(A very personal view on) non-commutative Gröbner bases for Weyl, shift and their homogenized algebras

Viktor Levandovskyy

RWTH Aachen, Germany

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## Plan of Attack

### Roadmap

- Monomial orderings on $\mathbb{K}[\mathbf{x}]$ and $\mathbb{N}^n$
- Gröbner bases in $\mathbb{K}[\mathbf{x}]$
- Weyl, shift and homogenized algebras
- Generalized framework: $G$-algebras
- Left Gröbner bases in $G$-algebras
- Different notations concerning GB
- Application: GK dimension
Preliminaries: Monomials and Monoideals

Let $\mathbb{K}$ be a field and $R$ be a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$. $R$ is infinite dimensional over $\mathbb{K}$, the $\mathbb{K}$–basis of $R$ consists of $\{x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} | \alpha_i \in \mathbb{N}\}$. We call such elements monomials of $R$. There is 1–1 correspondence

$$\text{Mon}(R) \ni x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n} \mapsto (\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n.$$

$\mathbb{N}^n$ is a monoid with the neutral element $\overline{0} = (0, \ldots, 0)$ and the only operation $\oplus$. A subset $S \subseteq \mathbb{N}^n$ is called a (additive) monoid ideal (monoideal), if $\forall \alpha \in S, \forall \beta \in \mathbb{N}^n$ we have $\alpha + \beta \in S$.

**Lemma (Dixon, 1913)**

*Every monoideal in $\mathbb{N}^n$ is finitely generated. That is, for any $S \subseteq \mathbb{N}^n$ there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{N}^n$, such that $S = \mathbb{N}^n\langle \alpha_1, \ldots, \alpha_m \rangle$.***
Orderings

Definition

1. A total ordering $\prec$ on $\mathbb{N}^n$ is called a **well–ordering**, if
   - $\forall F \subseteq \mathbb{N}^n$ there exists a minimal element of $F$,
   - in particular $\forall a \in \mathbb{N}^n$, $0 \prec a$

2. An ordering $\prec$ is called a **monomial ordering on** $R$, if
   - $\forall \alpha, \beta \in \mathbb{N}^n \alpha \prec \beta \Rightarrow x^\alpha \prec x^\beta$
   - $\forall \alpha, \beta, \gamma \in \mathbb{N}^n$ such that $x^\alpha \prec x^\beta$ we have $x^{\alpha+\gamma} \prec x^{\beta+\gamma}$.

3. Any $f \in R \setminus \{0\}$ can be written uniquely as $f = cx^\alpha + f'$, with
   - $c \in K^*$ and $x^{\alpha'} \prec x^\alpha$ for any non–zero term $c'x^{\alpha'}$ of $f'$.
   - We define $\text{lm}(f) = x^\alpha$, the **leading monomial** of $f$
   - $\text{lc}(f) = c$, the **leading coefficient** of $f$
   - $\text{lex}(f) = \alpha$, the **leading exponent** of $f$. 
Gröbner Basis: Preparations

From now on, we assume that a given ordering is a well-ordering.

**Definition**

We say that monomial $x^\alpha$ **divides** monomial $x^\beta$, if $\alpha_i \leq \beta_i \ \forall i = 1 \ldots n$. We use the notation $x^\alpha | x^\beta$.

It means that $x^\beta$ is **reducible** by $x^\alpha$, that is $\beta \subset \mathbb{N}^n\langle \alpha \rangle$. Equivalently, there exists $\gamma \in \mathbb{N}^n$, such that $\beta = \alpha + \gamma$. It also means that $x^\beta = x^\alpha x^\gamma$.

**Definition**

Let $\prec$ be a monomial ordering on $R$, $I \subset R$ be an ideal and $G \subset I$ be a finite subset. $G$ is called a **Gröbner basis** of $I$, if $\forall f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\text{lm}(g) | \text{lm}(f)$.
Characterizations of Gröbner Bases

**Definition**

Let $S$ be any subset of $R$.

- We define a **monoideal of leading exponents** $\mathcal{L}(S) \subseteq \mathbb{N}^n$ to be a $\mathbb{N}^n$–monoideal $\mathcal{L}(S) = \mathbb{N}^n \langle \alpha \mid \exists s \in S, \text{lex}(s) = \alpha \rangle$, generated by the leading exponents of elements of $S$.

- $L(S)$, the **span of leading monomials of** $S$, is defined to be the $\mathbb{K}$–vector space, spanned by the set $\{x^\alpha \mid \alpha \in \mathcal{L}(S)\} \subseteq R$.

**Equivalences**

- $G$ is a Gröbner basis of $I \iff \forall f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\text{lm}(g) \mid \text{lm}(f)$,

- $G$ is a Gröbner basis of $I \iff L(G) = L(I)$ as $\mathbb{K}$–vector spaces,

- $G$ is a Gröbner basis of $I \iff \mathcal{L}(G) = \mathcal{L}(I)$ as $\mathbb{N}^n$–monoideals.
Weyl and shift algebras

Let $\mathbb{K}$ be a field and $R$ be a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$.

**Weyl** $D = D(R) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{ \partial_j x_i = x_i \partial_j + \delta_{ij} \}\rangle$.

The $\mathbb{K}$–basis of $D$ is
\[
\{ x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_n^{\beta_n} \mid \alpha_i \geq 0, \beta_j \geq 0 \}
\]

**Shift** $S = S(R) = \mathbb{K}\langle y_1, \ldots, y_n, s_1, \ldots, s_n \mid \{ s_j y_i = y_i s_j + \delta_{ij} \cdot s_j \}\rangle$.

The $\mathbb{K}$–basis of $S$ is
\[
\{ y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} s_1^{\beta_1} s_2^{\beta_2} \ldots s_n^{\beta_n} \mid \alpha_i \geq 0, \beta_j \geq 0 \}
\]
Weyl and shift algebras under homogenization

Let $w$ be the weight vector $(u_1, \ldots, u_n, v_1, \ldots, v_n)$, $u_i + v_i \geq 0$. Assigning weights $u_i$ to $x_i$ and $v_i$ to $\partial_i$, we introduce a new commutative variable $h$ and homogenize the relation into $\partial_j x_j = x_j \partial_j + h^{u_i + v_i}$.

$$D_{w}^{(h)} (R) = \mathbb{K} \langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, h \mid \{ \partial_j x_i = x_i \partial_j + \delta_{ij} h^{u_i + v_i} \} \rangle.$$  

The $\mathbb{K}$–basis of $D$ is

$$\{ x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_n^{\beta_n} h^\gamma \mid \alpha_i \geq 0, \beta_j \geq 0, \gamma \geq 0 \}$$

Assigning weights $u_i$ to $y_i$ and $v_i$ to $s_i$, we introduce a new commutative variable $h$ and homogenize the relation into $s_j y_i = y_i s_j + \delta_{ij} \cdot s_j h^{u_i}$.

$$S_{w}^{(h)} (R) = \mathbb{K} \langle y_1, \ldots, y_n, s_1, \ldots, s_n, h \mid \{ s_j y_i = y_i s_j + \delta_{ij} \cdot s_j h^{u_i} \} \rangle.$$  

The $\mathbb{K}$–basis of $S$ is

$$\{ y_1^{\alpha_1} y_2^{\alpha_2} \ldots y_n^{\alpha_n} s_1^{\beta_1} s_2^{\beta_2} \ldots s_n^{\beta_n} h^\gamma \mid \alpha_i \geq 0, \beta_j \geq 0, \gamma \geq 0 \}$$
Yet another homogenization

Let $w$ be the weight vector $(u_1, \ldots, u_n, v_1, \ldots, v_n)$, such that $u_i + v_i = 0$, in other words $u_i = -w_i$, $v_i = w_i$.

Since we need nonnegative weights for Gröbner basis, we do the following. We introduce a new commutative variable $h$ and homogenize the relation into $\partial_j(x_j h^{w_j}) = (x_j h^{w_j}) \partial_j + h^{w_j}$. In what follows, we denote $x_j h^{w_j}$ by $x_j$, it has weight 0.

The examples before suggest a more general framework.
Computational Objects

Suppose we are given the following data

1. a field $\mathbb{K}$ and a commutative ring $R = \mathbb{K}[x_1, \ldots, x_n]$,
2. a set $C = \{c_{ij}\} \subset \mathbb{K}^*$, $1 \leq i < j \leq n$
3. a set $D = \{d_{ij}\} \subset R$, $1 \leq i < j \leq n$

Assume, that there exists a monomial well–ordering $\prec$ on $R$ such that

$$\forall 1 \leq i < j \leq n, \text{ lm}(d_{ij}) \prec x_i x_j.$$

The Construction

To the data $(R, C, D, \prec)$ we associate an algebra

$$A = \mathbb{K}\langle x_1, \ldots, x_n \mid \{x_j x_i = c_{ij} x_i x_j + d_{ij} \} \forall 1 \leq i < j \leq n \rangle$$
PBW Bases and $G$–algebras

Define the $(i, j, k)$–nondegeneracy condition to be the polynomial

$$NDC_{ijk} := c_{ik}c_{jk} \cdot d_{ij}x_k - x_kd_{ij} + c_{jk} \cdot x_jd_{ik} - c_{ij} \cdot d_{ik}x_j + d_{jk}x_i - c_{ij}c_{ik} \cdot x_id_{jk}.$$ 

**Theorem (Levandovskyy)**

$A = A(R, C, D, \prec)$ has a PBW basis $\{x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_n^{\alpha_n}\}$ if and only if

$$\forall \ 1 \leq i < j < k \leq n, \ NDC_{ijk} \text{ reduces to 0 w.r.t. relations}$$

**Easy Check** \ $NDC_{ijk} = x_k(x_jx_i) - (x_kx_j)x_i$.

**Definition**

An algebra $A = A(R, C, D, \prec)$, where nondegeneracy conditions vanish, is called a $G$–algebra (in $n$ variables).
We call $A$ a $G$–algebra of Lie type, if the relations of $A$ are of the form
\[ x_j x_i = x_i x_j + d_{ij} \] \( \forall 1 \leq i < j \leq n \) and the conditions above hold.

**Theorem (Properties of $G$–algebras)**

Let $A$ be a $G$–algebra in $n$ variables. Then
- $A$ is left and right Noetherian,
- $A$ is an integral domain,
- the Gel’fand–Kirillov dimension over $\mathbb{K}$ is $\text{GK. dim}(A) = n$,
- the global homological dimension $\text{gl. dim}(A) \leq n$,
- the Krull dimension $\text{Kr. dim}(A) \leq n$. 
Gröbner Bases for Modules I

Let $S \subseteq R^r$ be a left submodule of the free module $R^r$. Then, it is given via its generators (vectors of $R^r$), or via a matrix with $r$ rows.

**Definition**

- $x^\alpha e_i$ divides $x^\beta e_j$, iff $i = j$ and $x^\alpha | x^\beta$.
- Let $\prec$ be a monomial module ordering on $R^r$, $I \subseteq R$ be a submodule and $G \subseteq I$ be a finite subset. $G$ is called a Gröbner basis of $I$, if $\forall f \in I \setminus \{0\}$, $\exists g \in G$ satisfying $\text{lm}(g) | \text{lm}(f)$.

Denote $\mathbb{N}_r := \{1, 2, \ldots, r\} \subseteq \mathbb{N}$. The action of $\mathbb{N}^n$ on $\mathbb{N}_r \times \mathbb{N}^n$, given by $\gamma : (i, \alpha) \mapsto (i, \alpha + \gamma)$ makes $\mathbb{N}_r \times \mathbb{N}^n$ an $\mathbb{N}^n$–monoideal (wrt addition).

**Definition. Let $S$ be any subset of $R$.**

- We define a **monoid of leading exponents** $L(S) \subseteq \mathbb{N}_r \times \mathbb{N}^n$ to be a $\mathbb{N}^n$–monoideal $L(S) = \mathbb{N}^n \langle (i, \alpha) \mid \exists s \in S, \leq (s) = x^\alpha e_i \rangle$.
- $L(S)$, the **span of leading monomials** of $S$, is defined to be the $\mathbb{K}$–vector space, spanned by the set $\{x^\alpha e_i \mid (i, \alpha) \in L(S)\} \subseteq R^r$. 
Gröbner Bases for Modules II

$G$ is a Gröbner basis of $I$ if:

- $\forall f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $\text{lm}(g) \mid \text{lm}(f)$,
- $\mathcal{L}(G) = \mathcal{L}(I)$ as $K$–vector spaces,
- $\mathcal{L}(G) = \mathcal{L}(I)$ as $\mathbb{N}^n$–monoideals.

A subset $S \subset R^r$ is called **minimal**, if $0 \notin S$ and $\text{lm}(s) \notin \mathcal{L}(S \setminus \{s\})$ for all $s \in S$.

A subset $S \subset R^r$ is called **reduced**, if $0 \notin S$, and if for each $s \in S$, $s$ is reduced with respect to $S \setminus \{s\}$, and, moreover, $s - \text{lc}(s) \text{lm}(s)$ is reduced with respect to $S$.

It means that for each $s \in S \subset R^r$, $\text{lm}(s)$ does not divide any monomial of every element of $S$ except itself.
Gröbner Bases for Modules III

Definition

Denote by \( \mathcal{G} \) the set of all finite ordered subsets of \( R^r \).

1. A map \( \text{NF} : R^r \times \mathcal{G} \rightarrow R^r, \quad (f, G) \mapsto \text{NF}(f \mid G) \), is called a **left normal form** on \( R^r \) if, for all \( f \in R^r \), \( G \in \mathcal{G} \),
   
   (i) \( \text{NF}(0 \mid G) = 0 \),
   
   (ii) \( \text{NF}(f \mid G) \neq 0 \) \( \Rightarrow \) \( \text{lm}(\text{NF}(f \mid G)) \notin L(G) \),
   
   (iii) \( f - \text{NF}(f \mid G) \in R \langle G \rangle \).

\( \text{NF} \) is called a **reduced n. f.** if \( \text{NF}(f \mid G) \) is reduced wrt \( G \).

2. Let \( G = \{g_1, \ldots, g_s\} \in \mathcal{G} \). A representation of \( f \in R \),

\[
f - \text{NF}(f \mid G) = \sum_{i=1}^{s} a_i g_i, \quad a_i \in R,
\]

satisfying \( \text{lm}(\sum_{i=1}^{s} a_i g_i) \geq \text{lm}(a_i g_i) \) for all \( i = 1 \ldots s \) such that \( a_i g_i \neq 0 \) is called a **left standard representation** of \( f \) (wrt \( G \)).
Normal Form: Properties

Let $A$ be a $G$-algebra.

**Lemma**

Let $I \subset A^r$ be a left submodule, $G \subset I$ be a Gröbner basis of $I$ and $\text{NF}(\cdot | G)$ be a left normal form on $A^r$ with respect to $G$.

1. For any $f \in A^r$ we have $f \in I \iff \text{NF}(f | G) = 0$.
2. If $J \subset A^r$ is a left submodule with $I \subset J$, then $L(I) = L(J)$ implies $I = J$. In particular, $G$ generates $I$ as a left $A$–module.
3. If $\text{NF}(\cdot | G)$ is a reduced left normal form, then it is unique.
Buchberger’s Criterion Theorem

Let $A$ be a $G$-algebra of Lie type.

**Definition**

Let $f, g \in A^r$ with $\text{lm}(f) = x^\alpha e_i$ and $\text{lm}(g) = x^\beta e_j$. Set $\gamma = \mu(\alpha, \beta)$, $\gamma_i := \max(\alpha_i, \beta_i)$ and define the left $s$–polynomial of $(f, g)$ to be

$$\text{LeftSpoly}(f, g) := x^{\gamma - \alpha} f - \frac{\text{lc}(f)}{\text{lc}(g)} x^{\gamma - \beta} g \text{ if } i = j \text{ and 0 otherwise.}$$

For a general $G$-algebra the formula for spoly is more involved.

**Theorem**

Let $I \subset A^r$ be a left submodule and $G = \{g_1, \ldots, g_s\}$, $g_i \in I$. Let $\text{LeftNF}(\cdot | G)$ be a left normal form on $A^r$ w.r.t $G$. Then the following are equivalent:

1. $G$ is a left Gröbner basis of $I$,
2. $\text{LeftNF}(f|G) = 0$ for all $f \in I$,
3. each $f \in I$ has a left standard representation with respect to $G$,
4. $\text{LeftNF}(\text{LeftSpoly}(g_i, g_j)|G) = 0$ for $1 \leq i, j \leq s$. 
**Left Normal Form: Algorithm**

\[ \text{LEFTNF}(f, G) \]

- **Input:** \( f \in A^r, \ G \in \mathcal{G} \);
- **Output:** \( h \in A^r \), a left normal form of \( f \) with respect to \( G \).

1. \( h := f \);
2. **while** (\( (h \neq 0) \) and (\( G_h = \{ g \in G : \text{lm}(g) \mid \text{lm}(h) \} \neq \emptyset \) )
   - choose any \( g \in G_h \);
   - \( h := \text{LeftSpoly}(h, g) \);
3. **return** \( h \);
Buchberger’s Gröbner Basis Algorithm

Let \( \prec \) be a fixed well-ordering on the \( G \)-algebra \( A \).

**\texttt{GröbnerBasis}\,(G,\texttt{LeftNF})**

- **Input:** Left generating set \( G \in \mathcal{G} \)
- **Output:** \( S \in \mathcal{G} \), a left Gröbner basis of \( I = A\langle G \rangle \subset A^r \).

\[ S = G; \]
\[ P = \{(f, g) | f, g \in S\} \subset S \times S; \]

while \( (P \neq \emptyset) \)

choose \((f, g) \in P;\)
\[ P = P \setminus \{(f, g)\}; \]
\[ h = \text{LEFTNF}(\text{LeftSpoly}(f, g)|S); \]
if \((h \neq 0)\)
\[ P = P \cup \{(h, f) | f \in S\}; \]
\[ S = S \cup h; \]

return \( S; \)
Criteria for detecting useless critical pairs

Let $A$ be an associative $K$–algebra. We use the following notations: $[a, b] := ab − ba$, a commutator or a Lie bracket of $a, b ∈ A$. For all $a, b, c ∈ A$ we have $[a, b] = −[b, a]$ and $[ab, c] = a[b, c] + [a, c]b$. The following result is due to Levandovskyy and Schönemann (2003).

**Generalized Product Criterion**

Let $A$ be a $G$–algebra of Lie type (that is, all $c_{ij} = 1$). Let $f, g ∈ A$. Suppose that $\text{lm}(f)$ and $\text{lm}(g)$ have no common factors, then $\text{spoly}(f, g) \rightarrow \{f,g\}[f, g]$.

The following classical criterion generalizes to $G$-algebras.

**Chain Criterion**

If $(f_i, f_j)$, $(f_i, f_k)$ and $(f_j, f_k)$ are in the set of pairs $P$, denote $\text{lm}(f_\nu) = x^{\alpha_\nu}$. If $x^{\alpha_j} | \text{lcm}(x^{\alpha_i}, x^{\alpha_k})$ holds, then we can delete $(f_i, f_k)$ from $P$. 

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Viktor Levandovskyy (RWTH)
Gel’fand–Kirillov dimension

Let $R$ be an associative $K$–algebra with generators $x_1, \ldots, x_m$.

A degree filtration

Consider the vector space $V = Kx_1 \oplus \ldots \oplus Kx_m$.
Set $V_0 = K$, $V_1 = K \oplus V$ and $V_{n+1} = V_n \oplus V^{n+1}$.
For any fin. gen. left $R$–module $M$, there exists a fin.–dim. subspace $M_0 \subset M$ such that $RM_0 = M$.
An ascending filtration on $M$ is defined by $\{H_n := V_n M_0, \ n \geq 0\}$.

Definition

The Gel’fand–Kirillov dimension of $M$ is defined to be

$$\text{GK. dim}(M) = \limsup_{n \to \infty} \log_n(\dim_K H_n)$$
Gel’fand–Kirillov Dimension: Examples

Let \( \deg x_i = 1 \), consider filtrations up to degree \( d \). We have \( V_d = \{ f \mid \deg f = d \} \) and \( V^d = \{ f \mid \deg f \leq d \} \).

**Lemma**

Let \( A \) be a \( \mathbb{K} \)-algebra with PBW basis \( \{ x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mid \alpha_i \geq 0 \} \). Then \( \text{GK. dim}(A) = n \).

**Proof.**

\[
\dim V_d = \binom{d+n-1}{n-1}, \quad \dim V^d = \binom{d+n}{n}.
\]

Thus \( \binom{d+n}{n} = \frac{(d+n)\ldots(d+1)}{n!} = \frac{d^n}{n!} + \text{l.o.t} \), so we have \( \text{GK. dim}(A) = \lim \sup_{d \to \infty} \log_d \binom{d+n}{n} = n. \)

\( T = \mathbb{K}\langle x_1, \ldots, x_n \rangle \)

\[
\dim V_d = n^d, \quad \dim V^d = \frac{n^{d+1} - 1}{n-1}.
\]

Since \( \frac{n^{d+1} - 1}{n-1} > n^d \), we are dealing with so–caled exponential growth. In particular, \( \log_d n^d = d \log_d n = \frac{d}{\log_n d} \to \infty \), \( d \to \infty \).

Hence, \( \text{GK. dim}(T) = \infty \).
Gel’fand–Kirillov Dimension for Modules

There is an algorithm by Gomez-Torrecillaz et.al., which computes Gel’fand–Kirillov dimension for finitely presented modules over $G$-algebras.

\[ \text{GK\text{DIM}}(F) \]

Let $A$ be a $G$–algebra in variables $x_1, \ldots, x_n$.

- Input: Left generating set $F = \{f_1, \ldots, f_m\} \subset A^{r}$
- Output: $k \in \mathbb{N}$, $k = \text{GK. dim}(A^{r}/M)$, where $M = A\langle F \rangle \subseteq A^{r}$.

- $G = \text{LEFTGRÖBNERBASIS}(F) = \{g_1, \ldots, g_t\}$;
- $L = \{\text{lm}(g_i) = x^{\alpha_i} e_s \mid 1 \leq i \leq t\}$;
- $N = \mathbb{K}[x_1, \ldots, x_n] \langle L \rangle$;
- return $\text{Kr. dim}(K[x_1, \ldots, x_n]^{r}/N)$;
Ring-theoretic Properties of Weyl and shift algebras

**gl. dim(A), the global homological dimension of A**

- \( \text{gl. dim}(S) = 2n \),
- if \( \text{char } \mathbb{K} = 0 \), \( \text{gl. dim}(D) = n \),
- if \( \text{char } \mathbb{K} = p > 0 \), \( \text{gl. dim}(D) = 2n \).

**Z(A) = \{z \in A \mid za = az \ \forall a \in A\}, the center of A**

- if \( \text{char } \mathbb{K} = 0 \), \( Z(D) = Z(S) = \mathbb{K} \),
- if \( \text{char } \mathbb{K} = p > 0 \), \( Z(D) = \{x_i^p, \partial_i^p\} \).
- if \( \text{char } \mathbb{K} = p > 0 \), \( Z(S) = \{y_i^p - y_i, s_i^p\} \).

If \( \text{char } \mathbb{K} = 0 \), \( D(R) \) has no proper two–sided ideals.

In \( S(R) \), \( I_\gamma = S\langle \{s_i, y_i - \gamma_i\}\rangle_S \) is a family of such ideals for \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{K}^n \).
Thank you for your attention!

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