^{w_j}$ by x_j, it has weight 0.

The examples before suggest a more general framework.

Computational Objects

Suppose we are given the following data

1. a field $\mathbb{K}$ and a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$,
2. a set $C = \{c_{ij}\} \subset \mathbb{K}^*$, $1 \leq i < j \leq n$
3. a set $D = \{d_{ij}\} \subset R, \quad 1 \leq i < j \leq n$

Assume, that there exists a monomial well–ordering $\prec$ on R such that

$$\forall 1 \leq i < j \leq n, \ \ \mathrm{lm}(d_{ij}) \prec x_i x_j.$$

The Construction

To the data $(R, C, D, \prec)$ we associate an algebra

$$A = \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij}\} \ \forall 1 \leq i < j \leq n \rangle$$

PBW Bases and G–algebras

Define the (i, j, k)–*nondegeneracy condition* to be the polynomial

$$NDC_{ijk} := c_{ik}c_{jk} \cdot d_{ij}x_k - x_k d_{ij} + c_{jk} \cdot x_j d_{ik} - c_{ij} \cdot d_{ik}x_j + d_{jk}x_i - c_{ij}c_{ik} \cdot x_i d_{jk}.$$

Theorem (Levandovskyy)

$A = A(R, C, D, \prec)$ *has a PBW basis* $\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}\}$ *if and only if*

$$\forall\ 1 \leq i < j < k \leq n, \ \ NDC_{ijk} \textit{ reduces to } 0 \textit{ w.r.t. relations}$$

Easy Check $NDC_{ijk} = x_k(x_j x_i) - (x_k x_j)x_i$.

Definition

An algebra $A = A(R, C, D, \prec)$, where nondegeneracy conditions vanish, is called **a G–algebra** (in n variables).

G–algebras

We call A a G–algebra of Lie type, if the relations of A are of the form $\{x_j x_i = x_i x_j + d_{ij}\}$ $\forall 1 \leq i < j \leq n$ and the conditions above hold.

Theorem (Properties of G–algebras)

Let A be a G–algebra in n variables. Then

- *A is left and right Noetherian,*
- *A is an integral domain,*
- *the Gel'fand–Kirillov dimension over $\mathbb{K}$ is* $\operatorname{GK.dim}(A) = n$,
- *the global homological dimension* $\operatorname{gl.dim}(A) \leq n$,
- *the Krull dimension* $Kr.dim(A) \leq n$.

Gröbner Bases for Modules I

Let $S \subseteq R^r$ be a left submodule of the free module R^r. Then, it is given via its generators (vectors of R^r), or via a matrix with r rows.

Definition

- $x^\alpha e_i$ **divides** $x^\beta e_j$, iff $i = j$ and $x^\alpha \mid x^\beta$.
- Let $\prec$ be a monomial module ordering on R^r, $I \subset R$ be a submodule and $G \subset I$ be a finite subset. G is called a **Gröbner basis** of I, if $\forall\ f \in I \setminus \{0\},\ \ \exists g \in G$ satisfying $\mathrm{lm}(g) \mid \mathrm{lm}(f)$.

Denote $\mathbb{N}_r := \{1, 2, \ldots, r\} \subset \mathbb{N}$. The action of $\mathbb{N}^n$ on $\mathbb{N}_r \times \mathbb{N}^n$, given by $\gamma : (i, \alpha) \mapsto (i, \alpha + \gamma)$ makes $\mathbb{N}_r \times \mathbb{N}^n$ an $\mathbb{N}^n$–monoideal (wrt addition).

Definition. Let S be any subset of R.

- We define a **monoideal of leading exponents** $\mathcal{L}(S) \subseteq \mathbb{N}_r \times \mathbb{N}^n$ to be a $\mathbb{N}^n$–monoideal $\mathcal{L}(S) = {}_{\mathbb{N}^n}\langle (i, \alpha) \mid \exists s \in S,\ \leq(s) = x^\alpha e_i \rangle$.
- $L(S)$, the **span of leading monomials of** S, is defined to be the $\mathbb{K}$–vector space, spanned by the set $\{x^\alpha e_i \mid (i, \alpha) \in \mathcal{L}(S)\} \subseteq R^r$.

Gröbner Bases for Modules II

G is a Gröbner basis of I $\Leftrightarrow$

- $\forall\, f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\text{lm}(g) \mid \text{lm}(f)$,
- $L(G) = L(I)$ as $\mathbb{K}$–vector spaces,
- $\mathcal{L}(G) = \mathcal{L}(I)$ as $\mathbb{N}^n$–monoideals.

A subset $S \subset R^r$ is called **minimal**, if $0 \notin S$ and $\text{lm}(s) \notin L(S \setminus \{s\})$ for all $s \in S$.

A subset $S \subset R^r$ is called **reduced**, if $0 \notin S$, and if for each $s \in S$, s is reduced with respect to $S \setminus \{s\}$, and, moreover, $s - \text{lc}(s)\,\text{lm}(s)$ is reduced with respect to S.

It means that for each $s \in S \subset R^r$, $\text{lm}(s)$ does not divide any monomial of every element of S except itself.

Gröbner Bases for Modules III

Definition

Denote by $\mathcal{G}$ the set of all finite ordered subsets of R^r.

1. A map $\mathrm{NF} : R^r \times \mathcal{G} \to R^r, \quad (f, G) \mapsto \mathrm{NF}(f|G)$, is called a **left normal form** on R^r if, for all $f \in R^r$, $G \in \mathcal{G}$,

   (i) $\mathrm{NF}(0 \mid G) = 0$,
   (ii) $\mathrm{NF}(f \mid G) \neq 0 \quad \Rightarrow \quad \mathrm{lm}\big(\mathrm{NF}(f|G)\big) \notin L(G)$,
   (iii) $f - \mathrm{NF}(f \mid G) \in {}_R\langle G \rangle$.

   NF is called a **reduced n. f.** if $\mathrm{NF}(f|G)$ is reduced wrt G.

2. Let $G = \{g_1, \dots, g_s\} \in \mathcal{G}$. A representation of $f \in R$,

$$f - \mathrm{NF}(f \mid G) = \sum_{i=1}^{s} a_i g_i, \quad a_i \in R,$$

   satisfying $\mathrm{lm}(\sum_{i=1}^{s} a_i g_i) \geq \mathrm{lm}(a_i g_i)$ for all $i = 1 \dots s$ such that $a_i g_i \neq 0$ is called a **left standard representation** of f (wrt G).

Normal Form: Properties

Let A be a G-algebra.

Lemma

Let $I \subset A^r$ be a left submodule, $G \subset I$ be a Gröbner basis of I and $\mathrm{NF}(\cdot|G)$ be a left normal form on A^r with respect to G.

1. *For any $f \in A^r$ we have $f \in I \iff \mathrm{NF}(f \mid G) = 0$.*
2. *If $J \subset A^r$ is a left submodule with $I \subset J$, then $L(I) = L(J)$ implies $I = J$. In particular, G generates I as a left A–module.*
3. *If $\mathrm{NF}(\cdot|G)$ is a reduced left normal form, then it is unique.*

Buchberger's Criterion Theorem

Let A be a G-algebra of Lie type.

Definition

Let $f, g \in A^r$ with $\mathrm{lm}(f) = x^\alpha e_i$ and $\mathrm{lm}(g) = x^\beta e_j$. Set $\gamma = \mu(\alpha, \beta)$, $\gamma_i := \max(\alpha_i, \beta_i)$ and define the left **s–polynomial** of (f, g) to be $\mathrm{LeftSpoly}(f, g) := x^{\gamma-\alpha} f - \frac{\mathrm{lc}(f)}{\mathrm{lc}(g)} x^{\gamma-\beta} g$ if $i = j$ and 0 otherwise.

For a general G-algebra the formula for spoly is more involved.

Theorem

Let $I \subset A^r$ be a left submodule and $G = \{g_1, \ldots, g_s\}$, $g_i \in I$. Let $\mathrm{LeftNF}(\cdot|G)$ be a left normal form on A^r w.r.t G. Then the following are equivalent:

1. G is a left Gröbner basis of I,
2. $\mathrm{LeftNF}(f|G) = 0$ for all $f \in I$,
3. each $f \in I$ has a left standard representation with respect to G,
4. $\mathrm{LeftNF}\big(\mathrm{LeftSpoly}(g_i, g_j)|G\big) = 0$ for $1 \leq i, j \leq s$.

Left Normal Form: Algorithm

LEFTNF(f, G)

- Input : $f \in A^r$, $G \in \mathcal{G}$;
- Output: $h \in A^r$, a left normal form of f with respect to G.

- $h := f$;
- **while** ($(h \neq 0)$ **and** $(G_h = \{g \in G : \text{lm}(g) \mid \text{lm}(h)\} \neq \emptyset)$)
  - **choose any** $g \in G_h$;
  - $h := \text{LeftSpoly}(h, g)$;
- **return** h;

Buchberger's Gröbner Basis Algorithm

Let $\prec$ be a fixed well-ordering on the G-algebra A.

GRÖBNERBASIS(G,LEFTNF)

- Input: Left generating set $G \in \mathcal{G}$
- Output: $S \in \mathcal{G}$, a left Gröbner basis of $I = {}_A\langle G \rangle \subset A^r$.

- $S = G$;
- $P = \{(f,g) | f,g \in S\} \subset S \times S$;
- **while** $(P \neq \emptyset)$
  - **choose** $(f,g) \in P$;
  - $P = P \setminus \{(f,g)\}$;
  - $h = \text{LEFTNF}\big(\text{LeftSpoly}(f,g) | S\big)$;
  - **if** $(h \neq 0)$
    - $P = P \cup \{(h,f) | f \in S\}$;
    - $S = S \cup h$;
- **return** S;

Criteria for detecting useless critical pairs

Let A be an associative $\mathbb{K}$–algebra. We use the following notations: $[a, b] := ab - ba$, a *commutator* or a *Lie bracket* of $a, b \in A$. $\forall a, b, c \in A$ we have $[a, b] = -[b, a]$ and $[ab, c] = a[b, c] + [a, c]b$. The following result is due to Levandovskyy and Schönemann (2003).

Generalized Product Criterion

Let A be a G–algebra of Lie type (that is, all $c_{ij} = 1$). Let $f, g \in A$. Suppose that $\mathrm{lm}(f)$ and $\mathrm{lm}(g)$ have no common factors, then $\mathrm{spoly}(f, g) \to_{\{f,g\}} [f, g]$.

The following classical criterion generalizes to G-algebras.

Chain Criterion

If (f_i, f_j), (f_i, f_k) and (f_j, f_k) are in the set of pairs P, denote $\mathrm{lm}(f_\nu) = x^{\alpha_\nu}$. If $x^{\alpha_j} \mid \mathrm{lcm}(x^{\alpha_i}, x^{\alpha_k})$ holds, then we can delete (f_i, f_k) from P.

Gel'fand–Kirillov dimension

Let R be an associative $\mathbb{K}$–algebra with generators $x_1, \ldots, x_m$.

A degree filtration

Consider the vector space $V = \mathbb{K}x_1 \oplus \ldots \oplus \mathbb{K}x_m$.
Set $V_0 = \mathbb{K}$, $V_1 = \mathbb{K} \oplus V$ and $V_{n+1} = V_n \oplus V^{n+1}$.
For any fin. gen. left R–module M, there exists a fin.–dim. subspace $M_0 \subset M$ such that $RM_0 = M$.
An ascending filtration on M is defined by $\{H_n := V_n M_0, n \geq 0\}$.

Definition

The **Gel'fand–Kirillov dimension** of M is defined to be

$$\mathrm{GK.dim}(M) \; = \; \limsup_{n\to\infty} \log_n (\dim_{\mathbb{K}} H_n)$$

Gel'fand–Kirillov Dimension: Examples

Let $\deg x_i = 1$, consider filtrations up to degree d. We have $V_d = \{f \mid \deg f = d\}$ and $V^d = \{f \mid \deg f \leq d\}$.

Lemma

Let A be a $\mathbb{K}$-algebra with PBW basis $\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mid \alpha_i \geq 0\}$. Then $\mathrm{GK.dim}(A) = n$.

Proof.

$\dim V_d = \binom{d+n-1}{n-1}$, $\dim V^d = \binom{d+n}{n}$. Thus $\binom{d+n}{n} = \frac{(d+n)\ldots(d+1)}{n!} = \frac{d^n}{n!} +$ l.o.t, so we have $\mathrm{GK.dim}(A) = \limsup_{d\to\infty} \log_d \binom{d+n}{n} = n$. □

$T = \mathbb{K}\langle x_1, \ldots, x_n \rangle$

$\dim V_d = n^d$, $\dim V^d = \frac{n^{d+1}-1}{n-1}$.

Since $\frac{n^{d+1}-1}{n-1} > n^d$, we are dealing with so–caled **exponential growth**.

In particular, $\log_d n^d = d \log_d n = \frac{d}{\log_n d} \to \infty, d \to \infty$.

Hence, $\mathrm{GK.dim}(T) = \infty$.

Gel'fand–Kirillov Dimension for Modules

There is an algorithm by Gomez-Torrecillaz et.al., which computes Gel'fand–Kirillov dimension for finitely presented modules over G-algebras.

GKDIM(F)

Let A be a G–algebra in variables $x_1, \ldots, x_n$.

- Input: Left generating set $F = \{f_1, \ldots, f_m\} \subset A^r$
- Output: $k \in \mathbb{N}$, $k = \text{GK}.\dim(A^r/M)$, where $M = {}_A\langle F \rangle \subseteq A^r$.

- $G = \text{LeftGröbnerBasis}(F) = \{g_1, \ldots, g_t\}$;
- $L = \{\text{lm}(g_i) = x^{\alpha_i} e_s \mid 1 \leq i \leq t\}$;
- $N = {}_{K[x_1,\ldots,x_n]}\langle L \rangle$;
- **return** $\text{Kr}.\dim(K[x_1, \ldots, x_n]^r/N)$;

Ring-theoretic Properties of Weyl and shift algebras

$\operatorname{gl.dim}(A)$, the global homological dimension of A

- $\operatorname{gl.dim}(S) = 2n$,
- if $\operatorname{char} \mathbb{K} = 0$, $\operatorname{gl.dim}(D) = n$,
- if $\operatorname{char} \mathbb{K} = p > 0$, $\operatorname{gl.dim}(D) = 2n$.

$Z(A) = \{z \in A \mid za = az \; \forall a \in A\}$, the center of A

- if $\operatorname{char} \mathbb{K} = 0$, $Z(D) = Z(S) = \mathbb{K}$,
- if $\operatorname{char} \mathbb{K} = p > 0$, $Z(D) = \{x_i^p, \partial_i^p\}$.
- if $\operatorname{char} \mathbb{K} = p > 0$, $Z(S) = \{y_i^p - y_i, s_i^p\}$.

If $\operatorname{char} \mathbb{K} = 0$, $D(R)$ has no proper two–sided ideals.
In $S(R)$, $I_\gamma = {}_S\langle \{s_i, y_i - \gamma_i\} \rangle_S$ is a family of such ideals for $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{K}^n$.

Thank you for your attention!

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